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SOME DISTRIBUTIONS OF SAMPLE MEANS

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1. **Summary.** It is shown that certain monomials in normally distributed quantities have stable distributions with index 2^{-k}. This provides, for $k > 1$, simple examples where the mean of a sample has a distribution equivalent to that of a fixed, arbitrarily large multiple of a single observation. These examples include distributions symmetrical about zero, and positive distributions.

Using these examples, it is shown that any distribution with a very long tail (of average order $\geq x^{-3/2}$) has the distributions of its sample means grow flatter and flatter as the sample size increases. Thus the sample mean provides *less* information than a single value. Stronger results are proved for still longer tails.

2 **Introduction.** This paper derives and exploits certain elementary expressions for stable distributions. The practicing statistician may be interested in the general discussion of results, going as far as Section 5. The reader interested in probability theory may be interested in

(i) the simple monomials in normally distributed quantities which are shown to be stable (Section 7)

(ii) the resulting bounds on the densities of these stable distributions (Section 8)

(iii) Theorem A, which forms a partial converse to the Central Limit Theorem.

It should be pointed out that examples of stable chance quantities arising from infinite series (Khintchine 1937, [2], [3]) and integrals (Levy 1935, [4]) are already known. These results form a natural part of broader investigations into

(i) the relative value of the mean, the median, and their competitors

(ii) the properties and distributions of simple functions of normally distributed quantities.

3. **Stable distributions.** One of the typical properties of the normal distribution with zero mean is that the distribution of the mean of a sample of n has the same shape but is compressed by the factor $\sqrt{n}$. The Cauchy distribution is well-known for the property that the mean of a sample of n has the same distribution as a single observation.

Statisticians have not widely appreciated the fact that there are symmetric, smooth distributions for every positive $\lambda \leq 2$, with the property that the distribution of the mean of a sample of n has the same shape as the original distribution but is spread out in the ratio $n^{(1-\lambda)/\lambda}$. These are the symmetric stable distributions of index λ.

It is interesting to note that if $\lambda = .001$, then the mean of a sample of two is 2^{999} times as variable as the mean of a sample of one For small λ the means become unduly variable with a rapidity which is difficult to comprehend.

4. **Outline of results.** Section 7 is devoted to the proof that certain monomials in normal variables are stable of index 2^{-k} for integral k. Both symmetrical and positive cases are shown to exist. For $k = 0$, the symmetrical case is the familiar Cauchy distribution, which is the distribution of Student's "t" on one degree of freedom, while the positive case for $k = 1$ is the distribution of Snedecor's "F" on ∞ and 1 degrees of freedom.

In Section 8 it is shown that the symmetrical stable distribution of index λ has a density which is

(i) bounded by a constant

(ii) bounded by a constant times $|x|^{-1-\lambda}$, for the values $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, for which elementary examples are available. It is conjectured that this is true for all $\lambda \leq 2$.

In section 9 it is shown that, if a distribution has one long tail in the sense that

$$\varliminf_{x \to \infty} |x|^{1+\lambda} P\{x < X \leq x + h\} > 0, \tag{1.1}$$

for some h and one of the above values of λ (the $\varliminf$ may be taken either as $x \to +\infty$ or as $x \to -\infty$), then the distribution of the sum of a sample of n spreads out as fast as for a stable distribution with the same value of λ. This may be restated for the mean as follows:

(i) *A distribution has a long tail of order* $|x|^{-(1+\lambda)}$ *if* (1.1) *holds for some* $h > 0$ *and choice of sign for* x.

(ii) *If the distribution has a density* $f(x)$, *then* (1.1) *is a consequence of*

$$f(x) \geq \frac{A}{1 + |x|^{1+\lambda}}, \qquad A > 0. \tag{1.2}$$

(iii) *The distribution of the mean of a sample of* n *will be said to spread out as fast as* n^k, *if the distance between any two percentage points for the mean of a sample of* n *is ultimately larger than a fixed multiple of* n^k.

(iv) THEOREM A. *If the distribution of* X *has at least one long tail of order* $|x|^{-(1+\lambda)}$, *where* $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \cdots$, *then the distribution of the mean of a sample of* n *values of* X *spreads out as fast as* $n^{(1-\lambda)/\lambda}$.

Section 10 presents a simple example of a distribution symmetric about zero with such long tails that

(i) the distribution of the sample mean spreads out faster than any power of n,

(ii) the median of a sample of any size fails to have finite moments of positive order, integral or fractional.

5. **Consequences for applied statistics.** The basic consequences of these results for applied statistics can be summarized in the following statements.

(a) The positions that the Cauchy distribution is an isolated case, or else an extreme example of pathology, are now untenable.

(b) The use of the mean of a sample as a measure of location (or, when dealing with positive distributions fixed at zero, as a measure of scale) implies a belief that the tails of the underlying distribution are not too long.

(c) It is probable that the relative efficiencies of mean and median are greatly affected by the length of the tail.

The importance of this last statement lies in the fact that direct empirical evidence about tail length is very hard to obtain. The mean is well known to be more efficient when the underlying distribution is normal. Normality of the tails of practical distributions is rarely based on firm empirical evidence. In these practical cases, greater efficiency of the mean should often not be assumed without empirical confirmation.

It may be argued that the results of this paper apply to the limit as $n \to \infty$ and to the behavior of the distribution near infinity, while the practical problems involve moderate values of n and the behavior of the distribution near its 5%, 1%, 0.1%, 95%, 99%, and 99.9% points. This is undoubtedly true, but the authors believe, and have some evidence to confirm, the following correspondence principle:

If certain mathematical tails imply certain asymptotic behavior, then similar practical tails imply similar behavior in moderate samples.

Here "mathematical tails" refers to behavior at infinity while practical tails run from the 5% to the 0.1% point and from the 95% to the 99.9% point.

It is of some interest to point out that Snedecor's "F" provides applications of Theorem A. If N values of F are averaged, where each was obtained on n_1 and n_2 degrees of freedom, then as N increases

(i) if $n_2 > 2$, the average converges to 1 (i.e. all percent points converge to 1), by the Central Limit Theorem

(ii) if $n_2 = 2$, the percent points of the average stay a finite distance away from each other, by Theorem A

(iii) if $n_2 = 1$, the percent points of the average separate from each other at least as fast as a constant times $\sqrt{N}$, by Theorem A.

The consequences of Theorem A follow from the asymptotic density of F, which is a constant times $F^{-1-\frac{1}{2}n_2}$.

6. **Notation and terminology.** Chance quantities (random variables) will be denoted by capitals and their values by lower case letters. The same letter will generally be used, so that x will frequently be a value of X.

The letter S, with or without indices, represents a standard deviate (normally distributed quantity with zero mean and unit variance). Unless otherwise specified all sets of chance quantities will be assumed to be independent.

Cumulative distribution functions will be referred to simply as "cumulatives" and will be denoted by capitals. Probability density functions will be referred to as "densities" and will be denoted by the corresponding lower case letters.

The convolution of two cumulatives F and G will be denoted by $F*G$. It is the cumulative of sums of two independent values, one from each distribution.

7 **Special stable distributions.** Cauchy (1853, [1]) recognized that distributions with characteristic functions of the form

$$e^{-\alpha|u|^\lambda}$$

were stable. A distribution is stable if whenever k and l are positive and A and B are independent chance quantities distributed according to the same law, then $kA + lB$ is distributed like a fixed multiple of A. It is known (Lévy 1937, [5], pp. 94 ff.) that any stable distribution has a characteristic function of the form

$$e^{-(\alpha+i\beta \operatorname{sgn} u)|u|^\lambda},$$

where $0 < \lambda \leq 2$, $\alpha > 0$, and $|\beta| \leq |\alpha \tan \frac{1}{2}\pi\lambda|$. Each stable distribution thus has an index λ such that $kA + lB$ and $(k^\lambda + l^\lambda)^{1/\lambda}A$ have the same distribution when A and B are a sample of two from the given distribution.

This section exhibits, for every integral k, simple monomials of standard deviates which have stable distributions of index 2^{-k}.

(7.1) THEOREM: *Let $S, S_0, S_1, S_2, \cdots$ be a sequence of independent standard deviates. Then*

(i) $C_0 = S/S_0$ *and* $P_0 = 1$
are stable of index $1 = 2^{-0}$.

(ii) $C_1 = S/S_0S_1^2 = C_0/S_1^2$ *and* $P_1 = 1/S_1^2 = P_0/S_1^2$
are stable of index $\frac{1}{2} = 2^{-1}$.

(iii) $C_2 = S/S_0S_1^2S_2^{2^2} = C_1/S_2^{2^2}$
and $P_2 = 1/S_1^2S_2^{2^2} = P_1/S_2^{2^2}$
are stable of index $\frac{1}{4} = 2^{-2}$.

(iv) *in general,* $C_k = C_{k-1}/S_k^{2^k}$ *and* $P_k = P_{k-1}/S_k^{2^k}$
are stable of index 2^{-k}.

The C_k are a sequence of symmetrically distributed chance quantities which are here presented as monomials in normally distributed chance quantities and whose stability properties imply for $k \geq 1$ that the distributions of means of samples spread out as the sample size increases. The P_k are a similar sequence, all of whose values are positive.

The stability properties of the C_k follow, directly, by means of elementary composition properties of characteritic functions, from

(7.2) LEMMA: *The characteristic function of C_k is*

$$E(e^{itC_k}) = \exp(-2|\tfrac{1}{2}t|^{2^{-k}}).$$

PROOF: The case $k = 0$ is the familiar Cauchy distribution. Denoting the normal cumulative by $N(s)$, it is seen that

$$\begin{aligned} E(e^{itC_0}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{its/s_0}\,dN(s)\,dN(s_0) \\ &= \int_{-\infty}^{\infty} \exp\left(-\tfrac{1}{2}t^2/s_0^2\right) dN(s_0) \\ &= e^{-|t|}. \end{aligned}$$

The second definite integral is well known (e.g. Formula 495 in B. O. Pierce's table). Assuming the result for $k-1$, write

$$E(e^{itC_k}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\,(itC_{k-1}/s_k^{2^k})\,dF_{k-1}(C_{k-1})\,dN(s_k)$$

$$= \int_{-\infty}^{\infty} \exp\,(-2\,|\,\tfrac{1}{2}t\,|^{2^{-k+1}}/s_k^2)\,dN(s_k)$$

$$= \exp\,(-2\,|\,\tfrac{1}{2}t\,|^{2^{-k}}),$$

precisely as in the derivation for $k = 0$.

The stability properties of the P_k follow, by completely analogous use of the moment generating function, from

(7.3) Lemma: *The moment generating function of P_k is*

$$E(e^{-tP_k}) = \exp(-2(\tfrac{1}{2}t)^{2^{-k}}),\ t \geq 0.$$

Proof: The trivial case $k = 0$ is verified directly, since $P_0 \equiv 1$. The induction from $k-1$ to k is identical with the derivation of (7.2), as is seen by writing

$$E(e^{-tP_k}) = \int_{-\infty}^{\infty}\int_{0}^{\infty} \exp\,(-tP_{k-1}/s_k^{2^k})\,dG_{k-1}(P_k)\,dN(s_k)$$

$$= \int_{-\infty}^{\infty} \exp\,(-2(\tfrac{1}{2}t)^{2^{-k+1}}/s_k^2)\,dN(s_k)$$

$$= \exp\,(-2(\tfrac{1}{2}t)^{2^{-k}})\,.$$

In order to verify the stability properties, consider distributions with characteristic functions of the form $exp(-d\,|\,t\,|^{\lambda})$. If A and B are independently distributed according to this distribution, then

$$E(e^{it(lA+mB)}) = E(e^{itlA})E(e^{itmB}) = e^{-d(l^\lambda+m^\lambda)|t|^\lambda}$$

for $l,\ m \geq 0$. Parallel application of the moment generating function yields precisely analogous results.

8. **Some auxiliary results.** It is the purpose of this section to establish some results concerning stable distributions. It will be convenient to state and prove some of these lemmas in general form.

(8.1) Lemma: *If X has a density $f(x)$ satisfying*

$$f(x) \leq A\,|\,x\,|^{-\alpha},$$

then X has finite negative moments of orders down to $-(1-\alpha)$.

Proof: If $-(1-\alpha) < \beta < 0$, then

$$|\,x\,|^{\beta}f(x) \leq A\,|\,x\,|^{-\alpha+\beta},$$

with $-\alpha+\beta > -1$. Now

$$\int_{-\infty}^{\infty} |\,x\,|^{\beta}\,f(x)\,dx \leq \int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{1} |\,x\,|^{\beta}\,f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$\leq \int_{-\infty}^{\infty} f(x)\,dx + \int_{-1}^{1} A\,|\,x\,|^{-\alpha+\beta}\,dx < \infty,$$

which proves the lemma.

(8.2) LEMMA: *If X has a density $f(x)$ satisfying*

$$f(x) \leq A \mid x \mid^{-\alpha}$$

and if Y has a density $g(y)$ and a finite negative moment of order $-(1-\alpha)$, then the density $h(x)$ of XY satisfies

$$h(x) \leq A_1 \mid x \mid^{-\alpha}.$$

PROOF: The density $h(x)$ satisfies

$$\begin{aligned} h(x) &= \int_{-\infty}^{\infty} \{f(x/t)g(t)/\mid t \mid\} \, dt \\ &\leq \int_{-\infty}^{\infty} A \mid t \mid^{\alpha} \mid x \mid^{-\alpha} g(t) \mid t \mid^{-1} dt \\ &= \left\{ \int_{-\infty}^{\infty} A \mid t \mid^{-(1-\alpha)} g(t) \, dt \right\} \mid x \mid^{-\alpha} = A_1 \mid x \mid^{-\alpha}. \end{aligned}$$

(8.3) LEMMA: *The density $h_k(y)$ of*

$$Y_k = S(S_1)^2(S_2)^{2^2} \cdots (S_k)^{2^k},$$

where, S, S_1, S_2, $\cdots$ S_k are independent standard deviates, satisfies

$$h_k(y) \leq A \mid y \mid^{-1+2^{-k}},$$

and hence Y_k has finite negative moments of all orders down to -2^{-k}.
PROOF: Let $g_k(x)$ be the density of

$$X_k = (S_k)^{2^k},$$

then

$$g_k(x) = (2\pi)^{-\frac{1}{2}} 2^{-k} \exp(-\tfrac{1}{2} x^{2^{1-k}}) x^{-1+2^{-k}},$$

whence

$$g_k(x) \leq A_1 \mid x \mid^{-1+2^{-k}}.$$

For $k = 0$ this is the desired result; the other cases follow by induction, using $Y_k = X_k Y_{k-1}$ and lemma (8.2). The final statement of the lemma then follows from lemma (8.1).
(8.4) THEOREM: *For $\lambda = 2^{-k}$, the density $m_\lambda(x)$ of C_k satisfies*

$$(*) \quad m_\lambda(x) \leq A \mid x \mid^{-(1+2^{-k})} = A \mid x \mid^{-(1+\lambda)},$$

and also

$$(**) \quad m_\lambda(x) \leq A_2 .$$

PROOF: By definition, $C_k = S/Y_k$. By lemma (8.3) the density of Y_k satisfies

$$h_k(y) \leq A_1 \mid y \mid^{-1+2^{-k}}.$$

The density of $1/Y_k$ satisfies

$$l_k(z) = \frac{1}{z^2} h_k(1/z)$$
$$\leq |z|^{-2} A_1 |z|^{1-2^{-k}} = A_1 |z|^{-(1+2^{-k})}.$$

Since S has a finite moment of order 2^{-k}, it follows from lemma (8.2) that the density of S/Y_k satisfies the desired relation (*). Since S has finite moments of all positive orders, so does S^{2^k} and therefore Y_k. Thus $1/Y_k$ has moments of all negative orders, including -1. Since the density of S is bounded, lemma (8.2) implies the same for S/Y_k and hence for C_k. This completes the proof of the theorem.

9. **Distributions with a long tail.** The purpose of this section is to prove

(9.1) THEOREM: *If D has a cumulative $F(x)$ such that for some $h > 0$, either*

$$\varliminf_{x\to+\infty} \frac{F(x+h) - F(x)}{|x|^{-(1+\lambda)}} > 0, \quad or \quad \varliminf_{x\to-\infty} \frac{F(x+h) - F(x)}{|x|^{-(1+\lambda)}} > 0,$$

where $\lambda = 2^{-k}$ for $k = 0, 1, 2, \cdots$, and if $k_n(\alpha)$ is the α-point (100α percent point) of the distribution of sums of n independent values of D, then

$$\varliminf_{n} \frac{K_n(\alpha_1) - K_n(\alpha_2)}{n^{1/\lambda}} > 0,$$

whenever $\alpha_1 > \alpha_2$.

We begin with some lemmas.

(9.2) LEMMA: *If*

$$\left.\begin{aligned} F(x) &= \beta F'(x) + (1-\beta)F''(x), \\ G(x) &= \beta F'(x) + (1-\beta)\mathbf{1}(x), \end{aligned}\right\} \qquad 0 \leq \beta \leq 1$$

where $F'(x)$ is a cumulative symmetric about zero and unimodal, $F''(x)$ is a cumulative symmetric about zero, and $\mathbf{1}(x)$ is the cumulative concentrated at zero (whence $F(x)$ and $G(x)$ are cumulatives), and if $F_n(x)$ and $G_n(x)$ are the cumulatives of sums of samples of n from $F(x)$ and $G(x)$ respectively, then

$$F_n(x) \leq G_n(x), \quad x > 0,$$
$$F_n(x) \geq G_n(x), \quad x < 0.$$

PROOF: We begin with the case $n = 2$, where

$$F_2 = \beta^2 F'*F' + 2\beta(1-\beta)F'*F'' + (1-\beta)^2 F''*F''$$

and

$$G_2 = \beta^2 F'*F' + 2\beta(1-\beta)F' + (1-\beta)^2\mathbf{1}.$$

The lemma will have been proved for $n = 2$ if we can show that

$$F'*F''(x) \leq F'(x), x > 0,$$
$$F'*F''(x) \geq F'(x), x < 0.$$

Now, if $x > 0$,

$$F'*F''(x) = \int_{-\infty}^{\infty} F'(x - s)\, dF''(s)$$

$$= \int_0^{\infty} \{F'(x - s) + F'(x + s)\}\, dF''(s)$$

$$\leq 2 \int_0^{\infty} F'(x)\, dF''(s) = F'(x),$$

where the first equality follows from the symmetry of F'', the inequality follows from the unimodality of F', and the last equality follows from the symmetry of F''. The inequality is reversed if $x < 0$.

For general n,

$$F_n = \sum_k \binom{n}{k} \beta^k (1 - \beta)^{n-k} F'_k * F''_{n-k},$$

$$G_n = \sum_k \binom{n}{k} \beta^k (1 - \beta)^{n-k} F'_k,$$

where F'_k (the convolution of k copies of F') is the cumulative for sums of k independent values from F', and F''_k is similarly related to F''. Since F'_k is unimodal and symmetric and since F''_{n-k} is symmetric, the same argument can be applied term by term to complete the proof of the lemma. The requirement that F'' be symmetric could be replaced by the formally weaker condition that $F''_k(0) = \frac{1}{2}$ for all k.

(9.3) Lemma: *If*

$$F(x) = \beta F_{(\lambda)}(x) + (1 - \beta) 1(x), \qquad 0 \leq \beta \leq 1,$$

where $F_{(\lambda)}(x)$ *is the cumulative of* C_k, *with* $\lambda = 2^{-k}$, *and if* $K_n(\alpha)$ *is as defined in* (9.1), *then*

$$\lim n^{-1/\lambda} K_n(\alpha) = \beta^{1/\lambda} K_{(\lambda)}(\alpha),$$

where $K_{(\lambda)}(\alpha)$ *is the* α*-point for* $F_{(\lambda)}(x)$.

Proof: Let F_n and $F_{(\lambda)n}$ be the cumulatives of sums of n from F and $F_{(\lambda)}$ respectively, whence

$$F_{(\lambda)n}(x) = F_{(\lambda)}(n^{1/\lambda} x).$$

Then

$$F_n(x) = \sum_k \binom{n}{k} \beta^k (1 - \beta)^{n-k} F_{(\lambda)k}(x)$$

$$= \sum_k \binom{n}{k} \beta^k (1 - \beta)^{n-k} F_{(\lambda)}(k^{1/\lambda} x).$$

The characteristic function of $(n\beta)^{1/\lambda}x$ is

$$E(e^{it(n\beta)^{1/\lambda}x}) = \sum_k \binom{n}{k}\beta^k(1-\beta)^{n-k}\exp(-d\,|\,(n\beta)^{-1/\lambda}k^{1/\lambda}t\,|^\lambda,$$
$$= \sum_k \binom{n}{k}\beta^k(1-\beta)^{n-k}\exp\left(-\frac{dk}{n\beta}|\,t\,|^\lambda\right),$$

where the characteristic function associated with $F_{(\lambda)}(x)$ is $exp(-d\,|\,t\,|^\lambda)$. Thus we have to deal with

$$\exp\left(-\frac{dk}{n\beta}|\,t\,|^\lambda\right)$$

where k has a binomial distribution with mean $n\beta$ and variance $n\beta(1-\beta)$, so that $k/n\beta$ converges stochastically to unity. This implies that

$$\lim E(e^{it(n\beta)^{1/\lambda}x}) = e^{-d|t|^\lambda}$$

uniformly in every finite interval, whence $(n\beta)^{1/\lambda}X$ converges stochastically to C_k, which completes the proof of the lemma.

(9.4) LEMMA: *If the symmetric cumulative $F(x)$ has a density $f(x)$, and if constants c_1 and c_2 exist such that*

$$f(x) \geq \min(c_1, c_2\,|\,x\,|^{-(1+\lambda)}),$$

where $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, then, if $\alpha \neq \frac{1}{2}$,

$$\underline{\lim}\,|\,n^{-1/\lambda}K_n(\alpha)\,| > 0.$$

PROOF: According to theorem (8.4) there are constants d_1 and d_2 such that the density of C_k is bounded by $\min(d_1, d_2\,|\,x\,|^{-(1+\lambda)})$. Hence

$$\frac{F(x) - \beta F_{(\lambda)}(x)}{1-\beta}$$

is monotone when $\beta = \min(c_1/d_1, c_2/d_2)$, and hence is a distribution function. By lemma (9.2) the α-points of F lie outside those of $\beta F_{(\lambda)}(x) + (1-\beta)1(x)$, and these, by lemma (9.3), increase at least as fast as $An^{-1/\lambda}$.

(9.5) LEMMA: *If the density of D exists and equals $f(x)$, and if either*

$$\underline{\lim}_{x\to+\infty}|\,x\,|^{1+\lambda}f(x) > 0,$$

or

$$\underline{\lim}_{x\to-\infty}|\,x\,|^{1+\lambda}f(x) > 0,$$

where $\lambda = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$, then, for $\alpha_1 > \alpha_2$,

$$\underline{\lim}_{n\to\infty}n^{-1/\lambda}\{K_n(\alpha_1) - K_n(\alpha_2)\} > 0.$$

PROOF: Let D_1 and D_2 be independent with the distribution of D. Then $D_1 - D_2$ has a symmetric density given by

$$g(x) = \int_{-\infty}^{\infty} f(x + s) f(s) ds.$$

If

$$\varliminf_{x \to +\infty} | x |^{1+\lambda} f(x) > 0,$$

then for suitable h and $\epsilon > 0$,

$$f(x) \geq \epsilon | x |^{-(1+\lambda)}, \text{ for all } x \geq h.$$

Therefore, for $x \geq 0$, writing $\gamma = -(1 + \lambda)$,

$$g(x) \geq \int_{h}^{h+1} f(x + s) f(s) ds \geq \epsilon^2 | h + 1 + x |^\gamma | h + 1 |^\gamma = b_1 | b_2 + x |^\gamma.$$

Now

$$b_1 | b_2 + x |^\gamma \geq \min \{ b_1 2^\gamma b_2^\gamma , \; b_1 2^\gamma | x |^\gamma \}$$

and hence, for $x \geq 0$ and suitable $c_1 > 0$, $c_2 > 0$,

$$g(x) \geq \min \{ c_1 , c_2 | x |^\gamma \}.$$

Since $g(x)$ is symmetric, this is also true for $x < 0$. If

$$\varliminf_{x \to -\infty} | x |^{1+\lambda} f(x) > 0,$$

then a similar argument proves the same result.

Let $K_{\delta_n}(\alpha)$ be the α-point for the sum of n values of $D_1 - D_2$ and $K_n(\alpha)$ be the α-point for the sum of n values of D. The most elementary relation between these functions is

$$| K_{\delta_n}(\tfrac{1}{2} \pm \tfrac{1}{2}(\alpha_1 - A_2)^2) | \leq | K_n(\alpha_1) - K_n(\alpha_2) |.$$

To see this, observe that the sum of a sample of n values of $D_1 - D_2$ is the difference of the sums of two independent samples of n values of D, and that there is a probability of $(\alpha_1 - \alpha_2)^2$ that both of these sums will fall between $K_n(\alpha_1)$ and $K_n(\alpha_2)$. Thus the intervals $(-| K_n(\alpha_1) - K_n(\alpha_2) |, 0)$ and $(0, | K_n(\alpha_1) - K_n(\alpha_2) |)$ are each occupied by the difference with probability $\geq \frac{1}{2}(\alpha_1) - \alpha_2)^2$. Since $K_{\delta_n}(\frac{1}{2}) = 0$, the relation follows. Hence, if $\alpha_1 > \alpha_2$,

$$\varliminf n^{-1/\lambda} \{ K_n(\alpha_1) - K_n(\alpha_2) \} \geq \varliminf n^{-1/\lambda} K_{\delta_n}(\tfrac{1}{2} \pm \tfrac{1}{2}(\alpha_1 - \alpha_2)^2)$$

and by lemma (9.4) applied to the distribution of $D_1 - D_2$ this latter $\varliminf$ is positive, which completes the proof of the lemma.

With the ground prepared, it is now possible to complete the

PROOF OF THE THEOREM: Let h be chosen so that

$$\varliminf_{x \to +\infty} | x |^{1+\lambda} (F(x + h) - F(x)) > 0.$$

This can always be done, if X is replaced by $-X$ when necessary. Let U have the uniform distribution on the interval (0, 1) and consider the variable $D + hU$. This variable has a density given by

$$g(x) = \frac{F(x + h) - F(x)}{h},$$

and, therefore,

$$\lim_{x \to +\infty} |x|^{1+\lambda} g(x) > 0.$$

Let $K_n(\alpha)$ be the α-point for the sum of a sample of n values of D, and let $K_n^*(\alpha)$ be the α-point for the sum of a sample of n values of $D + hU$. Since $|hU| \leq h$, it follows that

$$|K_n(\alpha) - K_n^*(\alpha)| \leq nh.$$

Therefore, if $1/\lambda > 1$ and $\alpha_1 > \alpha_2$,

$$\underline{\lim}_{x \to \infty} n^{-1/\lambda} \{K_n(\alpha_1) - K_n(\alpha_2)\} = \underline{\lim}_{x \to \infty} n^{-1/\lambda} \{K_n^*(\alpha_1) - K_n^*(\alpha_2)\},$$

and by lemma (9.5) the latter $\underline{\lim}$ is positive.

The case of $\lambda = 1$ requires a slightly more delicate argument. The sum of a sample of n values of hU is asymptotically normally distributed, and hence it is less than $A_\beta n^{\frac{1}{2}}$, for a suitable A_β, with probability β. Therefore

$$K_n(\alpha\beta) \leq K_n^*(\alpha\beta) \leq K_n(\alpha) + A_\alpha n^{\frac{1}{2}}$$

and the same process yields the desired conclusion.

10. A distribution with very long tails. A somewhat pathological example is provided by the symmetric cumulative

$$F(x) = \frac{1}{ln(e^2 + |x|)}, \qquad x \leq 0,$$

$$F(x) = 1 - \frac{1}{ln(e^2 + |x|)}, \; x \geq 0,$$

which has the density

$$f(x) = \frac{1}{(e^2 + |x|)\{ln(e^2 + |x|)\}^2}.$$

Since

$$\lim_{x \to \infty} |x|^{1+\lambda} f(x) = \infty \quad \text{for all} \quad \lambda > 0,$$

it follows from theorem (9.1) that the distribution of the sum of a sample of n values of X spreads out faster than any power of n. The same must therefore

be true of the mean of a sample of n. There is clearly no use in taking any kind of mean of such a sample.

There will, of course, be something to gain by taking the median of a sample of $2n + 1$, since the distribution of the median always shrinks together as $n \to \infty$, and whenever, as is true here, the density is finite and continuous at the population median, the distributions of the sample medians shrink toward the population median

This does not prevent some pathology, however, since the cumulative for the median of $2n + 1$ takes the form

$$\frac{(2n+1)!}{(n!)^2(n+1)} \{F(x)\}^{n+1}\{1 + P(F(x))\},$$

where $P(t)$ is a polynomial of degree n with no constant term. Thus, for large negative values of x, the cumulative for the median is asymptotically

$$\frac{(2n+1)!}{(n!)^2(n+1)} \cdot \frac{1}{\{ln(e^2 + |x|)\}^n}$$

and the corresponding density is asymptotically

$$\frac{(2n+1)!\,n}{(n!)^2(n+1)\{ln(e^2+|x|)\}^{n+1}(e^2+|x|)}$$

and it follows that the median has no moments of any positive order, integral or fractional. This is true no matter how large the sample used!

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UNBIASED ESTIMATES FOR CERTAIN BINOMIAL SAMPLING PROBLEMS WITH APPLICATIONS[1]

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1. Introduction. The purpose of this paper is to present some theorems with applications concerning unbiased estimation of the parameter p (fraction defective) for samples drawn from a binomial distribution. The estimate constructed is applicable to samples whose items are drawn and classified one at a time until the number of defectives i, and the number of nondefectives j, simultaneously agree with one of a set of preassigned number pairs. When this agreement takes place, the sampling operation ceases and an unbiased estimate of the proportion p of defectives in the population may be made. Some examples of this kind of sampling are ordinary single sampling in which n items are observed and classified as defective or nondefective; curtailed single sampling where it is desired to cease sampling as soon as the decision regarding the lot being inspected can be made, that is as soon as the number of defectives or nondefectives attain one of a fixed pair of preassigned values, double, multiple, and sequential sampling. In the cases of double and multiple sampling the subsamples may be curtailed when a decision is reached, while for sequential sampling the process may be truncated, i e an upper bound may be set on the amount of sampling to be done. In section 3 expressions are given for the unique unbiased estimates of p for single, curtailed single, curtailed double, and sequential sampling.

One or two of the illustrative examples of section 3 may be of interest because their rather bizarre results suggest that some estimate other than an unbiased estimate may be preferable; but the discussion of estimates other than unbiased ones is outside the scope of this paper.

2. The estimate $\hat{p}$. For the purposes of the present paper the word point will refer only to points in the xy-plane with nonnegative integral coordinates.

We shall need the following nomenclature. A region R is a set of points containing (0, 0). The point (x_2, y_2) is *immediately beyond* (x_1, y_1) if either $x_2 = x_1 + 1$, $y_2 = y_1$ or $x_2 = x_1$, $y_2 = y_1 + 1$. A *path in R* from the point α_0 to the point α_n is a finite sequence of points $\alpha_0, \alpha_1, \cdots, \alpha_n$ such that α_i $(i > 0)$ is immediately beyond α_{i-1}, and $\alpha_i \epsilon R$ with the possible exception of α_n. A *boundary* point, that is, an element of the boundary B of R, is a point not in R which is the last point α_n of a path from the origin. *Accessible* points are the points in R which can be reached by paths from the origin, while *inaccessible* points are the points which cannot be reached by any path from the origin.

[1] This paper was originally written by Mosteller and Savage. A communication from M A Girshick revealed that he had independently discovered for the sequential probability ratio test the estimate $\hat{p}(\alpha)$ given here and demonstrated its uniqueness. For purposes of publication it seemed appropriate to present the results in a single paper.

All points are thus divided into three mutually exclusive categories: accessible, inaccessible, and boundary points. The *index of a point* is the sum of its coordinates, and the *index of a region* is the least upper bound of the indices of its accessible points. A *finite* region is a region for which the indices of the accessible points are less than some number n. In particular a region containing only a finite number of points is finite.

Paths may be thought of as arising by a random process such that a path reaching $\alpha_i = (x, y)$, $\alpha_i \epsilon R$, will be extended to $\alpha_{i+1} = (x, y + 1)$ with probability p or to $\alpha'_{i+1} = (x + 1, y)$ with probability $q = 1 - p$. We exclude $p = 0, 1$ unless these values are specifically mentioned. When a path is extended to a boundary point of R the process ceases. It is clear from the definitions that for a finite region R, paths from the origin cannot include more points than $n + 2$ where n is the index of the region This means that a path from the origin cannot escape from a finite region and that the probability that it strikes some boundary point is unity. It is clear that each path from the origin to a boundary point or an accessible point has probability $p^y q^x$, if the point has coordinates (x, y). We will need the following statements which are immediate consequences of the discussion above:

A. *The probability of a boundary point or an accessible point being included in a path from the origin is* $P(\alpha) = k(\alpha)p^y q^x$, *where* $k(\alpha)$ *is the number of paths from the origin to the point. We shall call* $P(\alpha)$ *the probability of the point.*

B. *For a finite region* $\sum_{\alpha \epsilon B} P(\alpha) = 1$, i.e. *the sum of the probabilities of the boundary points is unity*

Any region for which $\sum_{\alpha \epsilon B} P(\alpha) = 1$ will be called a *closed* region.

Of course, all finite regions are closed; but it is convenient to have a condition such as that supplied by the following theorem guaranteeing the closure of some infinite regions as well

THEOREM 1. *A sufficient condition*[2] *that a region* R *be closed is that* $\liminf_{n \to \infty} A(n)/\sqrt{n} = 0$, *where* $A(n)$ *is the number of accessible points of index* n.

PROOF We consider the ascending sequence of finite regions R_n, each consisting of the points of R whose indices are less than n. The boundary B_n of R_n can be written as the set theoretic union $K_n \cup A_n$, where K_n is $B_n \cap B$, and A_n are the accessible points of R of index n. If $\alpha \epsilon B_n$ and $P_n(\alpha)$ is the probability of α with respect to R_n, it is easily seen that for $\alpha \epsilon K_n$, $P_n(\alpha) = P(\alpha)$. Since every point of B is ultimately contained in the ascending sequence K_n,

$$\sum_{\alpha \epsilon B} P(\alpha) = \lim_{n \to \infty} \sum_{\alpha \epsilon K_n} P(\alpha) = \lim_{n \to \infty} \sum_{\alpha \epsilon K_n} P_n(\alpha) \leq 1,$$

the inequality being a consequence of statement B. But $\sum_{\alpha \epsilon A_n} P_n(\alpha)$ is monotonically decreasing because $\sum_{\alpha \epsilon K_n} P_n(\alpha)$ is monotonically increasing with n while $\sum_{\alpha \epsilon B_n} P_n(\alpha) = 1$, from statement B.

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[2] If it is desired to admit $p = 0, 1$, the existence of boundary points $(x,0)$ or $(0,y)$ respectively must be postulated

If we can show $\lim_{n\to\infty} \sum_{\alpha \epsilon A_n} P_n(\alpha) = 0$ under the condition of the theorem, the proof is complete For any point $\alpha \epsilon A_n$, $P_n(\alpha) = k_n(\alpha)p^v q^{n-v}$ which for fixed p is $O(1/\sqrt{n})$. The sum over A_n is $O(A(n)/\sqrt{n})$ and therefore since the hypothesis of the theorem implies that $A(n)/\sqrt{n}$ attains arbitrarily small values for arbitrarily large values of n, the sum in question decreases monotonically to zero.

COROLLARY. *If the number of accessible points of R of index n is bounded, the region is closed.*

That the condition given in Theorem 1 is not a necessary condition may be seen by examining the region R consisting of all points except points of the form $(2x + 1, 2y + 1)$ and $(3, 0)$ and $(0, 3)$.

THEOREM 2. *If R is closed and R contains S, S is closed.*

PROOF. The proof is essentially similar to that of Theorem 1.

Any reasonable estimate of p will be a function defined on the boundary points, because the boundary points constitute, so to speak, a sufficient statistic for p. That is, the probability of any path from (0, 0) given the boundary point α at which it terminates is independent of p, and is in fact $1/k(\alpha)$.

We shall construct an unbiased estimate of p for closed regions R, that is a function $\hat{p}(\alpha)$, $\alpha \epsilon B$, such that $\sum_{\alpha \epsilon B} \hat{p}(\alpha)P(\alpha) = p$ (absolutely convergent).[3]

CONSTRUCTION. *Let $k^*(\alpha)$ be the number of paths in R from the point (0, 1) to the boundary point α, and let $\hat{p}(\alpha) = k^*(\alpha)/k(\alpha)$.* We remark that the definitions imply $k^*((0, 1)) = 1$, when (0, 1) is a boundary point.

THEOREM 3. *For any closed region R $\hat{p}(\alpha)$ is an unbiased estimate of p.*

PROOF:

$$\sum_{\alpha \epsilon B} \hat{p}(\alpha) P(\alpha) = \sum_{\alpha \epsilon B} \frac{k^*(\alpha)}{k(\alpha)} k(\alpha) p^y q^x$$
$$= \sum_{\alpha \epsilon B} k^*(\alpha) p^y q^x.$$

If (0, 1) is a boundary point, then $k^*((0, 1)) = 1$ and $k^*(\alpha) = 0$, $\alpha \neq (0, 1)$, in which case the sum in question consists of the single term p. If (0, 1) is not a boundary point, consider the region R' obtained by deleting (0,1) from R, and $k'(\alpha)$, the number of paths in R' from the origin to the boundary point α of R.

$$k^*(\alpha) = k(\alpha) - k'(\alpha)$$

$$\sum_{\alpha \epsilon B} k^*(\alpha) p^y q^x = \sum_{\alpha \epsilon B} k(\alpha) p^y q^x - \sum_{\alpha \epsilon B} k'(\alpha) p^y q^x$$
$$= 1 - \sum_{\alpha \epsilon B} k'(\alpha) p^y q^x.$$

Now R' is closed (Theorem 2); except for (0, 1) every boundary point of R' is

[3] Even if such a sum were p for a region which was not closed, we would not call the estimate an unbiased estimate

easily seen to be a boundary point of R; and $k'(\alpha)$ vanishes except for the boundary points of R'. Therefore

$$p + \sum_{\alpha \in B} k'(\alpha) p^y q^x = 1,$$

and the proof is complete.

It is clear from the construction that $0 \leq \hat{p}(\alpha) \leq 1$; this is rather satisfying, since an estimate of p outside of these bounds would be received with some misgivings.

Theorem 3 may be generalized to yield unbiased estimates of linear combinations of functions of the form $p^t q^u$ provided the points (u, t) are not inaccessible points We need only let the point (u, t) play the role of $(0, 1)$. Even though the point (u, t) is inaccessible it may be possible to represent $p^t q^u$ as a polynomial, none of whose terms correspond to inaccessible points.

It is clear from Theorem 1 that $\hat{p}(\alpha)$ is an unbiased estimate of p for the usual sequential binomial tests, but the computation may be quite heavy. It should be noted that the coordinate system used here differs slightly from the coordinate system customarily used in sequential analysis. The custom is to let the x coordinate represent the number of items inspected, whereas we use it to represent the number of nondefectives, this is the only difference between the coordinates. We understand that in applications the customary procedure seems preferable, but we find the present coordinates more convenient for the purposes of ths article.

In general $\hat{p}$ is not the only unbiased estimate of p. A necessary condition for uniqueness is that the region be *simple*, that is that all the points between any two accessible points on the line $x + y = n$ be accessible points. In other words no accessible points of index n shall be separated on the line $x + y = n$ by inaccessible points or boundary points.

THEOREM 4. *A necessary condition that the estimate $\hat{p}$ be the unique unbiased estimate for the closed region R is that R be simple.*

PROOF. For a region that is not simple we shall construct a function $m(\alpha)$ not identically zero, such that

$$\sum_{\alpha \in B} m(\alpha) P(\alpha) = 0. \tag{1}$$

But $\hat{p}(\alpha) + m(\alpha)$ will be an unbiased estimate of p different from $\hat{p}$.

Suppose we have a closed region R which is not simple. We consider the lowest index n where the accessible points are separated. There will be at least one uninterrupted sequence of points between some pair of accessible points that are not accessible points. It is easy to see that all the points of this uninterrupted sequence are boundary points of R. Let this sequence be the points $\alpha_i = (x_0 - i, y_0 + i)$, $i = 0, 1, \cdot \cdot, t$, $x_0 + y_0 = n$. To begin the construction of $m(\alpha)$ let $m(\alpha_j) = (-1)^j/k(\alpha)$, $0 \leqq j \leqq t$. The coordinates of the point α'' above the top point of the sequence are $(x_0 - t, y_0 + t + 1)$, and the number of paths from α'' to any point on the boundary is $l''(\alpha)$, where if α'' is a boundary point the number of paths $l''(\alpha'') = 1$; similarly $\alpha' = (x_0 + 1, y_0)$ and $l'(\alpha)$ is

the number of paths from α' to the boundary point α with the same convention if α' is a boundary point. To complete the construction of $m(\alpha)$, let $m(\alpha) = -[l'(\alpha) + (-1)^t l''(\alpha)]/k(\alpha)$ for boundary points not members of the sequence under consideration. Before proceeding to check equation (1), we show that

$$\sum_{\alpha \in B} l'(\alpha)\, p^y q^x = p^{y_0} q^{x_0+1}\,;\ \sum_{\alpha \in B} l''(\alpha)\, p^y q^x = p^{y_0+t+1} q^{x_0-t}. \tag{2}$$

Because of symmetry we need only carry out the demonstration for the first sum. If α' is a boundary point $l'(\alpha') = 1$, and for all other points α $l'(\alpha) = 0$, and the sum is the single term $p^{y_0} q^{x_0+1}$. If α' is not a boundary point consider the region obtained by deleting α' from R and the corresponding $k'(\alpha)$, the number of paths from (0, 0) to the boundary points of the new closed region R'. Every boundary of R' except α' is a boundary point of R Let us extend the definition of $k'(\alpha)$ to the whole boundary of R by defining $k'(\alpha) = 0$ for α not in the boundary B' of R'. Then it is easy to see that

$$k(\alpha) = k'(\alpha')l'(\alpha) + k'(\alpha).$$

Now

$$\begin{aligned}
1 &= \sum_{\alpha \in B} k(\alpha) p^y q^x \\
&= k'(\alpha') \sum_{\alpha \in B} l'(\alpha) p^y q^x + \sum_{\alpha \in B} k'(\alpha) p^y q^x \\
&= k'(\alpha') \sum_{\alpha \in B} l'(\alpha) p^y q^x + 1 - k'(\alpha') p^{y_0} q^{x_0+1}
\end{aligned}$$

establishing equation (2)

We now check that $m(\alpha)$ satisfies equation (1):

$$\begin{aligned}
\sum_{\alpha \in B} m(\alpha)k(\alpha)p^y q^x &= \sum_{j=0}^{t} (-1)^j p^{y_0+j} q^{x_0-j} - \sum_{\alpha \in B} l'(\alpha) p^y q^x - \sum_{\alpha \in B} (-1)^t l''(\alpha) p^y q^x \\
&= \sum_{j=0}^{t} (-1)^j p^{y_0+j} q^{x_0-j} - p^{y_0} q^{x_0+1} - (-1)^t p^{y_0+t+1} q^{x_0-t} \\
&= p^{y_0} q^{x_0-t} \left(\sum_{j=0}^{t} (-1)^j p^j q^{t-j} - q^{t+1} - (-1)^t p^{t+1} \right) \\
&= 0.
\end{aligned}$$

THEOREM 5. *A necessary condition that $\hat{p}(\alpha)$ be a unique unbiased estimate of p for the closed region R is that there be no closed region R' whose boundary is a proper subset of the boundary of R.*

PROOF. Again supposing that the condition is not satisfied we shall construct a function $m(\alpha)$ not identically zero such that equation (1) is satisfied. Let $k'(\alpha)$ be the number of paths in R' to α in B of R, understanding, of course, that $k'(\alpha) = 0$ if α is not in B' of R'. Consider $m(\alpha) = 1 - k'(\alpha)/k(\alpha)$, $m(\alpha)$ is not identically zero because $k'(\alpha)$ vanishes for at least one α, but $k(\alpha)$ does not. From the closure of R and R' it is obvious that $m(\alpha)$ satisfies equation (1).

Two simple examples will suffice to show that neither simplicity nor the condition of Theorem 5 is alone sufficient to insure the uniqueness of $\hat{p}$. The region consisting of the points whose coordinates are given in the following configuration and whose boundary points are

$$\begin{array}{lllll}
x & & & & \\
(0, 3) & x & & & \\
(0, 2) & x & & & \\
(0, 1) & (1, 1) & x & x & \\
(0, 0) & (1, 0) & (2, 0) & (3, 0) & x
\end{array}$$

indicated by the x's satisfies the condition of Theorem 5 but is not simple. On the other hand the region consisting of all points for which $y < 3$, except for the two points $(1, 0)$, $(1, 1)$ is simple but does not satisfy the conditions of Theorem 5, because the region consisting of all points except $(1, 0)$ with $y < 3$ can play the role of R'.

The authors are unable to decide whether the two conditions together guarantee the uniqueness of $\hat{p}$ as an unbiased estimate of p, and supply the following sufficient condition which is adequate for many practical purposes.

THEOREM 6. *A sufficient condition that a closed region have $\hat{p}(\alpha)$ a unique unbiased estimate of p is that the region be simple and that there exist g, h $(0 < g, h \leqq 1)$ such that for all boundary points $| gx - hy | < M$.*

PROOF. If there were an unbiased estimate of p different from $\hat{p}$, subtracting it from p would yield an equation of the form (sum absolutely convergent):

$$\sum_{\alpha \epsilon B} m(\alpha) p^y q^x = 0, \tag{3}$$

where $m(\alpha)$ is not identically zero. But this will be shown to be impossible. If $m(\alpha)$ were not identically zero, there would be an α_0 such that $m(\alpha_0) \neq 0$ and 1) $m(\alpha) = 0$ for all boundary points of index less than that of α_0, and 2) one of the coordinates of α_0 is less than the corresponding coordinate of any other boundary point for which $m(\alpha) \neq 0$. This follows easily from the simplicity requirement which implies that the boundary points of index n are broken into two sets a) those whose y coordinates are less than the y coordinates of the accessible points of index n, and b) those whose x coordinates are less than the x coordinates of the accessible points of index n.[4] Since the situations a) and b) are symmetrical we suppose without loss of generality that α_0 is a boundary point whose y coordinate is less than that of any other boundary point with $m(\alpha) \neq 0$. Equation (3) may be written

$$m(\alpha_0) p^{y_0} q^{x_0} + p^{y_0+1} \sum_{\substack{\alpha \epsilon B \\ \alpha \neq \alpha_0}} m(\alpha) p^{y-y_0-1} q^x = 0, \tag{4}$$

[4] It will be seen as the proof proceeds that if there are no boundary points to which alternative a) applies, the restriction $g > 0$ may be removed and replaced by $g \geqq 0$, similarly if there are no boundary points to which b) applies the condition $h > 0$ may be replaced by $h \geqq 0$.

where the exponents appearing in the sum are nonnegative. But it will be shown that for sufficiently small p

$$q^{x_0} \mid m(\alpha_0) \mid > p \mid \sum_{\substack{\alpha \in B \\ \alpha \neq \alpha_0}} m(\alpha) p^{\nu - \nu_0 - 1} q^x \mid, \tag{5}$$

which contradicts equation (4). Now

$$\begin{aligned} \mid \Sigma m(\alpha) p^{\nu-\nu_0-1} q^x \mid &\leq \Sigma \mid m(\alpha) \mid p^{\nu-\nu_0-1} q^x \\ &\leq \Sigma \mid m(\alpha) \mid p^{\nu-\nu_0-1} q^{x-(h\nu_0+h+M+gx-h\nu)/g} \\ &= q^{-M/g} \Sigma \mid m(\alpha) \mid (pq^{h/g})^{\nu-\nu_0-1} \\ &\leq q^{-[h(\nu_0+1)+2M]/g} \Sigma \mid m(\alpha) \mid p^{\nu-\nu_0-1} q^x, \end{aligned} \tag{6}$$

where all the summations range over the values indicated in (5). The summation indicated in (5) is thus seen to be dominated by a convergent power series in $pq^{h/g}$.

Thus Theorem 6 shows that $\hat{p}$ is a unique estimate for the sequential binomial tests.

THEOREM 7. *A necessary and sufficient condition that $\hat{p}$ be the unique unbiased estimate of p for a closed finite region R is that R be simple.*

PROOF. The proof follows immediately from Theorems 4 and 6.

3. Applications and illustrative examples.

A. Single sampling. In single sampling a random sample of n items is drawn from a lot containing items each of which is either defective or nondefective. It is customary to estimate p, the proportion defective by the unbiased estimate i/n, where i is the number of defectives observed. The boundary of the region defined by a single sampling plan consists of all points of index n. Now $k((n-i, i)) = \binom{n}{i}$ and $k^*((n-i, i-1)) = \binom{n-1}{i-1}$. *Consequently the unique unbiased estimate of p is*

$$\hat{p}((n-i, i)) = \binom{n-1}{i-1} \Big/ \binom{n}{i} = i/n,$$

the result above.

It may be of interest to note that an unbiased estimate of the variance pq/n of the proportion $\hat{p}$, is $\binom{n-2}{i-1} \Big/ \left[\binom{n}{i} n\right] = \dfrac{i(n-i)}{n^2(n-1)}$, $(n > 1)$; this estimate is obtained by the method suggested immediately following Theorem 3.

B. Curtailed single sampling. In single sampling schemes, there is usually given a rejection number c as well as the sample size n. If c or more defectives are found in the sample the lot is rejected, but if less than c defectives are found in the sample the lot is accepted. It is customary to inspect all the items in the sample even if the final decision to accept or reject the lot is known before the completion of the inspection of the sample. One reason sometimes men-

tioned for this procedure is that an unbiased estimate for p is not known when the inspection is halted as soon as a decision is reached. We provide the unbiased estimate in the following paragraph

In curtailed single sampling the boundary points when rejecting are (x, c), $c + x \leqq n$, when accepting $(n - c + 1, y)$, $y \leqq c - 1$. The region is a rectangular array and obviously simple. *The unique unbiased estimate along the horizontal line corresponding to rejection with* $c > 1$ *therefore is*

$$\hat{p}((x, c)) = \binom{c - 2 + x}{c - 2} \Big/ \binom{c + x - 1}{c - 1} = \frac{c - 1}{c + x - 1},$$

or in words, *one less than the number of defectives observed divided by one less than the number of observations. The unique unbiased estimate along the vertical line corresponding to acceptance for* $c > 1$ *is*

$$\hat{p}((n - c + 1, y)) = \binom{n - c + i - 1}{n - c} \Big/ \binom{n - c + i}{n - c} = \frac{i}{n - c + i}$$

that is, the number of defectives observed divided by one less than the number of observations. We reserved the case $c = 1$ because it is rather illuminating. The construction of Theorem 3 works as usual, and we note that $\hat{p}((0, 1)) = 1$, $\hat{p}((n, 0)) = 0$ as we might expect, but $\hat{p}((x, 1)) = 0$, $0 < x < n$.

It is somewhat startling to find that the only unbiased estimate of p for curtailed single sampling with $c = 1$ provides zero estimates unless a defective is observed on the first item We remark that the variance of this estimate is pq. In other words, curtailed single sampling with $c = 1$ is no better for estimation purposes than a sample of size one when the unbiased estimate $\hat{p}$ is used.

A limiting case of curtailed sampling when n is unbounded has been considered by Haldane[5] as a useful technique in connection with estimates of the frequency of occurrence of rare events The region would not be closed unless $p = 0$ were excluded In our nomenclature there is a "rejection number" c $(c > 1)$, and we continue sampling and inspecting until c defectives have been observed. The unbiased estimate[6] is $(c - 1)/(j - 1)$, where j is the total number of observations, and of course this is the estimate given by Haldane.

C A general curtailed double sampling plan The following example will illustrate the sort of calculations involved in computing p for multiple and sequential plans. A sample of size n_1 is drawn and items are inspected until 1) $r_1 (1 < r_1 \leqq n_1)$ defectives are found, or 2) $n_1 - a + 1$ $(a \geqq 0)$ nondefectives are found, or 3) the sample is exhausted with neither of these events occurring. If case 3) arises, a second sample of size n_2 is drawn and inspection proceeds until a grand total of $r_2 (r_1 \leqq r_2 \leqq n_1 + n_2)$ defectives is found or $n_1 + n_2 - r_2 + 1$

[5] J B. S. Haldane, *Nature*, Vol. 155 (1945), No. 3924.

[6] For the uniqueness, see footnote [4].

nondefectives are found. In this scheme we call r_1 and r_2 rejection numbers and a an acceptance number. *The unique unbiased estimate* $\hat{p}$ *is as follows:*

$$\text{(a)} \qquad \hat{p}((j, r_1)) = \frac{r_1 - 1}{r_1 + j - 1}, \qquad j = 0, 1, \cdots, n_1 - r_1;$$

$$\text{(b)} \qquad \hat{p}((n_1 - a + 1, i)) = \frac{i}{n_1 - a + i}, \qquad i = 0, 1, \cdots, a;$$

$$\text{(c)} \quad \hat{p}((x, r_2)) = \frac{\Sigma \binom{x_0 + y_0 - 1}{x_0} \binom{x - x_0 + r_2 - y_0 - 1}{r_2 - y_0 - 1}}{\Sigma \binom{x_0 + y_0}{x_0} \binom{x - x_0 + r_2 - y_0 - 1}{r_2 - y_0 - 1}}, \qquad n_1 - r_1 < x \leqq n_1 + n_2;$$

$$\text{(d)} \ \hat{p}((n_1 + n_2 - r_2 + 1, y)) = \frac{\Sigma \binom{x_0 + y_0 - 1}{x_0} \binom{n_1 + n_2 - r_2 + y - y_0 - x_0}{y - y_0}}{\Sigma \binom{x_0 + y_0}{x_0} \binom{n_1 + n_2 - r_2 + y - y_0 - x_0}{y - y_0}}, \qquad a < y \leqq n_1 + n_2;$$

where the summations extend from $y_0 = a + 1$ *to* $y_0 = r_1 - 1$, *and* $x_0 + y_0 = n_1$. In the above equations (a) and (b) are the estimates corresponding to rejection and acceptance on the basis of the first sample, while (c) and (d) correspond to rejection and acceptance when a second sample has been drawn. Rather than use the sums indicated in (c) and (d), some may find it preferable to make the estimation entirely on the basis of the first sample. If there is no curtailing, the procedure of estimation is equivalent to single sampling, and the estimate is again i/n_1 as mentioned in paragraph A above. *If the first sample is curtailed and the estimate is made on the basis of the results of the first sample only, the unique unbiased estimate is given by formula* (a) *when rejecting, by formula* (b) *when accepting, and by* i/n_1 *when a second sample is to be drawn.* It will be noted that (a) and (b) are identical with the expressions derived in paragraph B over the range of values for which they are valid.

D The sequential probability ratio test. Using the nomenclature of sequential analysis,[7] the criterion for a decision is given by two parallel straight lines in the *dn*-plane

$$(7) \qquad \begin{aligned} d_1 &= h_1 + sn \text{ (lower line)} \\ d_2 &= h_2 + sn \text{ (upper line)}, \end{aligned}$$

where d is the number of defectives and n is the number of observations. The acceptance and rejection numbers for any n are given by a_n and r_n, respectively,

[7] See, for example, *Sequential Analysis of Statistical Data: Applications*, Section 2, Columbia University Press, 1945

where a_n is the largest positive integer less than or equal to d_1, and r_n is the smallest integer greater than or equal to d_2. We let $k_a(n)$ be the number of paths from the origin which end in a decision to accept on the nth observation; $k_r(n)$ is similarly defined when rejection occurs on the nth observation. We also require an auxiliary sequential test with acceptance and rejection numbers $a'_{n-1} = a_n - 1$, $r'_{n-1} = r_n - 1$ (which is equivalent to replacing h_1 and h_2 by $h_1 + 1 - s$ and $h_2 - 1 + s$ in the equations (7)), and with $k'_a(n)$ and $k'_r(n)$ the number of paths from the origin which lead to acceptance or rejection on the nth observation for the new test. A graphical comparison of the two plans shows that: *The unique unbiased estimate of p is*

$$\hat{p}(n) = k'_a(n - 1)/k_a(n)$$

when the original test leads to a decision to accept, and

$$\hat{p}(n) = k'_r(n - 1)/k_a(n)$$

when the original test leads to a decision to reject on the nth observation.

E. Regions with narrow throats. Let us consider the case of a closed region which has only one accessible point of index n, $n > 0$ (n being the lowest index not zero at which this phenomenon occurs). The number of paths from the origin to this accessible point α' we will denote m, while the number of paths from α' to α, boundary points of index greater than n, will be denoted $l(\alpha)$. Then the total number of paths to α from the origin is $ml(\alpha)$. We use the construction preceding Theorem 3 to get $\hat{p}(\alpha)$. The number of paths from (0,1) to α is similarly $m^*l(\alpha)$, so for such points $\hat{p}(\alpha) = m^*/m$. In other words, if a closed region has a narrow throat such as that described, $\hat{p}(\alpha)$ for α of index higher than that of the accessible point α' are independent of the shape of the region beyond the line $x + y = n$, and in fact they are all identical. The curtailed single sample with $c = 1$ is a particular case of a region with a narrow throat.

4. Estimation based on data from several experiments. In the previous discussion we have been concerned with estimation based on the result of a single experiment. Various kinds of acceptance sampling plans have been suggested as examples of the possible experiments. Acceptance sampling is one of many activities where data toward the estimation of p are often accumulated in a series of experiments. It has been pointed out by John Tukey that when information is available from several experiments the estimate $\hat{p}$ will no longer be the unique unbiased estimate of p. Little has been done on this problem of combining information from several experiments, but to illustrate the point, we will discuss a very simple example in terms of acceptance sampling.

Let us suppose that two large lots of the same size are inspected according to the following curtailed single sampling plan: if a defective occurs at the first or second observation, sampling is stopped and the lot is rejected; if the first two items inspected are nondefective, we accept the lot.

The total number of defective and of nondefective items in the two samples form a sufficient statistic for p. In a single application of the sampling plan the boundary points with their probabilities are $(0, 1)$, p; $(1, 1)$, pq; $(2, 0)$, q^2. From this information we can generate the possible totals of defectives and of nondefectives which may arise when samples are drawn from two lots, with their probabilities by expanding

$$(8)\qquad (p + pq + q^2)^2 = p^2 + p^2q^2 + q^4 + 2\,p^2q + 2\,pq^2 + 2pq^3,$$

where a term on the right of the form mp^yq^x is the probability that in two samples there will be x nondefectives and y defectives altogether. On the basis of the observed number pair (x, y), which may be regarded as a possible terminal point α for the two experiments performed successively, we wish to form an unbiased estimate $e((x, y)) = e(\alpha)$. For the estimate e to be unbiased the condition $\Sigma\, e(\alpha)P(\alpha) = p$ must be satisfied, where in the present example the $P(\alpha)$ are the six terms on the right of equation (8), and the $e(\alpha)$ are the estimates with which the six probabilities are associated.

In the example under consideration the condition for unbiasedness will be satisfied if and only if $e((0, 2)) = 1$, $e((4, 0)) = 0$, $e((1, 2)) = \frac{1}{2}$, $e((2, 1)) = [1 - e((2, 2))]/2$, $e((3, 1)) = e((2, 2))/2$. Consequently a one parameter family of unbiased estimates is available. Unfortunately the popular condition that the variance be a minimum depends on the true value of p; in fact the variance is minimized just when $e((2, 2)) = 1/(2 + p)$. So an unbiased estimate of uniformly minimum variance does not exist. In practical applications to acceptance sampling one might meet this difficulty by choosing a value of p near zero for such a minimization scheme.

However it is clear that the last word has yet to be said about how best to estimate p when one is faced with the results of several experiments.

5. Conclusion. We would like to call attention to a few problems raised by but not solved in this paper: 1) find a necessary and sufficient condition that $\hat{p}$ be the unique unbiased estimate for p; 2) suggest criteria for selecting one unbiased estimate when more than one is possible; 3) evaluate the variance of $\hat{p}$.

In this connection, in a forthcoming paper by M. A. Girshick, it will be shown for certain regions, for example for those of the sequential probability ratio test, that the variance of $\hat{p}(\alpha)$,

$$\sigma_{\hat{p}}^2 \geq pq/E(x + y),$$

where $E(x + y)$ is the expected number of observations required to reach a boundary point.

DISTRIBUTION OF SAMPLE ARRANGEMENTS FOR RUNS UP AND DOWN

By P. S. Olmstead

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1. Summary. Using the notation of Levene and Wolfowitz [1], a new recursion formula is used to give the exact distribution of arrangements of n numbers, no two alike, with runs up or down of length p or more. These are tabled for n and p through $n = 14$. An exact solution is given for $p \geq n/2$. The average and variance determined by Levene and Wolfowitz are presented in a simplified form. The fraction of arrangements of n numbers with runs of length p or more are presented for the exact distributions, for the limiting Poisson Exponential, and for an extrapolation from the exact distributions. Agreement among the tables is discussed

2. Introduction. Assume that

$$x_1, x_2, \cdots x_n$$

represent a series of repetitive measurements. In engineering work, experience has shown that, when the values of these measurements exhibit changes in level, trends, cycles, etc., it is usually indicative of the presence of findable causes. In general, the engineer becomes more confident that a findable cause exists for a change in level, a trend, or a cycle, when the change is large, the trend is long, or the cycle is regular.

On the basis of this experience, the engineer selects particular measures of change in level, length of trend, etc., to guide him in deciding when it is profitable to look for a cause. Having selected the measure, he is interested in knowing how often he may have to look for a cause that does not exist. One such measure is the length of the longest run up or down in a sample of n values. The chart in Figure 1, based on the analysis given here, applies when no two values are alike and indicates the fraction of all nonidentical arrangements that have runs up or down of length p or more.

Attention is directed to the distribution of sample arrangements that have at least one run up or down of length p or more. The distribution and the variances and covariances for lengths of runs up and down are given by Levene and Wolfowitz [1] In addition, Wolfowitz [2] has shown that the limiting distribution for a particular length of run up or down is a Poisson Exponential.

The notation of Levene and Wolfowitz [1] will be used. Thus, let $a_1, a_2, \cdots, a_n$ be n numbers, no two alike, and let the sequence $S = (h_1, h_2, \cdots, h_n)$ be any permutation of $a_1, a_2, \cdots, a_n$, where S is to be considered a chance variable, and each of the $n!$ permutations of $a_1, a_2, \cdots, a_n$ is assigned the same

probability. Consider the derived sequence R whose ith element is the sign ($+$ or $-$) of $h_{i+1} - h_i$, ($i = 1, 2, \cdots, n - 1$). A sequence of p consecutive $+$ signs immediately preceded by a $-$ sign is called a run up of length p or more; a sequence of p consecutive $-$ signs immediately preceded by a $+$ sign is called a run down of length p or more. When such a run is both immediately preceded and immediately followed by an unlike sign, it is a run of length exactly p. The distribution of arrangements with at least one run up or down of length p or more is considered under five specific headings:

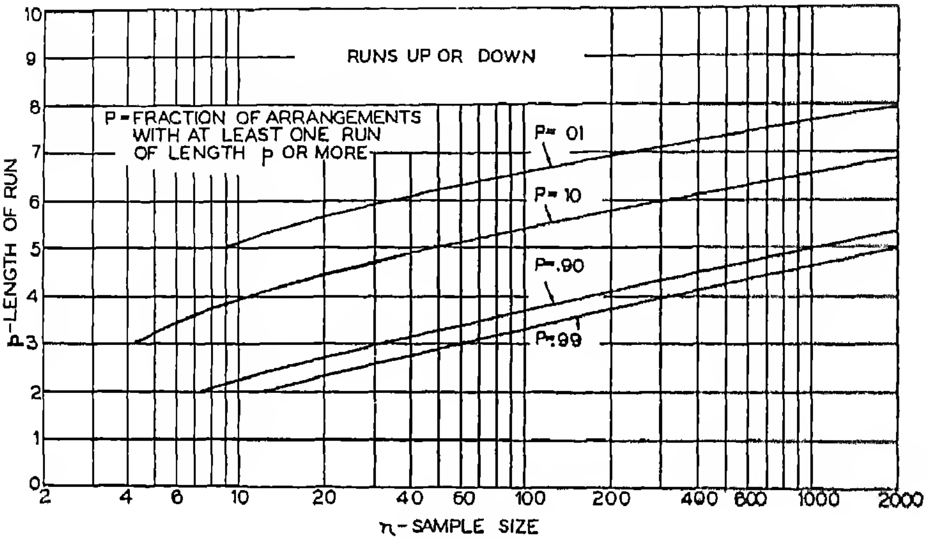

FIG. 1

1. An exact numerical solution for n small, i.e., computations have been completed up to and including $n = 14$.
2. An exact solution for $p \geq \frac{n}{2}$.
3. A limiting solution for $\frac{(p+1)!}{n} =$ constant.
4. An extrapolation from n small.
5. Constant probability relationships.

3. Solution for n small. Starting with a single number, a_1, a second number, $a_2 > a_1$, may be placed before or after it to obtain the two independent arrangements of one run of length exactly 1. A third number, $a_3 > a_2 > a_1$, may be placed before, between, or after the preceding pair to obtain two independent arrangements of one run of length exactly 2 and four of two runs of length exactly 1. Continuing this process it is seen that, on the assumption that the

distribution of independent arrangements for $(n-1)$ numbers, $a_1 < a_2 < a_3 < \cdots < a_{n-1}$, is known, the distribution of independent arrangements for n numbers, $a_1 < a_2 < a_3 < \cdots < a_n$, can be found by using the following recursion formula:

$$
\begin{aligned}
&F_n[r_{n-1}, r_{n-2}, \cdots, r_h, \cdots, r_i, \cdots, r_j, \cdots, r_1] \\
&= \sum_{i=2}^{n-1} (r_{i-1}+1)F_{n-1}[r_{n-2}, r_{n-3}, \cdots, (r_i - 1), (r_{i-1}+1), \cdots r_1] \\
&+ 2F_{n-1}[r_{n-2}, r_{n-3}, \cdots, (r_1 - 1)] \\
&+ 2\sum_{i=2}^{n-3}\sum_{j=1}^{i-1} (r_h + 1) \\
&\quad \cdot F_{n-1}[r_{n-3}, \cdots, (r_{h=i+j}+1), \cdots, (r_i - 1), \cdots, (r_j - 1), \cdots, (r_1 - 1)] \\
&+ \sum_{i=1}^{n-3} (r_h + 1)F_{n-1}[r_{n-3}, \cdots, (r_{h=2i}+1), \cdots, (r_i - 2), \cdots (r_1 - 1)]
\end{aligned} \tag{1}
$$

where r_i, etc., represents the number of runs either up or down of exactly length i in each arrangement of the n numbers designated F_n,

(2) $\Sigma_{i=1}^{n-1} r_i = r$, the total number of runs having lengths exactly i (from 1 to $n-1$) for each arrangement included in F_n,

(3) $\Sigma_{i=1}^{n-1} i r_i = n - 1$, that is, the sum of the lengths of all such runs in any arrangement is one less than the total number of numbers,

$F_n[r_{n-1}, r_{n-2}, \cdots, r_h, \cdots, r_i, \cdots, r_j, \cdots, r_1]$, the total number of nonidentical sequences of the n numbers with exactly r_{n-1} runs of length exactly $(n-1)$, $\cdots$ r_h runs of length exactly h, $\cdots$ r_i runs of length exactly i, $\cdots$ r_j runs of length exactly j, $\cdots$ r_1 runs of length exactly 1. Some of these r's are of course zero and their sum is that given in (2) above. Similar statements apply to the four F_{n-1}'s.

In the last two summations in (1), when $r_j = r_1$, $(r_j - 1)$ combines with $(r_1 - 1)$ to give $(r_1 - 2)$, and when $r_i = r_1$, $(r_i - 2)$ combines with $(r_1 - 1)$ to give $(r_1 - 3)$.

By using the above recursion formula, the exact number of arrangements with at least one run up or down of length p or more has been computed for $n = 2$ to $n = 14$, inclusive. This information is given in Table 1. In addition, it has been used to determine the probabilities of arrangements with runs up or down of length p or more as shown in Table 2. These tables provide a useful background for the limiting expressions considered in the next three sections.

TABLE 1

Exact Numbers of Arrangements of n numbers with Runs of Length p or More

n \ p	1	2	3	4	5	6	7	8	9	10	11	12	13
2	2												
3	6	2											
4	24	14	2										
5	120	88	18	2									
6	720	598	156	22	2								
7	5,040	4,496	1 388	224	26	2							
8	40,320	37,550	13,334	2,352	304	30	2						
9	362,880	347,008	138,422	26,068	3,600	396	34	2					
10	3,628,800	3,527,758	1,554,854	309,178	44,640	5,220	500	38	2				
11	39,916,800	39,209,216	18,835,878	3,926,538	585,576	71,280	7,260	616	42	2			
12	479,001,600	473,596,070	245,249,548	53,333,016	8,159,498	1,021,680	108,240	9,768	744	46	2		
13	6,227,020,800	6,182,284,288	3,419,024,924	772,958,890	120,760,922	15,442,152	1,681,680	157,872	12,792	884	50	2	
14	87,178,291,200	86,779,569,238	50,852,433,294	11,920,405,298	1,895,856,108	246,427,634	27,387,360	2,642,640	222,768	16,380	1,036	54	2

TABLE 2

Exact Fraction of Arrangements of n numbers with Runs of Length p or More

n \ p	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	1.00000000	.00000000												
3	"	.33333333	.00000000											
4	"	.58333333	.08333333	.00000000										
5	"	.73333333	.15000000	.01666667	.00000000									
6	"	.83055556	.21666667	.03055556	.00277778	.00000000								
7	"	.89206349	.27539683	.04444444	.00515873	.00039683	.00000000							
8	"	.93129960	.33070437	.05833333	.00753968	.00074405	.00004960	.00000000						
9	"	.95626102	.38145392	.07183642	.00992063	.00109127	.00009369	.00000531	.00000000					
10	"	.97215553	.42847608	.08520117	.01230159	.00143849	.00013779	.00001047	.00000055	.00000000				
11	"	.98227353	.47187846	.09836806	.01466991	.00178571	.00018188	.00001543	.00000105	.00000005	.00000000			
12	"	.98871501	.51200152	.11134204	.01703439	.00213294	.00022597	.00002039	.00000155	.00000010	.00000000	.00000000		
13	"	.99281574	.54906271	.12412981	.01939305	.00247986	.00027006	.00002535	.00000205	.00000014	.00000001	.00000000	.00000000	
14	"	.99542636	.58331533	.13673504	.02174688	.00282671	.00031415	.00003031	.00000256	.00000019	.00000001	.00000000	.00000000	.00000000

4. Solution for $p \geq \frac{n}{2}$. When $p \geq \frac{n}{2}$, it is clear that no sequence can contain more than one run of length p. Thus, the expected number of runs of length p or more in an arrangement is also the probability that an arrangement contains runs of length p or more. Writing Levene and Wolfowitz's [1] expression (4.2) in the simplified form previously published [3], we have

$$P(r_p') = E(r_p') = \frac{2[(n-p)(p+1)+1]}{(p+2)!} \quad \text{for} \quad \frac{n}{2} \leq p < n, \tag{4}$$

where r_p' represents the number of runs of length p or more. This expression checks exactly with Table 2 over the range to which it applies.

5. Solution for $\frac{(p+1)!}{n}$ = constant. As mentioned above, Wolfowitz [2] has shown that the limiting distribution for runs up and down is a Poisson Exponential. His proof applies specifically to the distribution of runs of length exactly p. However, the assumptions made in his derivation could have been applied to the distribution of runs of length p or more and would have led to identical conclusions for such runs. To see how closely this is approximated, it is possible to throw expression (4.17) for the variance of (r_p') derived by Levene and Wolfowitz [1] into the following simplified form:

$$\begin{aligned} \sigma^2(r_p') = \Bigg\{ & \frac{2[(n-p)(p+1)+1]}{(p+2)!}\left[1 - \frac{2(p+1)^2[6p^2+7(p-1)]}{(p+2)!(2p+3)(2p+1)} \right. \\ & \left. - \frac{4(p+2)!}{(2p+3)!}\right] + \left[\frac{(p+1)[(2p+3)p(p-1)-6]}{p!(p+2)!(2p+3)(2p+1)} \right. \\ & \left. + \frac{2(p+1)^2+1}{(2p+3)!}\right]\Bigg\} \doteq [E(r_p')]\left[1 - \frac{3}{p!} - \frac{p!}{(2p)!}\right] + \frac{1}{2}\left[\frac{1}{(p!)^2} + \frac{1}{(2p)!}\right]. \end{aligned} \tag{5}$$

Thus, $\sigma^2(r_p')$ is equal to $E(r_p')$ within one part in one thousand for $p \geq 7$ and it is apparent that the first two moments approximate those of a Poisson Exponential. Making use of this information, it is possible to prepare Table 3, which gives approximate values of the probabilities of arrangements with runs of length p or more based on

$$P(r_p') = 1 - e^{-E(r_p')} = 1 - e^{-(2[(n-p)(p+1)+1])/(p+2)!}. \tag{6}$$

Comparison of Tables 2 and 3 shows agreement to closer than .0001 for $p \geq 6$, .001 for $p \geq 5$, .01 for $p \geq 4$, and .1 for $p \geq 3$ when $n \leq 14$. Similarly, the agreement for $p = 1$ is within .1 at $n \geq 4$, within .01 at $n \geq 8$, within .001 at $n \geq 11$ and .0001 at $n \geq 14$, the agreement for $p = 2$ is within .1 at $n \geq 10$. Possible agreement beyond $n = 14$ is of course subject to conjecture. However, it may be observed that the maximum difference for a given value of p was reduced from .2679 at $n = 2$, $p = 1$ to .1691 at $n = 6$, $p = 2$ indicating that closer agreement may be expected as p is increased.

6. Extrapolation from the exact solution for n small. Since the exponential in equation (6) may be written in the form:

$$e^{-(2[(n-p)(p+1)+1])/(p+2)!} = e^{(2[p(p+1)-1])/(p+2)!} \cdot e^{-(2n(p+1))/(p+2)!} \tag{7}$$

it follows that:

$$\frac{1 - P_{n+1}(r'_p)}{1 - P_n(r'_p)} = e^{-(2(p+1))/(p+2)!} \tag{8}$$

TABLE 3

Fraction of Arrangements of n numbers with Runs of Length p or More Based on Poisson Exponential

n \ p	1	2	3	4	5	6	7	8	9	10	>10
2	7321	0900									
3	8111	2835	0165								
4	9030	4220	.0900	0028							
5	9562	5654	1393	.0165	.0004						
6	9744	6615	.1949	0301	.0028	0001					
7	9869	7364	.2407	0435	.0052	.0004	.0000				
8	9943	7947	2953	.0567	.0075	.0007	.0001	.0000			
9	9965	8401	3408	.0697	.0099	.0011	.0001	.0000	.0000		
10	9982	8742	3833	.0825	.0122	.0014	.0001	.0000	.0000	.0000	
11	9991	9030	4230	.0952	.0146	.0018	.0002	.0000	.0000	0000	.0000
12	9995	9244	4603	.1076	.0169	.0021	.0002	.0000	.0000	.0000	.0000
13	9997	9412	4951	.1200	.0193	.0025	.0003	.0000	.0000	.0000	.0000
14	9999	9542	5276	.1321	.0216	.0028	.0003	.0000	.0000	.0000	0000
15	.9999	9643	.5581	.1411	.0239	.0032	.0004	.0000	.0000	.0000	.0000
20	1.0000	9998	.6834	.2015	.0355	.0049	.0006	.0001	.0000	.0000	0000
40	"	.9999	.9165	.3952	.0803	.0118	.0015	.0002	.0000	0000	.0000
60	"	1.0000	.9780	.5419	.1231	.0186	.0023	.0003	.0000	.0000	0000
80	"	"	.9942	.6530	.1639	.0254	.0032	.0004	0000	.0000	.0000
100	"	"	.9985	.7371	.2030	.0322	.0041	0005	.0000	.0000	0000
200	"	"	1.0000	.9345	.3717	.0652	.0085	.0010	.0001	.0000	.0000
500	"	"	"	.9990	6924	.1577	.0215	.0024	.0002	.0000	.0000
1000	"	"	"	1.0000	.9065	.2919	.0428	.0049	.0005	.0000	.0000
5000	"	"	"	"	1.0000	.8234	1976	.0245	.0025	.0002	.0000

showing that consecutive values of $1 - P(r'_p)$ are related by a constant of proportionality dependent only on p. Since this is true in the limit, Table 2 was examined to determine similar multipliers for extrapolation. The results of this examination are shown in Table 4 together with the values of (8) This table shows that the agreement between the value of $\dfrac{1 - P_{n+1}(r'_p)}{1 - P_n(r'_p)}$ for $n = 12$, e.g., and $e^{(-2(p+1))/(p+2)!}$ becomes closer the larger the value of p. The con-

TABLE 4

Determination of Extrapolation Constant

n \ p	2		3		4		5		6	
	$1-P_n(r_2')$	$\frac{1-P_{n+1}(r_2')}{1-P_n(r_2')}$	$1-P_n(r_3')$	$\frac{1-P_{n+1}(r_3')}{1-P_n(r_3')}$	$1-P_n(r_4')$	$\frac{1-P_{n+1}(r_4')}{1-P_n(r_4')}$	$1-P_n(r_5')$	$\frac{1-P_{n+1}(r_5')}{1-P_n(r_5')}$	$1-P_n(r_6')$	$\frac{1-P_{n+1}(r_6')}{1-P_n(r_6')}$
2	1.00000000	.66666667								
3	.66666667	.62500000	1 00000000	.91666667						
4	41666667	.64000000	.91666667	.92727273	1 00000000	.98333333				
5	.26666667	.63541667	.85000000	.92156863	.98333333	.98587571	1.00000000	.99722222		
6	.16944444	.63700234	.78333333	92502533	.96944444	.98567336	.99722222	.99761242	1.00000000	.99960317
7	.10793651	.63648898	72460317	.92367196	.95555556	.98546511	.99484127	99760670	99960317	.99965264
8	.06870040	.63666266	.66929563	.92417469	.94166667	98566044	.99246032	.99760096	.99925595	.99965252
9	.04373898	.63660534	.61854608	.92397954	.92816358	.98560087	.99007937	.99759518	.99890873	.99965240
10	.02784447	.63662455	.57152392	92405851	.91479883	.98560679	.98769841	99760218	.99856151	.99965228
11	.01772647	.63661817	.52812154	.92402684	.90163194	.98561056	.98533009	.99760032	.99821429	.99965215
12	.01128499	.63662030	.48799848	.92405470	.88865796	.98561002	.98296561	.99760046	.99786706	.99965234
13	00718426	.63661959	.45093729	92404128	.87587019	.98560731	.98060695	.99759961	.99752014	.99965228
14	.00457364		.41668467		.86326406		.97825312		.99717329	
Chosen value of $\frac{1-P_{n+1}(r_p')}{1-P_n(r_p')}$		$.63662 \doteq \frac{2}{\pi}$		.92404		.98561		.99760		.999652
Value of $e^{-(2(p+1))/(p+2)^2}$		.77880		.93550		.98629		.99762		.9996528

stancy of the ratio for a given value of p is such as to permit calculation of probabilities for any value of n to a minimum of three or possibly four decimal places. Such calculations have been made and recorded in Table 5. The following formulae[1] were used for these calculations:

$$
\begin{aligned}
P_n(r_1') &= 1 \\
P_n(r_2') &= 1 - (.00437364)\left(\frac{2}{\pi}\right)^{n-14} \\
P_n(r_3') &= 1 - (.45093729)(.92404)^{n-13} \qquad (9)\\
P_n(r_4') &= 1 - (.87587019)(.98561)^{n-13} \\
P_n(r_5') &= 1 - (.98060695)(.99760)^{n-13} \\
P_n(r_6') &= 1 - (.99752014)(.999652)^{n-13}
\end{aligned}
$$

or in general

$$P_n(r_p') = 1 - [1 - P_{n_0}(r_p')][\text{Constant}_p]^{n-n_0}. \tag{10}$$

Comparison of Table 3 with Tables 2 and 5 shows that the difference for given p and n has a maximum for each value of p and that this maximum decreases with increase in p. The maximum values of the difference shown in the tables are: $p = 1, n = 2$, .2079; $p = 2, n = 6$, .1691; $p = 3, n = 20$, .0572; $p = 4, n = 80$, .0154; $p = 5, n = 500$, .0033; and $p = 6, n = 5000$, .0007. Thus, it is apparent that the agreement beyond $p = 6$ should be within .0001 and the method of Section 5 used for Table 3 is satisfactory for these probabilities.

7. Constant probability relationships. From Tables 2, 3 and 5, it is possible to make interpolations for the values of n required to have a probability of at least $P(r_p')$ that an arrangement will have a run of length p or more. When the conditions of Section 5 apply, the value of n is, of course:

$$n = p - \frac{1}{p+1} - \frac{p+2}{2}\, p! \log_e [1 - P(r_p')]. \tag{11}$$

[1] It will be noted that the constant for $p = 2$ has been taken to be $\frac{2}{\pi}$, whereas the last value shown in Table 4 is .63661959. However, alternate values in this series are converging. Comparing these subseries shows that by $n = 16$, the values would agree with $\frac{2}{\pi}$ to eight decimal places. An analytic proof that $\frac{2}{\pi}$ is the limiting value of the constant has recently been found by J. W. Tukey.

While reading the manuscript J. Riordan observed that the number of arrangements with longest length 1, say $f(n, 1)$ has the generating function

$$\Sigma f(n, 1) \frac{t^n}{n!} = 2(\sec t + \tan t)$$

hence is twice the Euler number for n even and twice the tangent number for n odd, a result given essentially by Netto [4]. These observations lead directly to the limiting value, $\frac{2}{\pi}$ noted above.

TABLE 5

Fraction of Arrangements of n Numbers with Runs of Length p or More Based on Extrapolation with Extrapolation Constant

n \ p	1	2	3	4	5	6
14	1.0000	.9954	.5833	.1367	.0217	.0028
15	"	.9971	.6150	.1492	.0241	.0032
20	"	.9997	.7406	.2086	.0358	.0049
40	"	1.0000	.9466	.4078	.0810	.0118
60	"	"	.9890	.5568	.1241	.0187
80	"	"	.9977	.6684	.1652	.0255
100	"	"	.9995	.7518	.2044	.0322
200	"	"	1.0000	.9418	.3743	.0653
500	"	"	"	.9992	.6957	.1580
1000	"	"	"	1.0000	.9085	.2925
5000	"	"	"	"	1.0000	.8241

TABLE 6

Sample Size for Constant Probability Based on Poisson Exponential

P \ p	1	2	3	4	5	6	7	8
≤ .99	7	20	71	335	1939	13268		
≤ .95	5	13	47	219	1263	8633		
≤ .90	3	10	37	169	971	6637		
≤ .10	0	2	4	11	49	309	2296	
≤ .05	0	1	3	7	26	153	1170	10350
≤ .01	0	1	2	4	9	34	235	2036

TABLE 7

Sample Size for Constant Probability Based on Extrapolation

P \ p	1	2	3	4	5	6
≤ .99	—	12	61	321	1923	13239
≤ .95	—	8	40	210	1253	8614
≤ .90	—	7	32	162	964	6622
≤ .10	—	(2)	4	11	48	308
≤ .05	—	(2)	(3)	7	26	153
≤ .01	—	(2)	(3)	(4)	9	34

Similarly, it may be obtained from the extrapolation formulae of Section 6 in the form:

$$n = n_0 + \frac{\log [1 - P_n(r'_p)] - \log [1 - P_{n_0}(r'_p)]}{\log [\text{Constant}_p]}. \tag{12}$$

Results of computations based on (11) and (12), are given in Tables 6 and 7, respectively for particular values of $P(r'_p)$. It will be noted that Table 7 is in exact agreement with Table 2 and that it differs but little in a practical sense from Table 6.

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THE THEORY OF UNBIASED ESTIMATION

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1. Summary. Let $F(P)$ be a real valued function defined on a subset $\mathfrak{D}$ of the set $\mathfrak{D}^*$ of all probability distributions on the real line. A function f of n real variables is an unbiased estimate of F if for every system, $X_1, \cdots, X_n$, of independent random variables with the common distribution P, the expectation of $f(X_1 \cdots, X_n)$ exists and equals $F(P)$, for all P in $\mathfrak{D}$. A necessary and sufficient condition for the existence of an unbiased estimate is given (Theorem 1), and the way in which this condition applies to the moments of a distribution is described (Theorem 2). Under the assumptions that this condition is satisfied and that $\mathfrak{D}$ contains all purely discontinuous distributions it is shown that there is a unique symmetric unbiased estimate (Theorem 3); the most general (non symmetric) unbiased estimates are described (Theorem 4); and it is proved that among them the symmetric one is best in the sense of having the least variance (Theorem 5). Thus the classical estimates of the mean and the variance are justified from a new point of view, and also, from the theory, computable estimates of all higher moments are easily derived. It is interesting to note that for n greater than 3 neither the sample nth moment about the sample mean nor any constant multiple thereof is an unbiased estimate of the nth moment about the mean. Attention is called to a paradoxical situation arising in estimating such non linear functions as the square of the first moment.

2. Introduction. Consider the set $\mathfrak{D}^*$ of all probability distributions on the real line. The elements P of $\mathfrak{D}^*$ may be regarded as either set functions $P(E)$, defined for all Borel subsets E of the real line, (probability measures) or monotone non decreasing functions $P(x)$ of a real variable x, (cumulative distribution functions). Suppose that $F = F(P)$ is a real numerically valued function of distributions. For example $F(P)$ may be the expectation or the standard deviation of the distribution P, or it may be the amount of probability P assigns to some fixed set E_0. The problem of unbiased estimation is to find a function (statistic) of a sample of n from a population with distribution P, in such a way that the expected value of this function is equal to the value of $F(P)$ identically in P. More precisely, if $F(P)$ is defined on a subset $\mathfrak{D}$ of $\mathfrak{D}^*$, then an unbiased estimate of order n over $\mathfrak{D}$ is a real valued function $f = f(x_1 \cdots x_n)$ of n real variables, which is such that for every system $X_1, \cdots, X_n$ of independent random variables with the common distribution P (belonging to $\mathfrak{D}$), the expected value $E\{f(X_1, \cdots, X_n)\}$ exists and is equal to $F(P)$.

The problems posed in this paper are the following. (I) Which functions $F(P)$ admit an unbiased estimate? (II) What are all possible unbiased estimates of a given function $F(P)$? (III) Is there a reasonable definition of "best

unbiased estimate" which enables one to select from all unbiased estimates of a fixed function $F(P)$ a unique best one?[1]

I shall present below a complete solution of these problems, under the assumption that the domain of estimation, $\mathfrak{D}$, is sufficiently large. The results also shed light on some classical concepts. It is possible, for instance, to exhibit computable unbiased estimates for all moments of a distribution about its expected value, and to prove that the known estimates of the expectation and the variance are essentially unique.

The vague concept of sufficiently large estimation domain $\mathfrak{D}$ is easily made precise. For any Borel set E on the real line let $\mathfrak{D}^*(E)$ be the set of all those distributions which assign the probability 1 to some finite subset of E. Thus, for example, if E consists of exactly two points then $\mathfrak{D}^*(E)$ is the set of all possible probability distributions in a dichotomy. A subset $\mathfrak{D}$ of $\mathfrak{D}^*$ will be said to be finitely closed over E if $\mathfrak{D}^*(E) \subseteq \mathfrak{D}$. Finitely closed domains are "sufficiently large."

It is clear that some restriction (from below) on the size of $\mathfrak{D}$ is essential for a discussion of the characterization problem (II) and the uniqueness problem (III). For if, for example, the domain $\mathfrak{D}$ is artificially restricted to contain only one distribution, then there will always be a plethora of completely unrelated and uninteresting solutions of the problem of unbiased estimation, none of which can be said to be preferable to any other one. It is true, however, that the assumption of finite closure is too restrictive. The general problems of unbiased estimation are still unsolved over such interesting and useful domains as the set of all continuous distributions, and the set of all absolutely continuous distributions. There are also more special problems connected with special classes of distributions (e.g. the normal and the rectangular distributions), as well as the general problem of characterizing the domains which are sufficiently large to make a uniqueness theorem possible. I hope to return to these problems in the near future.

3. Existence. A function $F(P)$, defined on a domain $\mathfrak{D} \subseteq \mathfrak{D}^*$, will be called homogeneous over $\mathfrak{D}$, of degree $k = 1, 2, \cdots$, if there exists a real valued function $\varphi = \varphi(x_1, \cdots, x_k)$ of k real variables which is such that for every P in $\mathfrak{D}$ the Lebesgue-Stieltjes integral[2]

$$\int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k)$$

[1] My interest in these problems stems from conversations and correspondence with Reinhold Baer, who first called my attention to the problem of finding unbiased estimates for the moments about the expected value. The general questions of existence and uniqueness of unbiased estimates were raised explicitly by J. F. Steffensen in a footnote on p. 18 of his book, *Some Recent Researches in the Theory of Statistics and Actuarial Science*, Cambridge Univ. Press, 1930.

[2] All integrals in this paper are to be extended over the entire Euclidean space of indicated dimension.

exists and is equal to $F(P)$, and if the integer k is minimal with respect to the property of the existence of such a representation.

THEOREM 1. *A necessary and sufficient condition that F have an unbiased estimate of order n over $\mathfrak{D}$ is that it be homogeneous over $\mathfrak{D}$ of degree $k \leq n$.*

PROOF. To prove sufficiency, suppose that

$$F(P) = \int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k)$$

for all P in $\mathfrak{D}$, with $k \leq n$. Define f by

$$f(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n) = \varphi(x_1, \cdots, x_k).$$

Then if $X_1, \cdots, X_n$ are independent random variables with the same distribution P (belonging to $\mathfrak{D}$)

$$\begin{aligned} E\{f(X_1, \cdots, X_n)\} &= \int \cdots \int f(x_1, \cdots, x_n)\, dP(x_1) \cdots dP(x_n) \\ &= \int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_n) \\ &= \int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k) = F(P). \end{aligned}$$

The necessity of the condition is even more trivial: the definition of an unbiased estimate of order n is such that the existence of one is equivalent to homogeneity of degree $\leq n$.

As a special case, and an important illustration of how the degree is evaluated, consider the moments $F_m = F_m(P)$ of a distribution P about the origin,

$$F_m(P) = \int x^m\, dP(x),$$

and the moments $\bar{F}_m(P)$ about the expected value $F_1(P)$,

$$\bar{F}_m(P) = \int (x - F_1(P))^m\, dP(x).$$

THEOREM 2. *If $\mathfrak{D}$ is any subset of $\mathfrak{D}^*$ contained in the domain of definition of each of the functions $F_1, \cdots, F_r$, and finitely closed over $\{0, 1\}$ (where $\{0, 1\}$ denotes the set containing the two numbers 0 and 1 only), and if $k_1, \cdots, k_r$ are arbitrary non negative integers, then the function*

$$F(P) = F_1^{k_1}(P) \cdots F_r^{k_r}(P)$$

is homogeneous over $\mathfrak{D}$ of degree exactly $k = k_1 + \cdots + k_r$.

PROOF. The representation of F by a k-fold integral,

$$F(P) = \int \cdots \int x_1 \cdots x_{k_1} x_{k_1+1}^2 \cdots x_{k_1+k_2}^2 \cdots x_{k_1+\cdots+k_r}^r\, dP(x_1) \cdots dP(x_k),$$

shows that F is homogeneous of degree $\leq k$. That the degree of F is indeed equal to k is proved as follows. Suppose that

$$F(P) = \int \cdots \int \varphi(x_1, \cdots, x_h)\, dP(x_1) \cdots dP(x_h)$$

for all P in $\mathfrak{D}$. Observe that if P is the singular distribution which assigns probability 1 to the point 1 on the real line then the identity of the two representations of F reduces to $\varphi(1, \cdots, 1) = 1$; similarly assigning the total probability to 0 implies that $\varphi(0, \cdots, 0) = 0$. More generally, choose P so that it assigns the probability p, $(0 \leq p \leq 1)$, to the point 1, and the probability $q = 1 - p$ to 0. It follows that

$$p^k = p^h + p^{h-1}q\,\varphi_1 + \cdots + pq^{h-1}\,\varphi_{h-1},$$

where φ_i is the sum of all $\varphi(x_1, \cdots, x_h)$, over those $h$tuples $(x_1, \cdots, x_h)$ which contain exactly i 0's and $(h - i)$ 1's If q is replaced by $1 - p$ in the right side of the last equation, the resulting equation is supposed to be satisfied by all p, $0 \leq p \leq 1$. If, however, $h < k$, then the two sides of the equation are polynomials of different degrees; hence $h \geq k$.

Corollary. *If $\mathfrak{D}$ is any subset of $\mathfrak{D}^*$ contained in the domain of definition of the function $\bar{F}_m$ and finitely closed over $\{0, 1\}$ then $\bar{F}_m$ is homogeneous over $\mathfrak{D}$ of degree exactly m and, consequently, it has unbiased estimates over $\mathfrak{D}$ of order n if and only if $m \leq n$.*

Proof. Since

$$\begin{aligned}\bar{F}_m(P) &= \int (x - F_1(P))^m\, dP(x) \\ &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} F_1^j(P) \int x^{m-j}\, dP(x) \\ &= \sum_{j=0}^{m} (-1)^j \binom{m}{j} F_1^j(P) F_{m-j}(P),\end{aligned}$$

the conclusions of the corollary are implied by Theorems 1 and 2.

4. Symmetry. Theorem 1 may be regarded as a solution of the existence problem (I). An examination of its proof shows, however, that the estimates there constructed are very unsatisfactory indeed. In the special case $F = F_1$, for instance, the estimate becomes $f(x_1, \cdots, x_n) = x_1$. The first element of a sample of n is, to be sure, an unbiased estimate of the expectation of the distribution, but it is intuitively clear that, since it ignores most of the information at hand, it is not a good one. In order to exhibit the best estimates it becomes necessary to study the symmetric ones. Recall that a function $f = f(x_1, \cdots, x_n)$ is symmetric if it is invariant under all permutations of its arguments. The proof of the main theorem of this section, the theorem of uniqueness for symmetric unbiased estimates, is based on two lemmas.

LEMMA 1. *If $Q = Q(p_1, \cdots, p_n)$ is a homogeneous polynomial of degree > 0 in n real variables, such that whenever $0 \leq p_i \leq 1$, $i = 1, \cdots, n$, and $p_1 + \cdots + p_n = 1$ then $Q(p_1, \cdots, p_n) = 0$, then Q must be identically zero.*

PROOF. (Induction on n.) For $n = 1$ the lemma is trivial. Assume therefore that $n > 1$ and that the lemma is true for $n - 1$. Observe that the hypothesis is equivalent to the vanishing of Q for all systems of non negative arguments (without the restriction $p_1 + \cdots + p_n = 1$), since any such system $\{p_i\}$ can be replaced by $\{p_i/(p_1 + \cdots + p_n)\}$. If in Q the variables $p_1, \cdots, p_{n-1}$ are given any non negative values, then the hypothesis implies that the resulting polynomial in p_n vanishes for all non negative values of p_n, and therefore identically. Consequently the coefficients of the powers of p_n in Q, which are themselves homogeneous polynomials in $p_1, \cdots, p_{n-1}$, vanish for non negative arguments and therefore (by the induction hypothesis) identically.[3]

LEMMA 2. *If $\mathfrak{D}$ is a set of distributions finitely closed over a Borel set E of the real line and if the symmetric function $f(x_1, \cdots, x_n)$ is such that for every distribution P in $\mathfrak{D}$ the Lebesgue-Stieltjes integral*

$$\int \cdots \int f(x_1, \cdots, x_n)\, dP(x_1) \cdots dP(x_n)$$

exists and has the value zero, then $f(x_1, \cdots, x_n) = 0$ whenever $x_i \epsilon E$, $i = 1, \cdots, n$.

PROOF. Consider any point $(x_1^0, \cdots, x_n^0)$ with $x_i^0 \epsilon E$, $i = 1, \cdots, n$, and any distribution P (in $\mathfrak{D}^*(E)$) which assigns the probability 1 to the subset $\{x_1^0, \cdots, x_n^0\}$ of E. If the probability of x_i^0 is p_i, $i = 1, \cdots, n$, then the integral

$$\int \cdots \int f(x_1, \cdots, x_n)\, dP(x_1) \cdots dP(x_n)$$

is a homogeneous polynomial (of degree n) in the n variables $p_1, \cdots, p_n$. The hypotheses of Lemma 1 are satisfied—it follows that this polynomial vanishes identically. The symmetry of f implies that the coefficient of the term $p_1 \cdots p_n$ is exactly $n!f(x_1^0, \cdots, x_n^0)$, thereby establishing the conclusion of the lemma.

If $\varphi = \varphi(x_1, \cdots, x_k)$ is any function of k real variables and if n is a positive integer, $n \geq k$, it is convenient to write

$$\varphi^{[n]} = \varphi^{[n]}(x_1, \cdots, x_n)$$

for the average of the values of φ over all points obtained from $(x_1, \cdots, x_n)$ by extracting ordered subsets of k x's. Thus, for instance,

$$(x_1x_2)^{[3]} = \tfrac{1}{3}(x_1x_2 + x_1x_3 + x_2x_3)$$

and

$$(x_1)^{[n]} = \tfrac{1}{n}(x_1 + \cdots + x_n).$$

[3] I am indebted to J. B. Rosser and R. J. Walker for this proof; my original proof of Lemma 1 was more complicated.

THEOREM 3. *Let $\mathfrak{D}$ be a set of distributions finitely closed over a Borel set E of the real line and let F be a homogeneous function of degree k,*

$$F(P) = \int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k)$$

over $\mathfrak{D}$. If $f(x_1, \cdots, x_n)$ is a symmetric unbiased estimate of F over $\mathfrak{D}$, of order $n \geq k$, then for every point $(x_1, \cdots, x_n)$ with $x_i \in E$, $i = 1, \cdots, n$, $f(x_1, \cdots, x_n)$ is equal to the symmetrized function $\varphi^{[n]}(x_1, \cdots, x_n)$.

PROOF. Observe first that

$$\int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k)$$

remains invariant if $(x_1, \cdots, x_k)$ is replaced by $(x_{i_1}, \cdots, x_{i_k})$, where $\{i_1, \cdots, i_k\}$ is any subset of $\{1, \cdots, n\}$, since the change is merely a matter of notation. It follows that

$$F(P) = \int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k)$$

$$= \int \cdots \int \varphi^{[n]}(x_1, \cdots, x_n)\, dP(x_1) \cdots dP(x_n),$$

so that $\varphi^{[n]}$ is indeed an unbiased estimate of F. Since $\varphi^{[n]}$ is also symmetric, $f - \varphi^{[n]}$ satisfies the hypotheses of Lemma 2, and the desired conclusion follows from an application of that lemma.

5. Characterization. For any Borel set E on the real line let $\mathfrak{D}^*(E)$ be the set of all those distributions which assign the probability 0 to the complement of E. Thus, clearly, $\mathfrak{D}_*(E) \subseteq \mathfrak{D}^*(E)$; if E is the entire real line then $\mathfrak{D}^*(E) = \mathfrak{D}^*$; if E consists of a finite number of points then $\mathfrak{D}_*(E) = \mathfrak{D}^*(E)$.

THEOREM 4. *Let $\mathfrak{D}$ be a set of distributions finitely closed over a Borel set E of the real line and contained in $\mathfrak{D}^*(E)$, and let F be a homogeneous function of degree k,*

$$F(P) = \int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k)$$

over $\mathfrak{D}$. A necessary and sufficient condition that the function $f = f(x_1, \cdots, x_n)$ be an unbiased estimate of F over $\mathfrak{D}$, of order $n \geq k$, is that the Lebesgue-Stieltjes integral

$$\int \cdots \int f(x_1, \cdots, x_n)\, dP(x_1) \cdots dP(x_n)$$

exist for every P in $\mathfrak{D}$ and that for every point $(x_1, \cdots, x_n)$ with $x_i \in E$, $i = 1, \cdots, n$, the symmetrized function $f^{[n]}(x_1, \cdots, x_n)$ be equal to $\varphi^{[n]}(x_1, \cdots, x_n)$.

Proof. If f is an unbiased estimate then $f^{[n]}$ is a symmetric unbiased estimate and therefore, by Theorem 3, equal to $\varphi^{[n]}$; the converse follows from the facts that

$$\int \cdots \int f(x_1, \cdots, x_n)\, dP(x_1) \cdots dP(x_n)$$

$$= \int \cdots \int f^{[n]}(x_1, \cdots, x_n)\, dP(x_1) \cdots dP(x_n)$$

and that (as a consequence of the hypothesis $\mathfrak{D} \subseteq \mathfrak{D}^*(E)$) the equality of $f^{[n]}$ and $\varphi^{[n]}$ for points whose coordinates are in E implies the equality of their integrals.

Theorem 4 exhibits all possibilities for unbiased estimates (over domains satisfying the hypotheses). Given a point $(x_1, \cdots, x_n)$, suppose that the number of different points obtained from it by permutations of the coordinates is N. (If the x_i are all different then $N = n!$). An unbiased estimate is obtained if f is defined arbitrarily over $N - 1$ of these points and if its value on the Nth point is chosen so that the identity $f^{[n]} = \varphi^{[n]}$ is satisfied. As long as the arbitrary choices at the (possibly) uncountably infinite point groups are not too wild and not too large (i.e. are such that the resulting function f is measurable and integrable), f will indeed be an unbiased estimate. Typical nonpathological examples of unsymmetric unbiased estimates are weighted averages of the permuted values of $\varphi(x_1, \cdots, x_k)$, similar to the unweighted average $\varphi^{[n]}(x_1, \cdots, x_n)$.

6. Uniqueness. The assumption of symmetry is a rather natural one to require of an estimate: it amounts to requiring that the estimated value should be independent of the order in which the observations are made. Theorems 3 and 4 establish that the concept of symmetry is inherently associated with unbiased estimation and that, under this assumption, there is a unique unbiased estimate (whenever there is one at all). These theorems, therefore, constitute a partial answer to the uniqueness problem (III): symmetry, after all, is a possible interpretation of "good" estimate. From another point of view the answer to the problem of "best" estimate is contained in the following theorem.

Theorem 5. *Under the hypotheses of Theorem 4, among all unbiased estimates of*

$$F(P) = \int \cdots \int \varphi(x_1, \cdots, x_k)\, dP(x_1) \cdots dP(x_k)$$

the symmetric one, $\varphi^{[n]}(x_1, \cdots, x_n)$ is the one with least variance or, equivalently, the least second moment

$$\int \cdots \int \{\varphi^{[n]}(x_1, \cdots, x_n)\}^2\, dP(x_1) \cdots dP(x_n).$$

Proof. Observe first that if $X_1, \cdots, X_n$ are independent random variables

with the same distribution P then, if f is an unbiased estimate of $F(P)$, the variance of $f(X_1, \cdots, X_n)$ is given by

$$E\{f(X_1, \cdots, X_n)\}^2 - E^2\{f(X_1, \cdots, X_n)\}.$$

Since the second term is the same for all f, namely $F^2(P)$, minimizing the variance is indeed equivalent to minimizing

$$E\{f(X_1, \cdots, X_n)\}^2 = \int \cdots \int \{f(x_1, \cdots, x_n)\}^2 \, dP(x_1) \cdots dP(x_n).$$

This quantity need not be finite even for f's and P's for which $E\{f(X_1, \cdots, X_n)\}$ exists. It will be shown, however, to be minimized by $\varphi^{[n]}$ in the sense that

$$E\{\varphi^{[n]}(X_1, \cdots, X_n)\}^2 \leq E\{f(X_1, \cdots, X_n)\}^2$$

for all unbiased estimates f and all P, and that the inequality actually holds for some P.

For the proof consider any unbiased estimate f of F. For any given point $(x_1, \cdots, x_n)$ suppose that N is the number of different points obtained from it by permutations of the arguments, and denote by f_i, $i = 1, \cdots, N$, the values of f at these points Since, according to Theorem 4, $f^{[n]} = \varphi^{[n]}$, it follows that

$$(\varphi^{[n]})^2 = \left(\frac{1}{N} \sum_{i=1}^{N} f_i\right)^2 \leq \frac{1}{N} \sum_{i=1}^{n} f_i^2 = (f^2)^{[n]},$$

Hence

$$\int \cdots \int \{\varphi^{[n]}(x_1, \cdots, x_n)\}^2 \, dP(x_1) \cdots dP(x_n)$$
$$\leq \int \cdots \int \{f^2(x_1, \cdots, x_n)\}^{[n]} \, dP(x_1) \cdots dP(x_n)$$
$$= \int \cdots \int f^2(x_1, \cdots, x_n) \, dP(x_1) \cdots dP(x_n).$$

This already establishes the minimal property of $\varphi^{[n]}$ in the weak sense.

If the inequality were an equality for all P for which the terms are defined then, by Lemma 2, it would follow that

$$\{\varphi^{[n]}(x_1, \cdots, x_n)\}^2 = \{f^2(x_1, \cdots, x_n)\}^{[n]}$$

for all $(x_1, \cdots, x_n)$. Hence the Schwarz inequality, as applied above to the sum $\frac{1}{N} \sum_{i=1}^{N} f_i$, reduces to an equality; this can happen if and only if $(f_1, \cdots, f_N)$ is proportional to $\left(\frac{1}{N}, \cdots, \frac{1}{N}\right)$, i.e. if and only if all f_i are equal to each other. The validity of this statement for every point is equivalent to the symmetry of f and hence, by Theorem 3, to the statement $f = \varphi^{[n]}$. This concludes the proof of Theorem 5.

7. Concluding remarks. (1) The most obvious estimates of the moments, $F_m(P)$, of a distribution about the origin are the sample moments

$$\frac{1}{n}\sum_{i=1}^{n} x_i^m .$$

Their use is justified by the uniqueness theorems (3, 4, and 5) of this paper. Similarly one might think that the natural estimates of the moments, $\bar{F}_m(P)$, about the expected value $F_1(P)$, are best estimated by the sample moments

$$g_m(x_1, \cdots, x_n) = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^m$$

about the sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$. Denote by $f_m(x_1, \cdots, x_n)$ the estimate of $\bar{F}_m(P)$ obtained by expanding $\bar{F}_m(P)$ in terms of the $F_j(P)$, as in the proof of the corollary to Theorem 2, and then estimating each term by the symmetric estimate considered in Theorems 3 and 4. Then an easy calculation shows that

$$f_2(x_1, \cdots, x_n) = \frac{n}{n-1} g_2(x_1, \cdots, x_n)$$

and

$$f_3(x_1, \cdots, x_n) = \frac{n^2}{(n-1)(n-2)} g_3(x_1, \cdots, x_n).$$

(These functions are the classical estimates of $\bar{F}_2$ and $\bar{F}_3$.) For $m > 3$, f_m can still be expressed in terms of g's, but no longer as a constant multiple of g_m. It appears that in general f_m is a linear combination of $g_1, \cdots, g_m$ with coefficients which are rational numbers whose denominators are $(n-1)(n-2)\cdots(n-m+1)$. This fact is another aspect of the non existence of unbiased estimates of order n for $\bar{F}_m$ when $m > n$.

(2) For any Borel set E on the real line denote by $F_E(P)$ the probability, $P(E)$, assigned by P to E. If $\varphi_E(x)$ is the characteristic function of the set E, the representation

$$F_E(P) = \int \varphi_E(x)\, dP(x)$$

shows that $F_E(P)$ is homogeneous of degree 1, and therefore possesses unbiased estimates of all orders. The symmetric unbiased estimate of order n is given, in perfect accordance with intuitive demands, by the function $f_E(x_1, \cdots, x_n)$ whose value is $\frac{1}{n}$ times the number of those coordinates x_i which belong to E.

(3) The situation in estimating such "non linear" functions as $(F_1(P))^2$ is somewhat paradoxical. In the first place it appears strange that there should be

essentially different processes for estimating the expected value and the square of the expected value. (Recall that since

$$(F_1(P))^2 = \int\int x_1 x_2 \, dP(x_1) \, dP(x_2),$$

the symmetric unbiased estimate of $(F_1(P))^2$, of order n, is $(x_1x_2)^{[n]}$.) Consider, for instance, the distribution P which assigns probability $\frac{1}{2}$ to each of the points $+1$ and -1. The symmetric unbiased estimate of order 2 for $F_1(P)$ is $\frac{1}{2}(x_1 + x_2)$, and for $(F_1(P))^2$ it is x_1x_2. Hence in the four possible cases

$$(1, 1), (1, -1), (-1, 1), (-1, -1)$$

the biased, incorrect estimate $\{\frac{1}{2}(x_1 + x_2)\}^2$ for $(F_1(P))^2$ yields

$$1, \quad 0, \quad 0, \quad 1,$$

whereas the unbiased, correct estimate yields

$$1, \quad -1, \quad -1, \quad 1.$$

The actual value of $(F_1(P))^2$ is, of course, 0. Hence it is true in this case that whenever the biased estimate is in error, the unbiased one errs by the same amount. To add insult to injury, the unbiased procedure even yields negative estimates for the essentially non negative quantity $(F_1(P))^2$. These considerations seem to indicate the necessity for caution in using unbiased estimates of "non linear" quantities, such for instance as $\bar{F}_m(P)$.

SOME SIGNIFICANCE TESTS BASED ON ORDER STATISTICS

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1. Summary. In this paper significance tests are developed whose application requires only the determination of one order statistic and the computation of sums of sample values. The simplest case considered is that of testing a new sample value x on the basis of m previous sample values $y_1, \cdots, y_m$, all sample values being assumed from normal populations with the same variance. Two separate tests of whether the mean of the new population from which x was taken exceeds the mean of the population from which $y_1, \cdots, y_m$ were drawn consist in accepting the alternative that the new population mean exceeds the old population mean if

$$x > \left(\frac{\sqrt{m+1}+1}{m}\right)\sum_1^m y_i - \sqrt{m+1}\, y_{(u)} \tag{1}$$

$$x > \left(\frac{\sqrt{m+1}-1}{m}\right)\sum_1^m y_i + \sqrt{m+1}\, y_{(m+1-u)}, \tag{2}$$

where $y_{(u)}$ is the uth largest of $y_1, \cdots, y_m$. It can be shown that both of these tests have the same power so that either one might be equally well selected for use. In practical application, however, there may exist reasons for preferring one test to the other. Similarly, the alternative that the new population mean is less than the old population mean will be accepted if

$$x < \left(\frac{\sqrt{m+1}+1}{m}\right)\sum_1^m y_i - \sqrt{m+1}\, y_{(m+1-u)} \tag{3}$$

$$x < \left(\frac{\sqrt{m+1}-1}{m}\right)\sum_1^m y_i + \sqrt{m+1}\, y_{(u)}. \tag{4}$$

All four of these significance tests have the same power, also the same significance level $\alpha(u, m)$. By appropriate choice of u and m the significance level can be made to assume values suitable for significance tests. For example,

$$\alpha(1, 6) = .0156, \qquad \alpha(2, 10) = .0107$$
$$\alpha(3, 13) = .0110, \qquad \alpha(4, 16) = .0107.$$

The above tests are still valid if each of $x, y_1, \cdots, y_m$ equals a sum of r sample values.

These order statistic tests are generalized to the case where x is a sum of r new sample values; $y_1, \cdots, y_m$ each equals a sum of s past sample values and another sum of relatively weighted past sample values is utilized but not as an order statistic. The introduction of this relatively weighted sum allows less reliable past information to be lumped together and weighted according to its relative importance.

In comparing the order statistic tests with the most powerful tests which could be used for these alternatives it is found that the size of the samples used must be increased in order to bring the efficiency of the order statistic test up to that of the corresponding most powerful test. Thus the advisability of using the order statistic test will depend upon whether it is more desirable to take larger samples but have less computation.

2. Introduction. Many statistical problems are concerned with the determination of whether a new sample can be considered as having been drawn from the same population as that from which a previous sample was taken. Frequently this reduces to the question of whether the mean of the population from which the new sample came is greater than the mean of the past sample population. The problem of whether the new population mean is less than that of the old population is also occasionally investigated. If both populations can be considered normal with the same variance, it is well known that the most powerful Studentized test of each of these one-sided alternatives is furnished by use of the appropriate Student t-test. When the number of previous sample values from which the test is determined is large, however, the computation of the numerical value required for the application of the Student t-test becomes lengthy. This calculation difficulty can become very important if the test is to be applied repeatedly as, for example, in quality control work. It is desirable, therefore, to develop other Studentized tests which are easily calculated and whose efficiency with relation to the corresponding Student t-tests is reasonably high. It is the purpose of this paper to develop tests of this type by the use of order statistics.

The class of tests in which a new sample value x is tested on the basis of m previous sample values $y_1, \cdots, y_m$ used as order statistics is developed in detail. The significance tests arising are the ones given in the summary above. For a better intuitive understanding of what takes place rewrite (1) to (4) as

$$(1') \qquad x - \bar{y} > \sqrt{m+1}(\bar{y} - y_{(u)})$$

$$(2') \qquad x + \bar{y} > \sqrt{m+1}(\bar{y} + y_{(m+1-u)})$$

$$(3') \qquad x - \bar{y} < \sqrt{m+1}(\bar{y} - y_{(m+1-u)})$$

$$(4') \qquad x + \bar{y} < \sqrt{m+1}(\bar{y} + y_{(u)}),$$

where $\bar{y}$ is the average of the y_i. The relative efficiencies of these tests with respect to the corresponding Student t-tests are determined and the simplicity of the computation necessary for their application is outlined. The method of attack having been sufficiently indicated by the development of this special class of tests, more general tests based on order statistics are stated but not proved here.

3. Statement of the significance tests. Let each of $x, y_1, \ldots, y_m$ be distributed independently of all the others, x according to $N(\nu, \sigma^2)$ and the y_i, $(i = 1, \cdots, m)$, according to $N(\mu, \sigma^2)$, where the notation $N(\xi, \sigma^2)$ signifies the

normal distribution with mean ξ and variance σ^2. As above let $y_{(u)}$ denote the uth largest of $y_1, \cdots, y_m$. The one-sided significance tests are then stated as follows:
If

$$
(5) \qquad
\begin{aligned}
x &> \frac{1}{K_2}\sum_1^m y_i - \frac{K_1}{K_2} y_{(u)} & (K_2 > 0) \\
x &> \frac{1}{K_2}\sum_1^m y_i - \frac{K_1}{K_2} y_{(m+1-u)} & (K_2 < 0)
\end{aligned}
$$

accept the alternative $\mu < \nu$, otherwise accept the hypothesis tested, namely that $\mu = \nu$.
If

$$
(6) \qquad
\begin{aligned}
x &< \frac{1}{K_2}\sum_1^m y_i - \frac{K_1}{K_2} y_{(m+1-u)} & (K_2 > 0) \\
x &< \frac{1}{K_2}\sum_1^m y_i - \frac{K_1}{K_2} y_{(u)} & (K_2 < 0)
\end{aligned}
$$

accept $\nu < \mu$, otherwise accept $\nu = \mu$.

The constants K_1 and K_2 are given by

$$
(7) \qquad K_1 = m + 1 \pm \sqrt{m+1}, \qquad K_2 = -1 \mp \sqrt{m+1},
$$

where all upper signs or all lower signs will be chosen so that to a given value of K_1 there is but one value of K_2 This rule for the choice of signs will hold throughout the paper.

It is to be noted that (5) defines two separate significance tests of the hypothesis $\mu = \nu$ against the alternative $\mu < \nu$ depending upon whether it is decided to use the positive or the negative value given for K_2. A similar statement applies to the two significance tests defined by (6).

Each of these four significance tests can be shown to have the same significance level, which is determined by the values of u and m. Denote this significance level by $\alpha(u, m)$. Then it can be demonstrated that

$$
\alpha(1, m) = (\tfrac{1}{2})^m, \qquad \alpha(2, m) = (m + 1)(\tfrac{1}{2})^m
$$

$$
\alpha(3, m) = (m^2 + m + 2)(\tfrac{1}{2})^{m+1}, \qquad \alpha(4, m) = \tfrac{1}{3}(m^3 + 5m + 6)(\tfrac{1}{2})^{m+1}.
$$

The general expression for $\alpha(u, m)$ is given by (12).

It is to be observed that the application of these tests is independent of the parameters of the normal populations in question.

4. Analysis. An analysis will be given for the development of the significance test in which the alternative is $\mu < \nu$ and $K_2 > 0$. The developments of the properties of the other three tests are almost identical with that for this case and will not be given here.

Now consider this analysis. Let

$$x' = \frac{x - \nu}{\sigma}, \qquad y_i' = \frac{y_i - \mu}{\sigma}, \qquad (i = 1, \cdots, m).$$

Then x' and the y_i' are independently distributed according to $N(0, 1)$. Define

$$r_u = \frac{1}{K_1}\left(K_1 y_u' - \sum_1^m y_i' + K_2 x'\right), \qquad (u = 1, \cdots, m).$$

It is easily seen that

$$E(r_u) = 0, \qquad E(r_u^2) = \frac{1}{K_1^2}(K_2^2 + K_1^2 - 2K_1 + m),$$

$$E(r_u r_v) = \frac{1}{K_1^2}(K_2^2 - 2K_1 + m), \qquad (u \neq v).$$

Thus the condition which must be fulfilled in order that the r_u be independently distributed according to $N(0, 1)$ is that

$$K_2^2 - 2K_1 + m = 0. \tag{8}$$

To insure that the r_u are independent of μ when $\mu = \nu$ it is evidently necessary that

$$K_1 - m + K_2 = 0. \tag{9}$$

Solving (8) and (9) for K_1 and K_2 one obtains (7).

Restrict the r_u by conditions (8) and (9) and let $r_{(u)}$ be the uth largest of $r_1, \cdots, r_m$. From (8) $K_1 > 0$; therefore

$$r_{(u)} = \frac{1}{K_1}\left[K_1 y_{(u)}' - \sum_1^m y_i' + K_2 x'\right],$$

where $y_{(u)}'$ is the uth largest of $y_1', \cdots, y_m'$. Then using (9),

$$r_{(u)} = \frac{1}{K_1 \sigma}\left[K_1 y_{(u)} - \sum_1^m y_i + K_2 x + K_2(\mu - \nu)\right].$$

From the definition of the power function and (5) for $K_2 > 0$, it follows that the power function for this test is given by

$$\begin{aligned} \text{Power Function} &= Pr\left[x > \frac{1}{K_2}\sum_1^m y_i - \frac{K_1}{K_2} y_{(u)}\right] \\ &= Pr\left[0 < K_1 y_{(u)} - \sum_1^m y_i + K_2 x < \infty\right] \\ &= Pr\left[\frac{K_2}{K_1\sigma}(\mu - \nu) < \frac{1}{K_1\sigma}\left\{K_1 y_{(u)} - \sum_1^m y_i + K_2 x + K_2(\mu - \nu)\right\} < \infty\right] \\ &= Pr\left[\frac{K_2}{K_1\sigma}(\mu - \nu) < r_{(u)} < \infty\right]. \end{aligned} \tag{10}$$

The distribution function of the order statistic $r_{(u)}$ may be found in [1], from which it follows that

$$\text{Power Function} = \frac{m!}{(u-1)!\,(m-u)!} \tag{11}$$
$$\cdot \int_{(K_2/K_1\sigma)(\mu-\nu)}^{\infty} \left(\int_{-\infty}^{z} f(y)\,dy\right)^{u-1} \left(\int_{z}^{\infty} f(y)\,dy\right)^{m-u} f(z)\,dz,$$

where

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}.$$

Consider the value of the power function under the assumption that the hypothesis is true. Then $\mu = \nu$ and from (11) the significance level of the test is given by

$$\alpha(u, m) = \frac{m!}{(u-1)!\,(m-u)!} \tag{12}$$
$$\cdot \int_{0}^{\infty} \left(\int_{-\infty}^{z} f(y)\,dy\right)^{u-1} \left(\int_{z}^{\infty} f(y)\,dy\right)^{m-u} f(z)\,dz.$$

The method used to eliminate σ from the quantities required for the application of the significance test, therefore, is to have the limits 0 and ∞ in the probability expression (10) for the power function when the hypothesis is true. Suitable significance levels are obtained by varying the statistical function $r_{(u)}$ by means of the selection of the values of u and m.

5. Comparison with Student t-test. The test considered is that of a single sample value on the basis of m other sample values. Hence, the corresponding Student t-test has $m - 1$ degrees of freedom. The probabilities of Type II errors for the Student t-tests are calculated for values of

$$\delta = \frac{\mu - \nu}{\sigma\sqrt{1 + \dfrac{1}{m}}}$$

by use of the normal approximation given in [2].

Using this notation

$$\frac{K_2}{K_1\sigma}(\mu - \nu) = \delta\frac{K_2}{K_1}\sqrt{1 + \frac{1}{m}}$$

and from (11) the power function for the significance test for which the alternative is $\mu < \nu$ and $K_2 > 0$ is found to be

$$\frac{m!}{(u-1)!\,(m-u)!} \int_{\delta(K_2/K_1)\sqrt{1+(1/m)}}^{\infty} \left(\int_{-\infty}^{z} f(y)\,dy\right)^{u-1} \left(\int_{z}^{\infty} f(y)\,dy\right)^{m-u} f(z)\,dz.$$

The probability of a Type II error for a given value of δ is equal to one minus the value of the power function for this value of δ.

It can be proved that the other three significance tests have the same probabilities of Type II errors as the one considered above.

The numerical comparison of the two types of tests is contained in Table I. In each instance the significance level was chosen to be approximately 01.

The process of increasing the size of each sample by a given percentage has practical meaning if each of $x, y_1, \cdots, y_m$ equals the sum of r sample values For example, if $x, y_1, \cdots, y_m$ each consist of the sum of ten sample values, increasing the sample size by 30% would amount to letting $x, y_1, \cdots, y_m$ each equal the sum of thirteen sample values. The case where each of $x, y_1, \cdots, y_m$

TABLE I

Test	Degrees of Freedom	m	% Increase in Sample Size	Significance Level	Probability of Type II Error			
					$\delta = -1$	$\delta = -2$	$\delta = -3$	$\delta = -4$
t	5		0	.0156	.919	.750	.477	.215
O.S.		6	0	.0156	.919	.752	.506	.276
O.S.		6	5	.0156	.916	.742	.486	.256
O.S.		6	10	.0156	914	.732	.469	.239
t	9		0	.0107	.930	.735	.413	.142
O.S.		10	0	.0107	.936	.782	.527	.270
O.S.		10	20	.0107	.927	.738	.448	.191
O.S.		10	30	.0107	.921	.715	.411	.161
t	12		0	.0110	.920	.699	.358	.106
O.S.		13	0	.0110	.933	.771	.492	.245
O.S.		13	30	.0110	.919	717	.378	.139
O.S.		13	40	.0110	.913	.679	.353	.119
t	15		0	.0107	.919	.688	337	.092
O S.		16	0	.0107	.938	.765	.488	.234
O S.		16	40	.0107	.917	.687	.351	.111
O.S.		16	50	.0107	.912	.664	.310	.090

equals the sum of r sample values will be treated later and will be shown to be a particular case of the one analyzed above.

In Table I the order statistic tests (O.S.) are calculated for cases where the size of each sample is increased by the same percentage. This amounts to saying that the amount of information used for the test has been increased by this percentage. This method furnishes a quantitative estimate of the relative efficiency of the order statistic test as compared with the corresponding Student t-test. For example, if 30% more information is required for the order statistic test to have the same probabilities of Type II errors as the corresponding Student

t-test, then the order statistic test will be said to have a relative efficiency of $\frac{1}{1.3} = 77\%$.

Examination of Table I shows that the order statistic tests have the approximate relative efficiencies listed in Table II. These relative efficiencies can be shown to be approximately the same as those for other significance levels.

6. Computation required. Since application of the order statistic test requires only the determination of one order statistic, the calculation of one sum, the multiplication of each of these quantities by given constants and the subtraction of the resulting values, the amount of computation required for application of the order statistic test is obviously much less than is necessary for the application of the corresponding Student t-test.

If the test is applied continuously from one sample to the next, as in quality control work, the value of $\sum_1^m y_i$ can be calculated by a continuous process. For let the sample values be taken in the order $y_1, \cdots, y_m, x$, where x is the new

TABLE II

m	Significance Level	% Increase in Sample Size	Relative Efficiency
6	.0156	5	95%
10	.0107	25	80%
13	.0110	35	74%
16	.0107	43	70%

sample value which is to be tested on the basis of the previous m sample values $y_1, \cdots, y_m$. Then x for the present test becomes y_m for the next test; y_m becomes $y_{m-1}; \cdots; y_2$ becomes y_1, and y_1 for the present test is no longer used. The value of x will be furnished by the next sample value drawn. Thus, $\sum_1^m y_i$ for the next test is calculated by adding $x - y_1$ for the present test to $\sum_1^m y_i$ for the present test. The order statistic can be easily determined from a plot of the sample values which is also applied continuously from one sample to the next.

7. Generalization of results. The derivations given above are immediately applicable to the case where x represents the sum of r sample values from a population with distribution $N(\nu', \sigma'^2)$, and each y_i, $(i = 1, \cdots, m)$, equals the sum of r sample values from a population with distribution $N(\mu', \sigma'^2)$. Then x would be distributed according to $N(r\nu', r\sigma'^2)$ and the y_i would be distributed according to $N(r\mu', r\sigma'^2)$. These distributions are of the form $N(\nu, \sigma^2)$ and $N(\mu, \sigma^2)$, where $\mu = r\mu'$, $\nu = r\nu'$ and $\sigma^2 = r\sigma'^2$.

If x equals the sum of r sample values from a population with distribution $N(\nu, \sigma^2)$ and each y_i, $(i = 1, \cdots, m)$, equals the sum of s sample values from a population with distribution $N(\mu, \sigma^2)$, the significance tests are derived in a similar manner and can still be stated in the forms (5) and (6), but the values of K_1 and K_2 become

$$K_1 = m + \frac{r}{s} \pm \sqrt{\frac{r}{s}\left(m + \frac{r}{s}\right)}, \qquad K_2 = -\sqrt{\frac{r}{s}} \mp \sqrt{m + \frac{r}{s}}.$$

The power function for the test in which the alternative is $\mu < \nu$ and $K_2 > 0$ is found by replacing $\frac{K_2}{K_1 \sigma}(\mu - \nu)$ by $\frac{K_2\sqrt{r}}{K_1\sigma}(\mu - \nu)$ in (11). The significance level of each of the four tests is again furnished by (12) and it can be shown that each test has the same power.

To this point all significance tests considered have consisted of testing a new sample on the basis of m previous samples used as order statistics. In some cases, however, it may be desirable to utilize additional samples in the test but not as order statistics. These sample values can be gathered together in a summation term in which values from different samples are given relative numerical weighting. This procedure can be used to emphasize those sample values which appear to be more important from practical consideration with relation to those which seem to have less importance. The determination of what relative weighting scheme to use is to be decided by the person applying the test and is not considered as a problem of this paper. The significance tests with this property can be stated as follows:

Let each of x_a, y_{ib}, z_{jc}, $(a = 1, \cdots, r; b = 1, \cdots, s; c = 1, \cdots, n_j; i = 1, \cdots, m; j = 1, \cdots, n)$, be distributed independently of all the others, the x_a according to $N(\nu, \sigma^2)$ and the y_{ib} and z_{jc} according to $N(\mu, \sigma^2)$. Define $y_u = \sum_{b=1}^{s} y_{ub}$, $(u = 1, \cdots, m)$, and let $y_{(u)}$ be the uth largest of $y_1, \cdots, y_m$. The one-sided significance tests are then given by

If

$$\sum_1^r x_a > \frac{-\sqrt{r}}{K_2} V_{m+1-u} \qquad (K_2 > 0)$$

$$\sum_1^r x_a > \frac{-\sqrt{r}}{K_2} V_u \qquad (K_2 < 0)$$

accept the alternative $\mu < \nu$, otherwise accept $\mu = \nu$.

If

$$\sum_1^r x_a < \frac{-\sqrt{r}}{K_2} V_u \qquad (K_2 > 0)$$

$$\sum_1^r x_a < \frac{-\sqrt{r}}{K_2} V_{m+1-u} \qquad (K_2 < 0),$$

accept $\mu > \nu$, otherwise accept $\mu = \nu$.

The quantity V_u is given by

$$V_u = \frac{1}{\sqrt{s}}\sum_{i=1}^{m}\sum_{b=1}^{s} y_{ib} - \frac{K_1}{\sqrt{s}} y_{(u)} + \sum_{j=1}^{n} \frac{C_j}{\sqrt{n_j}}\left(\sum_{c=1}^{n_j} Z_{jc}\right),$$

where the constants C_j, $(j = 1, \cdots, n)$, are defined by $C_j = w_j\eta$, the w_j being given positive weights. The values of η, K_1 and K_2 are

$$\eta = \frac{\frac{A}{B}\sqrt{\frac{r}{s}}K_2}{m + A^2/B}, \qquad K_1 = \frac{m}{m + A^2/B}\left(m + \frac{A^2}{B} + \sqrt{\frac{r}{s}}K_2\right),$$

$$K_2 = \frac{m + A^2/B}{\left[B(m + A^2/B)^2 + A^2\left(\frac{r}{s}\right)\right]} \cdot \left\{Bm\sqrt{\frac{r}{s}} \pm \sqrt{B^2m^2\left(\frac{r}{s}\right) + Bm\left[B(m + A^2/B)^2 + A^2\left(\frac{r}{s}\right)\right]}\right\},$$

where

$$A = \frac{1}{\sqrt{s}}\sum_{1}^{n} w_j\sqrt{n_j}, \qquad B = \sum_{1}^{n} w_j^2.$$

The quantity η in the expressions for the C_j is not considered given but is determined in the derivation of the tests. The two equations corresponding to (8) and (9) then contain three undetermined quantities η, K_1 and K_2. Thus there are infinitely many possible selections of these quantities, each selection resulting in a valid significance test. The values of η, K_1 and K_2 given above, however, are the ones which result in the maximum power function and consequently the smallest probabilities of Type II errors. The power function for the test in which the alternative is $\mu < \nu$ and $K_2 > 0$ is that given in (11) with $\frac{K_2}{K_1\sigma}\cdot(\mu - \nu)$ replaced by $\frac{K_2\sqrt{r}}{K_1\sigma}(\mu - \nu)$. The significance level of each of the four tests given above is still that of (12). It can also be shown that each of the tests has the same probabilities of Type II errors.

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CHAINS OF RARE EVENTS

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1. Summary. The negative binomial distribution of Greenwood and Yule is generalized and modified in order to obtain distribution curves which could be used in many concrete cases of chains of rare events. Assuming that the numbers of single, double, triple, and so on, events are distributed according to Poisson's law with parameters $\lambda_1, \lambda_2, \lambda_3 \cdots$ respectively, and that λ_s is given by $\lambda_s = \lambda_1 \frac{a^{s-1}}{s!}$, the probability of obtaining M successful events is studied. In the considered relation λ_s, for convenient values of a, first increases with s and after a certain saturation value of s starts to decrease. A relation of this type is very suitable for studying the distribution of score in a match between two first class billiard players, the probability of accidents on a highway of dense traffic, etc. The general methods of finding the distribution curves for arbitrary relations between the λ's are indicated. The method of steepest descent is applied to find an acceptable approximation of the distribution function; and the advantage of this method is pointed out for other similar cases, in addition to the concrete one which was developed, in which the method of direct expansion into power series becomes inapplicable.

2. Introduction. M. Greenwood and G. U. Yule [1] have deduced the negative binomial distribution from a compound Poisson law:

$$P(m, \lambda) = \frac{\lambda^m}{m!} e^{-\lambda},$$

where λ itself is a random variable distributed according to Pearson's law of type III:

$$P(\lambda)\, d\lambda = \beta^{\alpha+1} \frac{\lambda^\alpha}{\alpha!} e^{-\beta\lambda}\, d\lambda.$$

They obtained the distribution

$$P(m) = (1 - a)^{\alpha+1} \frac{(\alpha + m)!}{\alpha!\, m!} a^m,$$

where $1 - a = \frac{\beta}{\beta + 1}$. As is easily seen, $P(m)$ is given by the coefficient of x^m in the expansion of:

$$(1 - a)^{\alpha+1}\left(1 - \frac{x}{\beta + 1}\right)^{-(\alpha+1)}.$$

[1] Research done at Harvard Astronomical Observatory as Guggenheim Fellow.

R. Lüders [2] has arrived at a negative binomial law by the following considerations. Certain events, like automobile accidents, can be classified as simple or multiple according to the number of units involved. Assume that the numbers of single, double, triple, and so on, events are distributed according to Poisson's law with the parameters $\lambda_1, \lambda_2, \lambda_3, \cdots$, respectively. The probability of obtaining n_1 single, n_2 double, n_3 triple, $\cdots$ successful events is (assuming mutual independence)

$$P(n_1, n_2, n_3, \cdots; \lambda_1, \lambda_2, \lambda_3, \cdots) = \frac{\lambda_1^{n_1}}{n_1!} \frac{\lambda_2^{n_2}}{n_2!} \cdots e^{-(\lambda_1+\lambda_2+\cdots)}. \tag{1}$$

The total number of successful events is

$$n = n_1 + 2n_2 + 3n_3 + \cdots + in_i + \cdots . \tag{2}$$

The probability of obtaining n successful events is given by the sum of all expressions (1) subject to the condition (2). This sum is given by the coefficient of x^n in the expansion

$$f(x) = e^{-(\lambda_1+\lambda_2+\cdots)} e^{\lambda_1 x+\lambda_2 x^2+\cdots}. \tag{3}$$

Now if the parameters λ_s satisfy

$$\lambda_s = \lambda_1 \frac{a^{s-1}}{s} \tag{4}$$

one finds

$$f(x) = \left(\frac{1-a}{1-ax}\right)^{\frac{\lambda_1}{a}} \tag{5}$$

and

$$P(n) = (1-a)^{\frac{\lambda_1}{a}} \frac{\lambda_1}{a}\left(\frac{\lambda_1}{a}+1\right)\cdots\left(\frac{\lambda_1}{a}+n-1\right)\frac{a^n}{n!}. \tag{6}$$

Taking $\frac{\lambda_1}{a}$ equal to $\alpha + 1$ one gets Greenwood and Yule's distribution in the form given above [3]. The negative binomial law has useful applications, for instance in some cases of accidents of workers in factories. It is proved that with values of a near 1, the most probable value for n is $n = 0$ and the average value is a finite number different from zero. Therefore the distribution will be in some way similar to the distribution of the scores in a match between two first class billiard players whose most frequent scores are zero and their average may be, say, 50. In the case of the Poisson distribution the most frequent score and the average score should be nearly the same. The relation (4) does not provide an adequate description of many practical distributions. For instance, in a match between two first class billiard players, the probability of making a second,

third, $\cdots$, point will be considerably greater than the probability of making the first. With the relation (4) λ_s is a decreasing function of s, while we shall investigate cases in which λ_s first increases with s and after a certain value of s starts to decrease. As other examples of distributions of similar types we shall mention the following: On a highway with dense traffic at high speeds the probability of only one car being involved in an accident may be smaller than the probability of having several cars involved. Something similar may be said for the cases of work accidents in factories where the work of one is interconnected with the work of others. In many cases of telephone calls (business transactions, organization of meetings, etc.) the sample Poisson law is not suitable to interpret the distribution of calls, since one call may increase the probability that the called person makes one or more calls.

The purpose of this paper is to treat the problem when, instead of (4), we take other expressions which may in a better way describe some processes such as the ones which we have referred to.

3. Modification and generalization of the scheme of Greenwood-Yule and Lüders. According to the relation (4) λ_s is a decreasing function of s and the parameter a must be in the interval $0 \leq a < 1$. Instead of (4) we shall use

$$\lambda_s = \lambda_1 \frac{a^{s-1}}{s!}, \tag{7}$$

where a may have any positive value. In particular for $a = 0$ our case reduces to the Poisson case.

From (7) it follows that

$$\frac{\lambda_{s+1}}{\lambda_s} = \frac{a}{s+1} \tag{8}$$

and we see that λ_s increases with s for $1 \leq s < a$ and decreases for $s + 1 > a$. Substituting from (7) in (3) we get

$$f(x) = e^{-(\lambda_1/a)e^a} e^{(\lambda_1/a)e^{ax}}. \tag{9}$$

As the probability of obtaining n successful events is given by the coefficient of x^n in (9), we shall expand $e^{\alpha e^{\beta x}}$ in power series (α, β being two arbitrary constants). We have

$$e^{\alpha e^{\beta x}} = 1 + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} e^{n\beta x} = e^{\alpha} + \sum_{n=1}^{\infty} \frac{\beta^n}{n!} x^n \sum_{m=1}^{\infty} \frac{m^n \alpha^m}{m!}. \tag{10}$$

Now

$$\sum_{m=1}^{\infty} \frac{m^n}{m!} \alpha^m = y_n(\alpha) e^{\alpha} \tag{11}$$

where [4]

$$
\begin{aligned}
y_1(\alpha) &= \alpha \\
y_2(\alpha) &= \alpha^2 + \alpha \\
y_3(\alpha) &= \alpha^3 + 3\alpha^2 + \alpha \qquad (12) \\
&\cdots\cdots\cdots\cdots \\
y_n(\alpha) &= \sum_{i=1}^{n} \frac{\Delta^i 0^n}{i!} \alpha^i.
\end{aligned}
$$

Here we use the notation of differences of zero: $\Delta^i 0^n$. We have

$$e^{\alpha e^{\beta x}} = e^{\alpha}\left[1 + \sum_{n=1}^{\infty} \frac{y_n(\alpha)\beta^n}{n!} x^n\right] \tag{13}$$

or

$$e^{\alpha e^{\beta x}} = e^{\alpha}\left[1 + \sum_{n=1}^{\infty} \frac{\beta^n}{n!} x^n \sum_{i=1}^{n} \frac{\Delta^i 0^n}{i!} \alpha^i\right]. \tag{14}$$

Now in our case

$$\alpha = \frac{\lambda_1}{a}, \qquad \beta = a, \tag{15}$$

whence

$$P(n) = e^{-(\lambda_1/a)(e^a-1)} \frac{a^n}{n!} \sum_{i=1}^{n} \frac{\Delta^i 0^n}{i!} \left(\frac{\lambda_1}{a}\right)^i, \qquad n > 0. \tag{16}$$

$$P(0) = e^{-(\lambda_1/a)(e^a-1)}, \qquad \text{for } n = 0. \tag{17}$$

We have in particular

$$
\begin{aligned}
P(1) &= \lambda_1 P(0) \\
P(2) &= \frac{1}{2!}(\lambda_1^2 + a\lambda_1)P(0) \\
P(3) &= \frac{1}{3!}(\lambda_1^3 + 3\lambda_1^2 a + \lambda_1 a^2)P(0) \\
P(4) &= \frac{1}{4!}(\lambda_1^4 + 6\lambda_1^3 a + 7\lambda_1^2 a^2 + \lambda_1 a^3)P(0) \\
P(5) &= \frac{1}{5!}(\lambda_1^5 + 10\lambda_1^4 a + 25\lambda_1^3 a^2 + 15\lambda_1^2 a^3 + \lambda_1 a^4)P(0) \\
P(6) &= \frac{1}{6!}(\lambda_1^6 + 15\lambda_1^5 a + 65\lambda_1^4 a^2 + 90\lambda_1^3 a^3 + 31\lambda_1^2 a^4 + \lambda_1 a^5)P(0) \\
P(7) &= \frac{1}{7!}(\lambda_1^7 + 21\lambda_1^6 a + 140\lambda_1^5 a^2 + 350\lambda_1^4 a^3 + 301\lambda_1^3 a^4 + 63\lambda_1^2 a^5 \\
&\qquad + \lambda_1 a^6)P(0) \qquad (18) \\
P(8) &= \frac{1}{8!}(\lambda_1^8 + 28\lambda_1^7 a + 266\lambda_1^6 a^2 + 1050\lambda_1^5 a^3 + 1701\lambda_1^4 a^4 + 966\lambda_1^3 a^5 \\
&\qquad + 127\lambda_1^2 a^6 + \lambda_1 a^7)P(0)
\end{aligned}
$$

$$\mathrm{P}(9) = \frac{1}{9!}(\lambda_1^9 + 36\lambda_1^8 a + 462\,\lambda_1^7 a^2 + 2646\lambda_1^6 a^3 + 6951\lambda_1^5 a^4 + 7770\lambda_1^4 a^5 + 3025\lambda_1^3 a^6 + 255\lambda_1^2 a^7 + \lambda_1 a^8)P(0)$$

$$\mathrm{P}(10) = \frac{1}{10!}(\lambda_1^{10} + 45\lambda_1^9 a + 750\lambda_1^8 a^2 + 5880\lambda_1^7 a^3 + 28827\lambda_1^6 a^4 + 42525\lambda_5^1 a^5 + 34105\lambda_1^4 a^6 + 9330\lambda_1^3 a^7 + 511\lambda_1^2 a^8 + \lambda_1 a^9)P(0).$$

For $\lambda_1 = a$ it follows that

$$P(0) = e^{-e^a+1} \tag{19}$$

$$P(n) = e^{-e^a+1}\frac{a^n}{n!}\,y_n(1) \tag{20}$$

$$\sum_{n=0}^{\infty} P(n) = ee^{-e^a}\left[1 + \sum_{n=1}^{\infty}\frac{a^n}{n!}\,y_n(1)\right].$$

Particular values of (20) are

$$\begin{aligned}
P(1) &= aP(0)\\
P(2) &= a^2 P(0)\\
P(3) &= \frac{5a^3}{3!}P(0)\\
P(4) &= \frac{15a^4}{4!}P(0)\\
P(5) &= \frac{52a^5}{5!}P(0)\\
P(6) &= \frac{203a^6}{6!}P(0)\\
P(7) &= \frac{877a^7}{7!}P(0)\\
P(8) &= \frac{4140a^8}{8!}P(0)\\
P(9) &= \frac{21147a^9}{9!}P(0)\\
P(10) &= \frac{115975a^{10}}{10!}P(0).
\end{aligned} \tag{21}$$

In Figure 1 we have graphed the curves $P(n)$ for the values $\frac{\lambda_1}{a} = 1$; $\lambda_1 = 0.1$, $\lambda_1 = 1$, $\lambda_1 = 2$. We see in particular, that for $\lambda_1 = 1$ we have $P(0) = P(1)$ and for $\lambda_1 = a = 1$ we have $P(0) = P(1) = P(2)$.

4. Application of the method of steepest descent. If λ_r is not given by (7) the above method of direct expansion of $f(x)$ into a power series, usually becomes

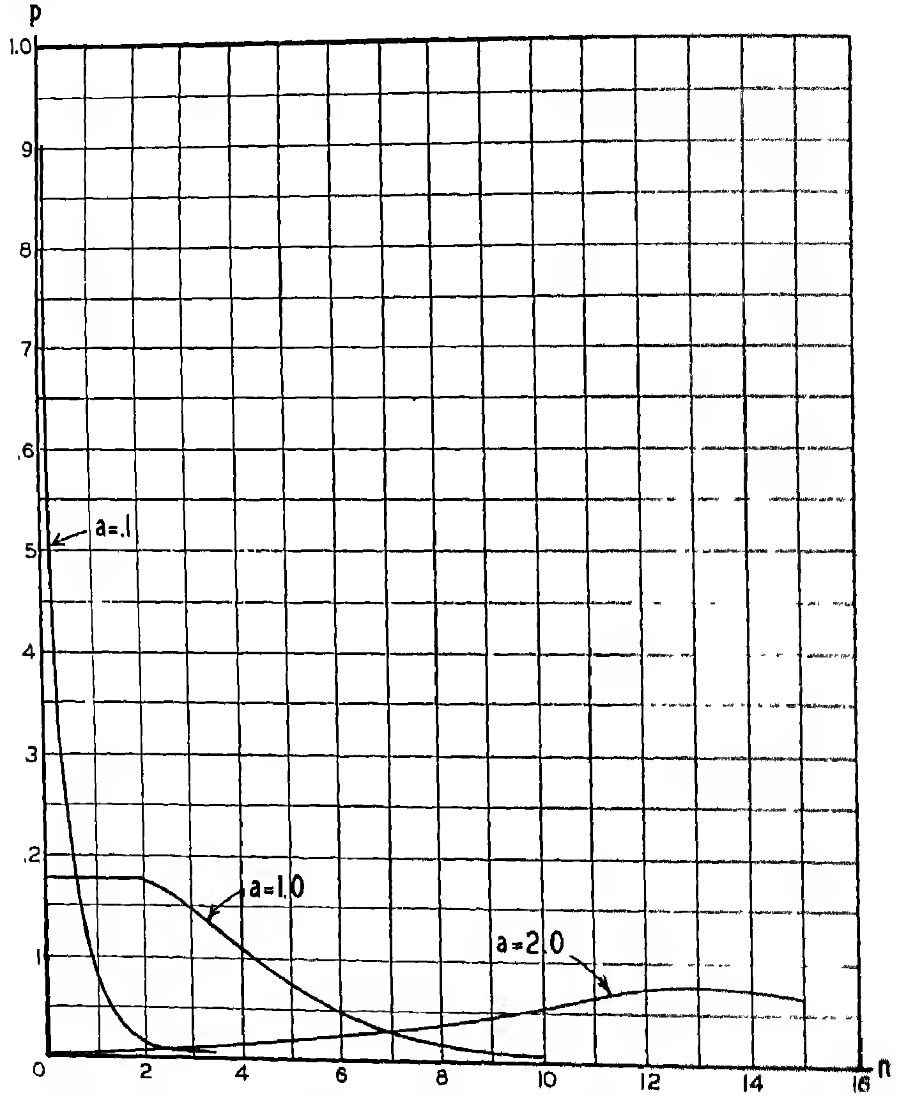

FIG. 1 DISTRIBUTION CURVES FOR $\frac{\lambda_1}{a} = 1, a = 0.1; \frac{\lambda_1}{a} = 1, a = 1; \frac{\lambda_1}{a} = 1, a = 2$

inapplicable. In many cases it is possible to use instead the method of steepest descent [5] in order to obtain approximate values for the coefficients of x^n in the relation (3).

As is well known, if $f(z)$ is an analytical function we have

$$\text{coeff. of } z^n = \frac{1}{2i\pi} \oint \frac{f(z)}{z^{n+1}}\, dz = \frac{1}{2i\pi} \oint e^{X(x,y)+iY(x,y)}\, dz \tag{22}$$

where $X + iY = \log \dfrac{f(z)}{z^{n+1}}$ and the integral is taken along any closed path around the origin.

To evaluate the integral (22) we shall follow a method similar to the one used by R. H. Fowler [6]. Putting $z = \rho e^{i\alpha}$ the relation (22) may be written:

$$\text{Coeff. of } z^n = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{f(\rho e^{i\alpha})}{\rho^n e^{in\alpha}}\, d\alpha \tag{23}$$

where the value of ρ is arbitrary. We shall put in particular $\rho = x_0$ where x_0 is the root of

$$\frac{x_0 f'(x_0)}{f(x_0)} = n. \tag{24}$$

For most functions which interest us $\dfrac{f(x)}{x^n} \to \infty$ as $x \to 0$ and as $x \to K$ (a positive number which in some cases may be infinite) and the second derivative is always positive. Consequently $f(x)/x^n$ has only one minimum between 0 and K, and (24) has therefore only one root x_0. Developing $\log \dfrac{f(x_0 e^{i\alpha})}{x_0^n e^{in\alpha}}$ into powers of α, (24) becomes

$$\text{coeff. of } x^n = \frac{1}{2\pi} \frac{f(x_0)}{x_0^n} \int_{-\pi}^{+\pi} e^{-(x_0^2\varphi''(x_0)/2)\alpha^2 + ig(x_0)\alpha^3 + h(x_0)\alpha^4 + \cdots}\, d\alpha, \tag{25}$$

where

$$\varphi(x) = \log \frac{f(x)}{x^n}.$$

In the case where $\varphi''(x_0) \dfrac{x_0^2}{2} \gg 1$ the first term in the exponent in (25) increases in absolute value very rapidly in the neighborhood of x_0. For small values of α we may therefore in a first approximation drop all other terms. Also, as this first term tends rapidly towards zero one does not appreciably increase the error by replacing the integral from $-\pi$ to $+\pi$ by the integral from $-\infty$ to $+\infty$.

In such cases we have, therefore, the approximate formula

$$\text{coeff. of } z^n \sim \frac{1}{2i\pi} \frac{f(x_0)}{x_0^n} \int_{\infty}^{+\infty} e^{-(\varphi''(x_0)x_0^2/2)\alpha^2}\, d\alpha = \frac{f(x_0)}{x_0^{n+1}\sqrt{2\pi\varphi''(x_0)}}. \tag{26}$$

We are now in a position to deduce asymptotic values for the probabilities $P(n)$

which we have previously calculated directly. In fact, for $f(x)$ defined by (9) we obtain from (26) for large n

$$P(n) \sim \frac{e^{-(\lambda_1/a)e^a}}{\sqrt{2\pi}} \frac{e^{n/ax_0}}{x_0^n\sqrt{n(ax_0+1)}}, \tag{27}$$

where x_0 is given by

$$e^{ax_0} = \frac{n}{\lambda_1 x_0}.$$

In particular for $\lambda_1 = a$ and putting $ax_0 = y_0$ it follows that

$$P(n) \sim 0.3989\left(\frac{\lambda_1}{y_0}\right)^n e^{-e^{\lambda_1}} \frac{e^{e^{y_0}}}{\sqrt{n(y_0+1)}}. \tag{28}$$

Comparing the numerical values given by the relation (28) with the exact values we find that even for $n = 4$ and $\lambda_1 = 1$ (28) gives an approximation with an error of about 5%.

Formula[2] (26) can also be used to evaluate the numbers $y_n(1)$ defined by (12) for $\alpha = 1$. Relation (13) gives for $\alpha = \beta = 1$

$$e^{e^x} = e\left[1 + \sum_{n=1}^{\infty} \frac{y_n(1)}{n!} x^n\right]$$

and therefore

$$\text{Coeff. of } x^n \text{ in expansion of } e^{e^x} = \frac{ey_n(1)}{n!}.$$

Putting $f(z) = e^{e^z}$ and using Stirling's formula for $n!$ we have from (26)

$$y_n(1) \sim \frac{e^{n\left[x_0+\frac{1}{x_0}-\left(1+\frac{1}{n}\right)\right]}}{\sqrt{x_0+1}},$$

[2] Applying this relation to $f(z) = e^z$ one obtains immediately Stirling's Formula:

$$\varphi(z) = \log\frac{f(z)}{z^n} = z - n\log z$$

$$\varphi'(z) = 1 - \frac{n}{z}, \qquad x_0 = n,$$

$$\varphi''(z) = \frac{n}{z^2}, \qquad \varphi''(x_0)\frac{x_0^2}{2} = \frac{n}{2},$$

$$\frac{1}{n!} \sim \left(\frac{e}{n}\right)^n \frac{1}{\sqrt{2\pi n}}.$$

Also relation (26) is useful to find other symptotic expressions; e g. for $f(z) = (pz + 9)^n$ one obtains for $n \to \infty$ the Laplace-Gauss formula.

where x_0 is given by

$$e^{x_0} = \frac{n}{x_0}.$$

For $n = 4$, $x_0 = 1.202$ and $y_4(1) \sim 15.56$. As the exact value of $y_4(1)$ is 15 we obtain in this case an error of less than 4%.

Repeating the calculations for $n = 6$, $x_0 = 1.432$, we find that $y_6(1)$ is given with an error of less than 3%.

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NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

A NOTE ON SOME SINGLE SAMPLING PLANS REQUIRING THE INSPECTION OF A SMALL NUMBER OF ITEMS

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In the practical application of sampling inspection plans it is often necessary to restrict the number of items (pieces, samples) inspected from each inspection lot to a relatively small number. For example, if many vendors are supplying a manufacturer with small lots of various kinds of material, the manufacturer will usually wish to have some check on his suppliers; however, he cannot afford to inspect large numbers of items from each lot. If sampling plans requiring the inspection of a small number of items are used, it is advantageous to know the characteristics of such plans. The present note offers several single sampling plans with sample size $n \leq 25$, together with their operating characteristic curves (OC curves) and average outgoing quality curves (AOQ curves).[2]

Single sampling plans for large lots may be described by the number n of items to be inspected, and the rejection number r. If r or more of the items inspected fail to meet some predetermined standard the lot is rejected; if less than r items fail to meet the standard the lot is accepted.

The OC curve (see Figures 1, 1A, 3 and 5) shows the relationship between the probability of rejecting a lot and the true quality of the lot. The quality of the lot is often measured by the "percent defective" in the lot; i.e., the proportion of material which does not meet some predetermined standard. It should be noted that the definition of OC curve given here is only one of several in common use. In particular, the vertical axis often gives the probability of "acceptance"; such a treatment would amount to an "inversion" of the curves given here. Another

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[1] The material in this note was originally prepared as an office memorandum for the use of engineering technical personnel in a Government Bureau. The author wishes to express his appreciation to Mr. C. F. Mosteller for extensive editorial work on the original memorandum which has resulted in a revision more suitable for publication in the *Annals*.

[2] The OC and AOQ curves are often adequate to analyze single sampling plans because it is not customary to curtail single sampling even when the outcome of the inspection (acceptance or rejection) is determined before all the items are inspected. In other kinds of sampling plans (double, multiple, and sequential) where curtailing is often used after the first sample, curves for the average amount of sampling are also useful. However, if one is interested in the average amount of inspection, including detailing, as a manufacturer inspecting his own product might be, curves for the average amount of inspection would be useful in connection with any sampling plan.

common form would have the "percentage of presented lots (of quality indicated on the horizontal axis) that will be rejected (accepted)" as its vertical scale.

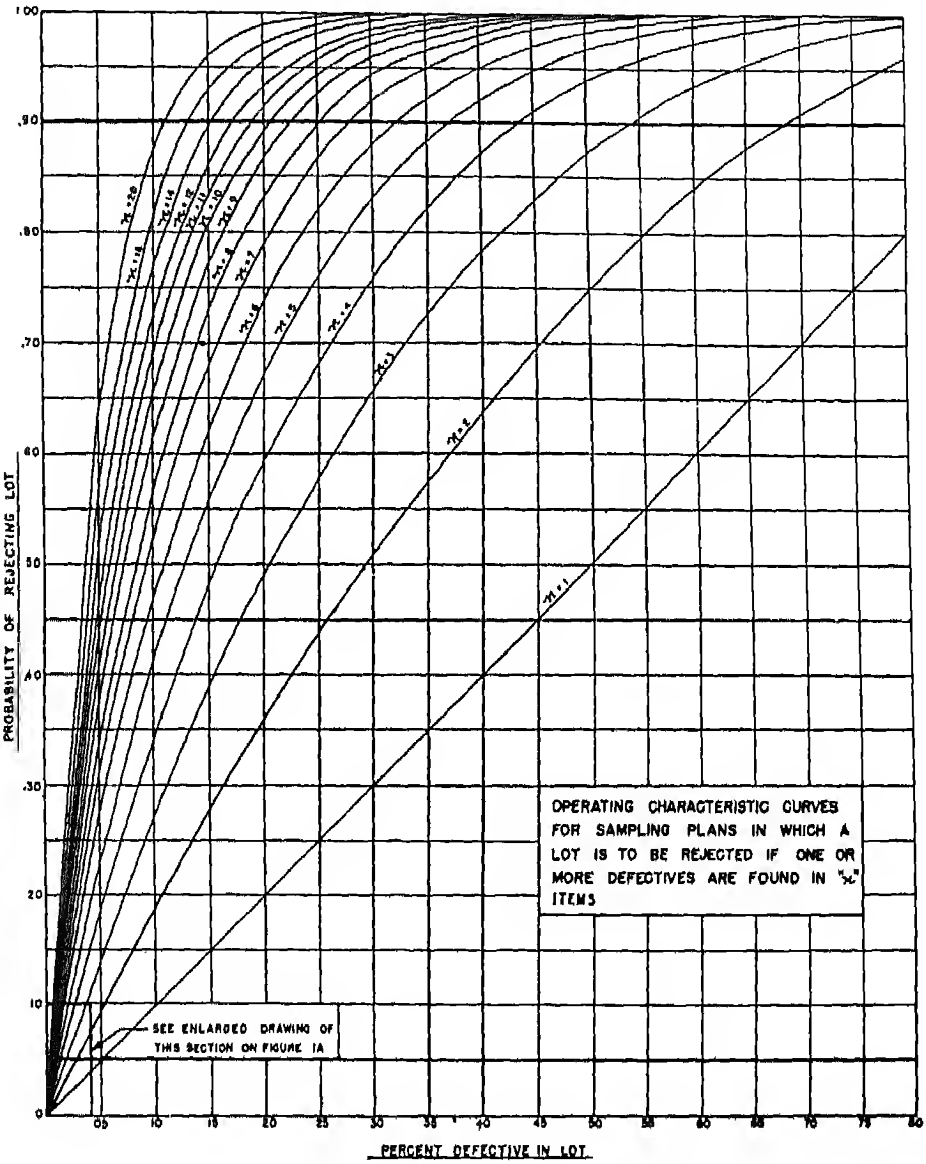

FIGURE 1

It has been assumed that the lots are so large that the samples can be regarded as being drawn from an infinite population, or to put it another way, that there

is no error in treating the samples as if they had been randomly drawn "with replacement".

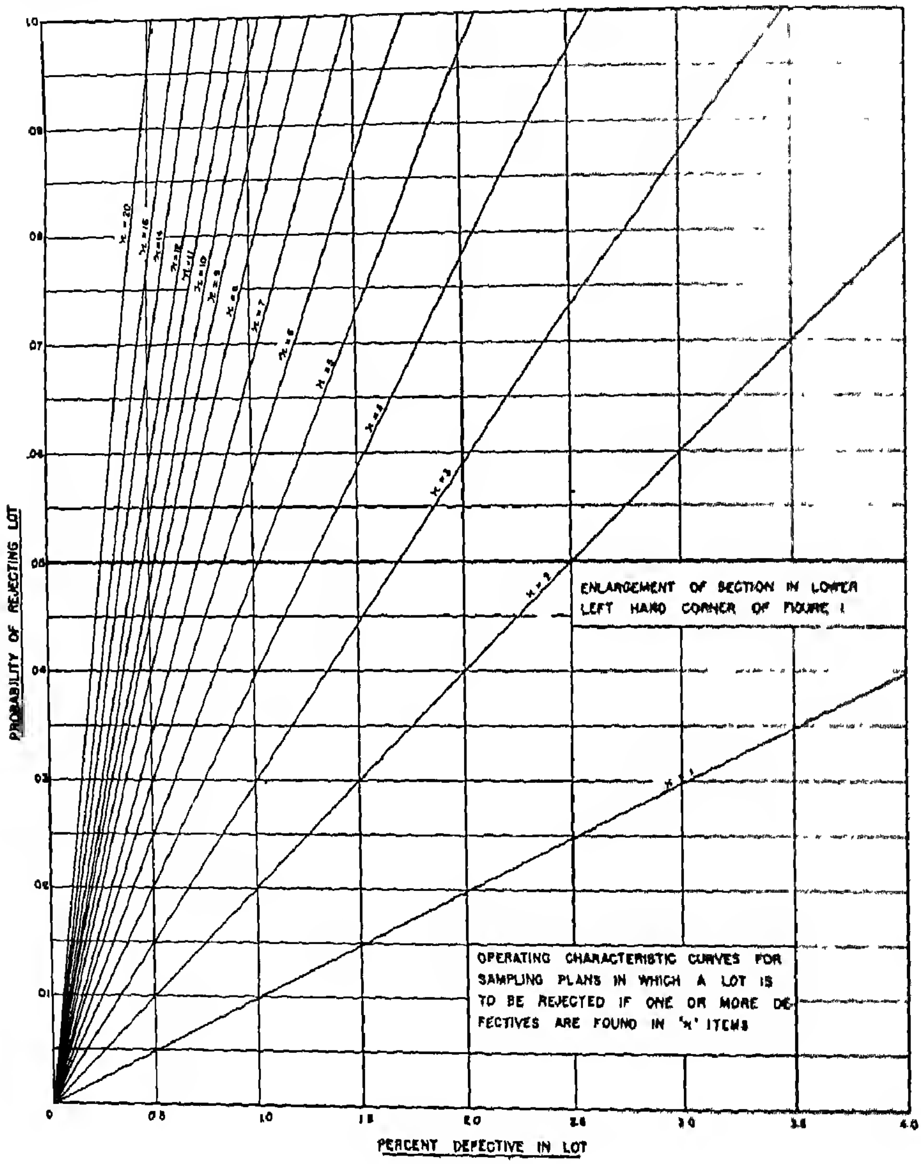

FIGURE 1A

Especial interest is often attached to the points where the curve crosses the 5% and the 90% probability levels. A rejection probability of 5% is frequently associated with a quality value that has been called the "acceptable quality level"

(abbreviated AQL), and in published sampling tables by Dodge and Romig,[3] a rejection probability of 90% is associated with a quality value which they call the "tolerance percent defective."

The average outgoing quality curve (AOQ curve, see Figures 2, 4 and 6) of a sampling plan shows the relationship between the long run average quality of the outgoing product *after* sampling inspection and the quality of the product as submitted for inspection. The quality of the product in each case is usually measured by the "percent defective" in the product.

SUPPLEMENT TO FIGURES 1 AND 1A.

Quality of Lot (measured in percent defective) corresponding to various probabilities of rejection, for sampling plans in which a lot is to be rejected if one or more defective items are found in a set of n random sample items

n	Probability of Rejection					
	.01	.05	.25	.50	.75	.90
	percent	*percent*	*percent*	*percent*	*percent*	*percent*
1	01.00	05.00	25.00	50.00	75.00	90.00
2	00.50	02.53	13.40	29.29	50.00	68.38
3	00.34	01.70	09.14	20.63	37.01	53.58
4	00.25	01.28	06.94	15.91	29.29	43.77
5	00.20	01.02	05.59	12.95	24.21	36.90
6	00.17	00.85	04.68	10.91	20.63	31.87
7	00.14	00.73	04.03	09.43	17.97	28.03
8	00.12	00.64	03.53	08.30	15.91	25.01
9	00.11	00.57	03.14	07.41	14.28	22.57
10	00.10	00.51	02.84	06.70	12.95	20.57
11	00.09	00.47	02.58	06.11	11.84	20.40
12	00.08	00.43	02.37	05.61	10.91	17.46
14	00.07	00.36	02.03	04.83	09.43	15.17
16	00.06	00.32	01.78	04.24	08.30	13.40
20	00.05	00.26	01.43	03.41	06.70	10.88

The average outgoing quality is dependent upon the treatment of rejected lots. If rejected lots are cast aside once and for all, and are never resubmitted with all deficiencies corrected, then the average quality of the outgoing product after the sampling inspection tends to be the same as the average quality of the product submitted for inspection (provided that the quality of individual lots does not fluctuate too wildly). The only direct effect that the sampling inspection has in this case is to reduce the amount of the product which is accepted. However,

[3] H. E. DODGE AND H. G. ROMIG, *Sampling Inspection Tables, Single and Double Sampling*, John Wiley and Sons, Inc., New York, 1944

the situation is very different if a rejected lot is always resubmitted with all defective material removed or replaced with non-defective material. In this case,

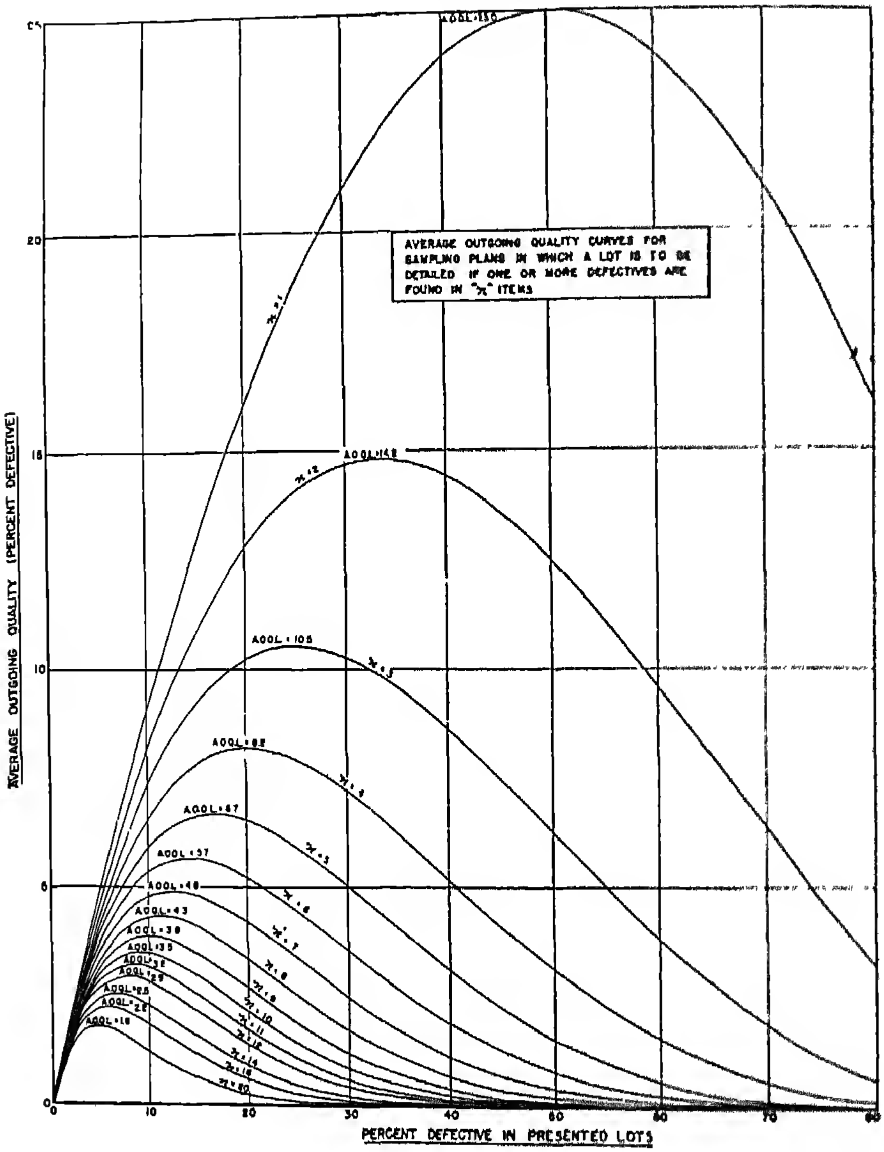

FIGURE 2

the average quality of the outgoing product after the sampling inspection will tend to be better than the average quality of the product submitted for inspection. In fact, if the submitted quality is very poor, the average outgoing quality

will theoretically tend to be very good, because so many of the lots are rejected and then detailed.

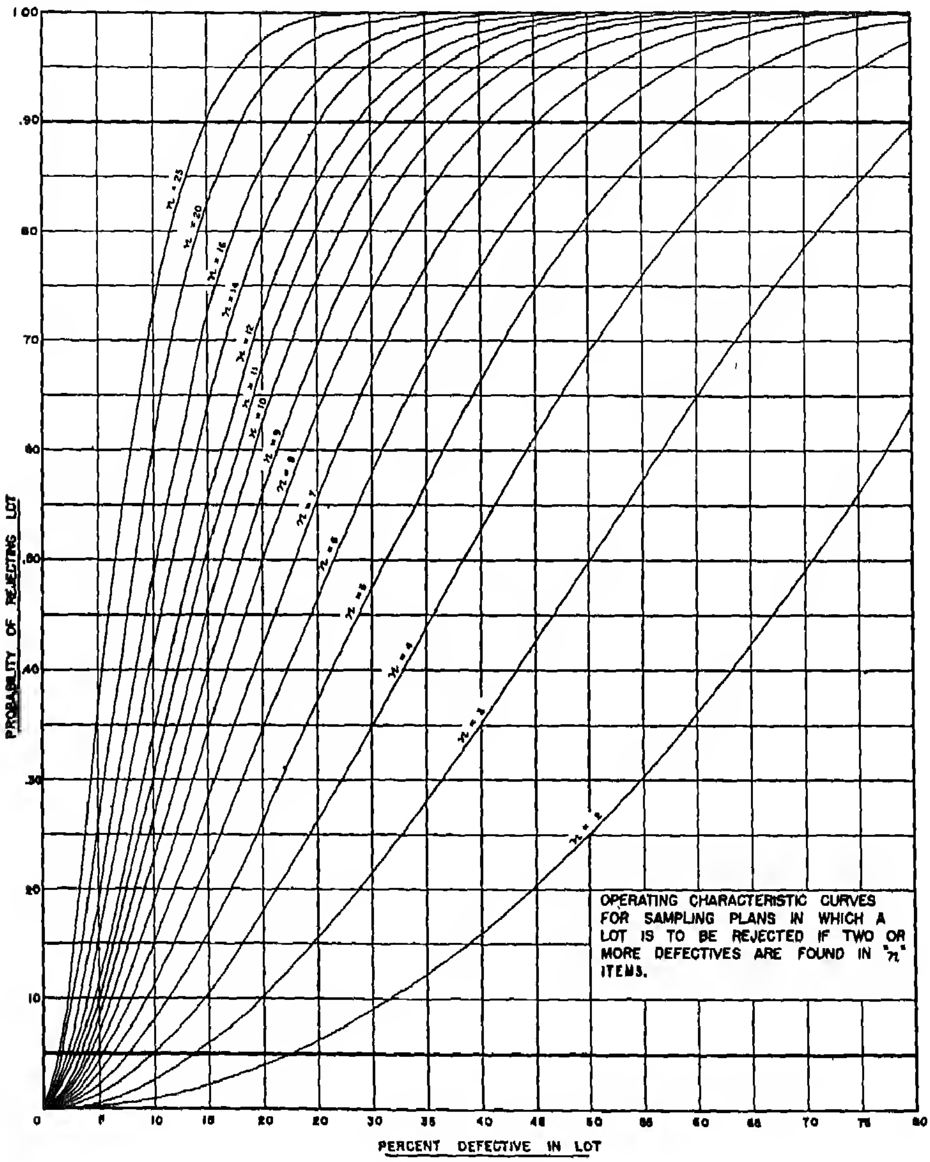

FIGURE 3

Under the assumption that each rejected lot will be detailed and resubmitted with all deficiencies corrected, a typical average outgoing quality curve starts at the origin, rises rapidly to a maximum, and falls off more slowly. The maxi-

mum average outgoing quality is called the average outgoing quality limit (AOQL) of the plan.

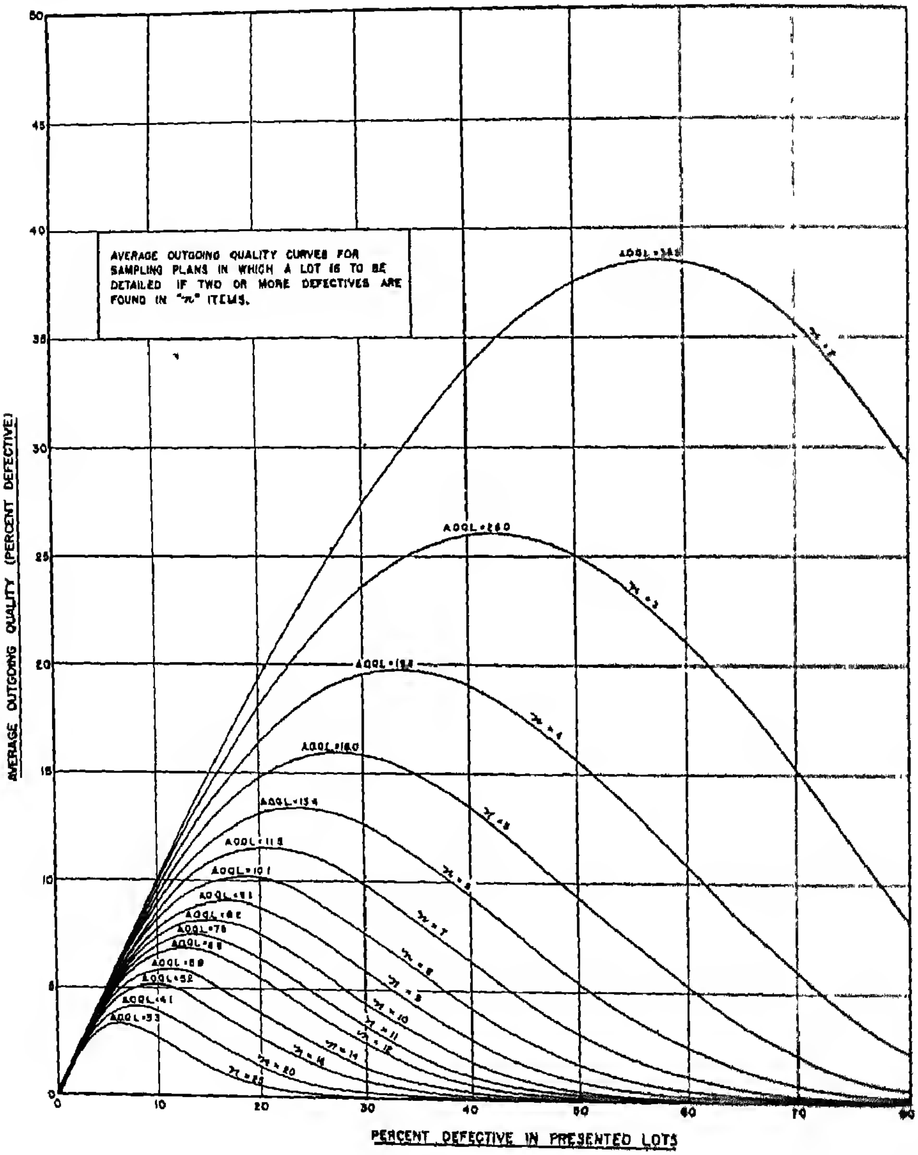

Figure 4

The graphs give the operating characteristic curves and average outgoing quality curves of certain single sampling plans. It is assumed the samples are taken at random without replacements from a lot which contains at least 10 times

the specified number of samples. In the case of the average outgoing quality curves, it is further assumed that rejected lots are always detailed and resub-

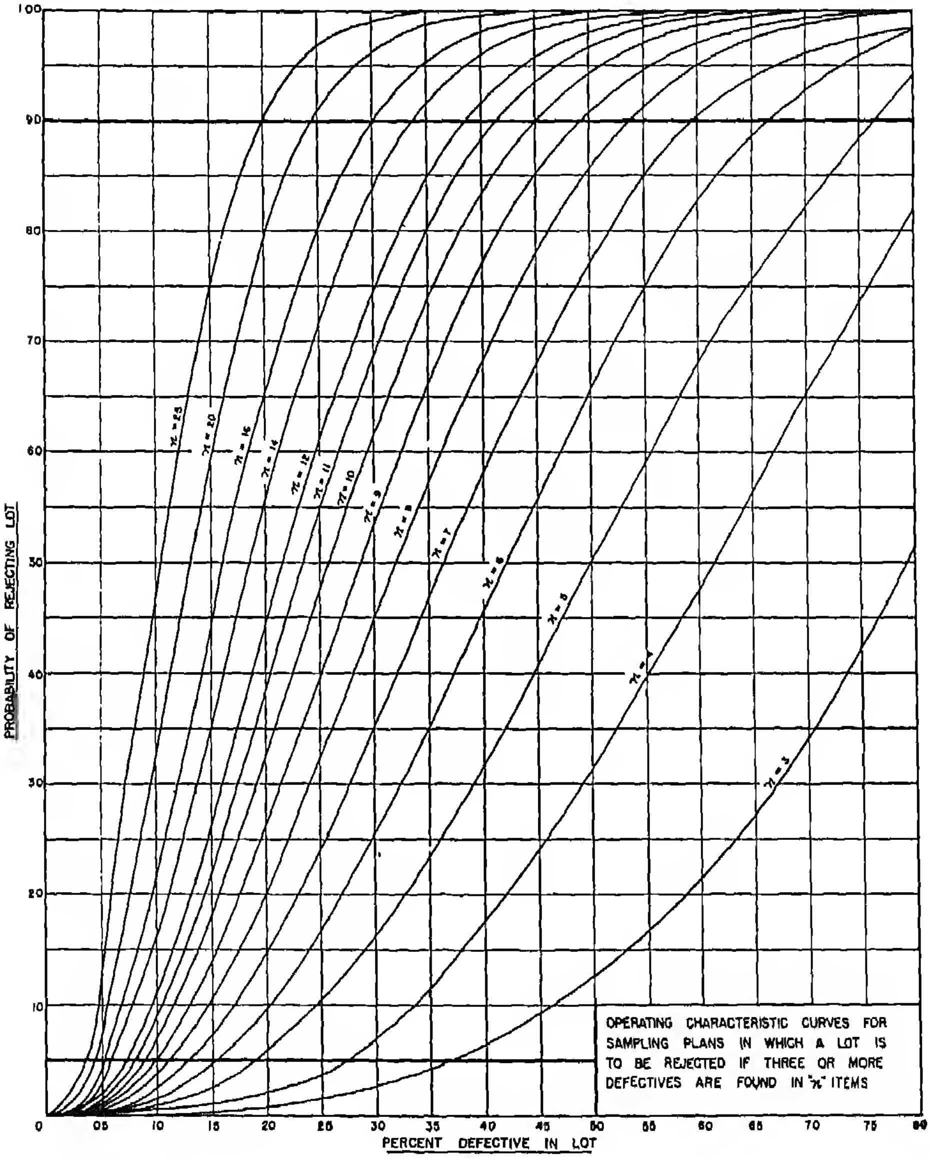

FIGURE 5

mitted with all the defective material replaced by non-defective material. An approximation has been made in the calculation of the AOQ curves which makes them upper bounds. If it is assumed that many lots of size N of exactly the

same quality of product p are being produced and that we are taking samples of size n from them, then it follows that AOQ $= p\, P_a\, (1 - n/N)$, where P_a is the probability of accepting a lot. The term n/N has been omitted; therefore these

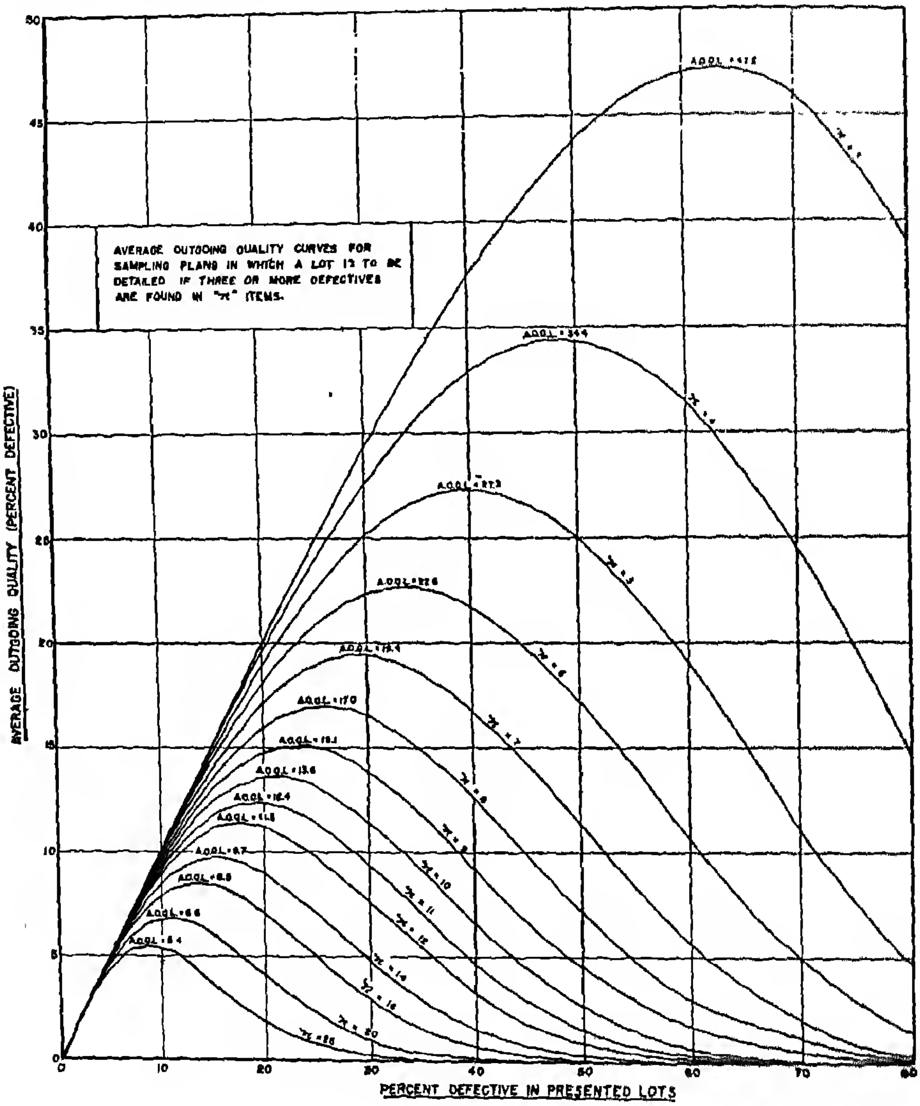

FIGURE 6

AOQ curves are too high, but are a good approximation provided only that the ratio of sample size to lot size is small. The condition mentioned earlier in this paragraph requires that $n/N \leq 0.1$.

ON THE USE OF THE SAMPLE RANGE IN AN ANALOGUE OF STUDENT'S t-TEST

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Let $x_1, \cdots, x_N$ represent independent observations on a variate x which is normally distributed with mean μ and variance σ^2. Assuming no prior information about the value of either parameter, let H_0 be the hypothesis that μ is equal to or less than a specified quantity μ_0. The classical test of this asymmetrical form of "Student's" hypothesis [1] is based upon the statistic

$$t = \sqrt{N}(\bar{x} - \mu_0) \Big/ \sqrt{\frac{\Sigma(x - \bar{x})^2}{N - 1}},$$

the region of rejection being defined by the relation $t > t_\epsilon$.

For certain applications of a routine nature, however, such as production line inspection, the usefulness of this test is rather seriously impaired by the arithmetical work involved in the computation of t. For this reason Dodge [2] and Knudsen [3] among others have proposed tests of H_0 based on a statistic of the form

$$G = \frac{\bar{x} - \mu_0}{w}$$

where w is the sample range. It is the object of this note to show how the probability distribution of G can be obtained with the aid of the distribution law of w tabulated by Pearson and Hartley [4], and to present some numerical results which indicate that the power of the resulting test is the same for all practical purposes as that of "Student's" t-test for sample sizes $N \leq 10$.

The calculation of the percent points of the G distribution is greatly facilitated by the following result, which does not appear to be generally known:

Lemma: *If $\bar{x}$ and w represent respectively the average and the range of a sample of N independent observations on a normally distributed variate x, then $\bar{x}$ and w are statistically independent.*

Proof: No generality is lost by putting $\mu = 0$, $\sigma^2 = 1$. The joint characteristic function of $\bar{x}$ and the $\frac{1}{2}N(N - 1)$ differences $x_j - x_k$, $(j < k)$, is then

$$\varphi(t, t_{jk}) = (2\pi)^{-(N/2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_j x_j^2 + i\frac{t}{N}\sum_j x_j + i\sum_{j,k} t_{jk}(x_j - x_k)} \, dx_1 \cdots dx_N$$

where the summation runs from 1 to N on each index with the understanding that $t_{jk} \equiv 0$ for $j \geq k$. The usual process of completing the square in the exponent then yields

$$\varphi(t, t_{jk}) = e^{-\frac{1}{2}\sum_j \left[\frac{t}{N} + \sum_k (t_{jk} - t_{kj})\right]^2} \cdot (2\pi)^{-(N/2)} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum_j \left\{x_j - i\left[\frac{t}{N} + \sum_k (t_{jk} - t_{kj})\right]\right\}^2} dx_1 \cdots dx_N.$$

Since

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+i\theta)^2}\,dx = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx,$$

this reduces to

$$\varphi(t, t_{jk}) = e^{-\frac{1}{2}\sum_j\left[\frac{t}{N}+\sum_k(t_{jk}-t_{kj})\right]^2},$$

which readily factors into

$$\varphi_1(t)\cdot\varphi_2(t_{jk}) = e^{-(t^2/2N)}\cdot e^{-\frac{1}{2}\sum_j\left[\sum_k(t_{jk}-t_{kj})\right]^2}.$$

Hence the differences $x_j - x_k$ are jointly independent of $\bar{x}$; and since the range w is a Borel measurable function of these differences (i.e., $w = \max|x_j - x_k|$) it follows that $\bar{x}$ and w are independently distributed.

The foregoing lemma is in fact capable of further generalization as follows: *Let $g(x_1, \cdots, x_N)$ be a function which, like the range, has the property that $g(x_1 + a, \cdots, x_N + a) \equiv g(x_1, \cdots, x_N)$. The characteristic function of $\bar{x}$ and g can then be written in the form*

$$\varphi(t,\lambda) = e^{-(t^2/2N)}\cdot(2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum(x-i(t/N))^2+i\lambda g(x)}\,dx_1\cdots dx_N = \varphi_1(t)\cdot\psi(t,\lambda).$$

Now if the second factor ψ is analytic in t, it must be a constant as far as variation with t is concerned; for by putting $t = iNa$ (a real) we have

$$\begin{aligned}\psi(iNa,\lambda) &= (2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum(x+a)^2+i\lambda g(x)}\,dx_1\cdots dx_N\\ &= (2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum(x+a)^2+i\lambda g(x+a)}\,dx_1\cdots dx_N\\ &= (2\pi)^{-(N/2)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\sum z^2+i\lambda g(z)}\,dz_1\cdots dz_N = \varphi_2(\lambda).\end{aligned}$$

Therefore $\psi(t, \lambda)$, being constant in t along the axis of imaginaries, must be free of t throughout the complex plane. The joint characteristic function of $\bar{x}$ and g is thus equal to the product of their respective characteristic functions, so that the two variates are independently distributed. In particular this result shows that in the normal case each of the moments about the sample mean is distributed independently of $\bar{x}$

Returning now to the distribution of G, we see that for $G_\epsilon > 0$

$$\begin{aligned}P\left\{\frac{\bar{x}-\mu}{w} > G_\epsilon\right\} &= P\left\{\frac{\sqrt{N}(\bar{x}-\mu)/\sigma}{\sqrt{N}G_\epsilon} > w/\sigma\right\}\\ &= \int_{z=0}^{\infty}\int_{w=0}^{z/\sqrt{N}G_\epsilon} f(z)h(w)\,dw\,dz\\ &= \int_0^{\infty} f(z)P(z/\sqrt{N}G_\epsilon)\,dz\end{aligned}$$

where $f(z)$ is the normal probability function for $\mu = 0$, $\sigma^2 = 1$, and $P(u)$ is the value [4] of the probability that the range of a sample of N observations

will be less than u standard units. For selected values of N Table I gives the value $G_{.05}$ such that

$$P_N\{(\bar{x} - \mu_0)/w > G_{.05} \mid \mu = \mu_0\} = .05.$$

TABLE I

Upper 5% points for distribution of G

N	$G_{.05}$
3	.88
5	.39
7	.26
10	.19

These values were calculated by Simpson's rule and checked by Weddle's rule.

To evaluate the probability that G will exceed G_ϵ when $\mu \neq \mu_0$ we may write, following Johnson and Welch [5]

$$\frac{\bar{x} - \mu_0}{w} = \frac{\sqrt{N}(\bar{x} - \mu)/\sigma + \sqrt{N}(\mu - \mu_0)/\sigma}{\sqrt{N}w/\sigma} = \frac{z + a}{\sqrt{N}w/\sigma}.$$

The required probability is then given by the integral

$$\int_{z=-a}^{\infty} f(z)P\left(\frac{z + a}{\sqrt{N}G_\epsilon}\right)dz, \quad a = \sqrt{N}(\mu - \mu_0)/\sigma.$$

Table II is a comparison of the probability that G will exceed $G_{.05}$ with the corresponding probability that "Student's" t will exceed $t_{.05}$ for various values of $(\mu - \mu_0)/\sigma$, the case $N = 3$ being chosen because the non-central t distribution is formally integrable in this case.

TABLE II

Probability of rejection for G and for t, (N = 3)

$(\mu - \mu_0)/\sigma$	P{G > .88}	P{t > 2.92}
.00	.050	.050
.50	.151	.151
.75	.229	.230
1.00	.322	.322

Similarly for $N = 10$ it was found that when $\mu - \mu_0 = .383\sigma$ (i.e., when $a = 1.21$) the probability that G will exceed $G_{.05}$ is .296; the corresponding probability for t is given by Neyman and Tokarska [1] as .30.

Pending the construction of more adequate tables of the percent points of the G distribution, it seems worthy of note that for $N \leq 10$ the values of $G_{.05}$ can be estimated quite accurately by multiplying the corresponding upper percent point $t_{.05}$ by the factor

$$k_N = \frac{E\left[\sqrt{\frac{\Sigma(x - \bar{x})^2}{N - 1}}\right]}{\sqrt{N}E[w]}$$

where $E[w]$ is obtainable from Tippett's table of the mean range [6]. Estimated values of $G_{.05}$ for sample sizes from 3 to 10 are listed for convenience in Table III. The approximate values of $G_{.05}$ proposed by Knudsen [3] were calculated in essentially this fashion, using however the square root of the expected value of $\Sigma(x - \bar{x})^2$ instead of the expected value of $\sqrt{\Sigma(x - \bar{x})^2}$, and employing percent points of the t distribution determined by the relation $P\{|t| > t^*_{.05}\} = .05$ instead of $P\{t > t_{.05}\} = .05$. Thus though the agreement between the values listed in Table III and the corresponding computed values shown in Table I is extremely good, the discrepancy between these values and those given by Knudsen is rather large. Any error committed by using Knudsen's table will,

TABLE III

Estimated upper 5% points for distribution of G

N	$G_{.05}$
3	.882
4	.526
5	.385
6	.309
7	.260
8	.227
9	.202
10	.183

however, be on the conservative side, in the sense that the probability of unjustly rejecting H_0 will have somewhat less than half the value indicated in that table.

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AN INEQUALITY FOR DEVIATIONS FROM MEDIANS

BY JOHN W. TUKEY

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In a recent note in these *Annals*, Birnbaum and Zuckerman [1] proved that if:

(1) $X_1, X_2, \cdots, X_n$ are independent random variables with the same distribution (i.e., form a sample),
(2) their common distribution is symmetric about zero,

then

$$E(|X_1 + X_2 + \cdots + X_n|) \geq \varphi(n) \cdot E(|X_1|),$$

where

$$\varphi(2k+1) = \varphi(2k+2) = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k+1)}{1 \cdot 2 \cdot 4 \cdot 6 \cdots (2k)}.$$

It is the purpose of the present note to extend this to the following, more general, result:

THEOREM. *If*

(i) $X_1, X_2, \cdots, X_n$ *are independent random variables,*
(ii) *the median of each* X_i *is zero,*

then

$$E(|X_1 + X_2 + \cdots + X_n|) \geq \frac{\varphi(n)}{n} E(|X_1| + |X_2| + \cdots + |X_n|).$$

It will be convenient to let $d_i = E(|X_i|)$ and

$$\bar{d} = \frac{1}{n} \sum d_i = \frac{1}{n} E(|X_1| + |X_2| + \cdots + |X_n|),$$

so that the desired inequality becomes

$$E(|X_1 + X_2 + \cdots + X_n|) \geq \varphi(n) \cdot \bar{d}.$$

Define e_i by

$$e_i = \int_0^\infty x dF_i(x),$$

where $F_i(x)$ is the cumulative distribution function of X_i. Since

$$d_i = E(|X_i|) = -\int_{-\infty}^0 x dF_i(x) + \int_0^\infty x dF_i(x),$$

it follows that

$$\int_{-\infty}^{0} x\,dF_i(x) = e_i - d_i\,.$$

The basic idea of the proof, which is common to both notes, is to divide the n-dimensional space of $x_1, x_2, \cdots, x_n$ into its 2^n "octants," break up the expectation of $| X_1 + X_2 + \cdots + X_n |$ into the corresponding parts, and apply elementary inequalities. Let O_S be the octant in which a set S of variables are ≤ 0. From (4), (5) and hypothesis (ii) it follows that

$$2^{n-1} \int_{O_S}\cdots\int x_i \prod dF_j(x_j) = \begin{cases} e_i\,, & \text{if } x_i \geq 0 \quad \text{in } O_S\,, \\ e_i - d_i\,, & \text{if } x_i \leq 0 \quad \text{in } O_S\,. \end{cases}$$

Hence

$$2^{n-1} \int_{O_S}\cdots\int \sum x_i \prod dF_j(x_j) = \sum_{i=1}^{n} e_i - \sum_{S} d_i = e - \sum_{S} d_i\,.$$

where $e = \Sigma\, e_i$, and the second and third sums are over all d_i for which $x_i \leq 0$ in the chosen octant O_S. The contribution of the octant O_S to $E(| X_1 + X_2 + \cdots X_n |)$ is

$$\int_{O_S}\cdots\int | \sum x_i | \prod dF_j(x_j) \geq \left| \int_{O_S}\cdots\int (\sum x_i) \prod dF_j(x_j) \right|$$

$$= 2^{-(n-1)} \left| e - \sum_{S} d_i \right|.$$

For each value of s, there will be $\binom{n}{s}$ octants with s variables ≤ 0. The sum of their contribution to $E(| X_1 + X_2 + \cdots X_n |)$ is

$$I_s = \frac{1}{2^{n-1}} \sum \left| e - \sum_{S} d_i \right| \geq \frac{1}{2^{n-1}} \left| \binom{n}{s} e - \binom{n-1}{s-1} \sum d_i \right|,$$

where the inequality follows from $\Sigma\, | a_s | \geq | \Sigma a_s |$, and it is noticed that each d_i occurs in $\binom{n-1}{s-1}$ different inner sums. Recalling that $\Sigma d_i = n\bar{d}$, this may be written

$$I_s \geq \frac{1}{2^{n-1}} \binom{n}{s} | e - s\bar{d} |\,.$$

Finally,

$$E(|X_1 - X_2 + \cdots + X_n|) = \sum_{s=0}^{n} I_s \geq 2^{-(n-1)} \sum_{s=0}^{n} \binom{n}{s} |e - s\bar{d}|$$

$$\geq 2^{-(n-1)} \sum_{2s<n} \binom{n}{s} \{|e - s\bar{d}| + |e - (n - s)\bar{d}|\}$$

$$\geq 2^{-(n-1)} \sum_{2s<n} \binom{n}{s} (n - 2s)\bar{d},$$

where the last inequality follows from $|a| + |b| \geq b - a$. To complete the proof, it is only necessary to evaluate the last sum One method of evaluation may be found in Birnbaum and Zuckerman's note.

If each $X_i = \pm 1$, each with probability one-half, then all of the inequalities of the proof become equalities. So that, in this case,

$$E(|X_1 + X_2 + \cdots + X_n|) = \varphi(n) \cdot \bar{d}.$$

Since the limiting distribution in this case is a normal distribution with standard deviation $n^{\frac{1}{2}}$ and $E(|X_1 + X_2 \cdots + X_n|) = (2n/\pi)^{\frac{1}{2}}$, it follows that this is the asymptotic value of $\varphi(n)$.

The inequality of the theorem is only efficient when the $E(|X_i|)$ are of nearly the same size. In other cases it can often be usefully supplemented by the

LEMMA. *If*

(i) $X_1, X_2, \cdots, X_n$ *are independent*

(ii) *for each i, either X_i has median zero, or the sum of the means of the other X_j is zero* (this is implied by either (a) *the median of each X_i is zero,* or (b) *the mean of each X_i is zero*), *then*

$$E(|X_1 + X_2 + \cdots + X_n|) \geq \operatorname{Max} E(|X_i|).$$

The lemma follows from the case where $n = 2$, by applying that case to

$$Y_1 = X_{i_0}, \qquad Y_2 = \sum_{i \neq i_0} X_i,$$

where the maximum of $E(|X_i|)$ is attained for $i = i_0$

The special case follows from the inequality

$$|x_1 + x_2| \geq |x_1| + x_2 \cdot \operatorname{sgn} x_1,$$

since this implies

$$E(|X_1 + X_2|) \geq E(|X_1|) + E(X_2) \cdot E(\operatorname{sgn} X_1) = E(X_1)$$

using first (i) and then (ii).

In conclusion, it is interesting to note that the mean cannot replace the median in the hypothesis of the theorem. For let X_1, X_2, X_3 be independent,

and take the values 1 (with probability 2/3) and -2 (with probability 1/3). $X_1 + X_2 + X_3$ takes the values 3 (with probability 8/27), 0 (with probability 12/27), -3 (with probability 6/27) and -6 (with probability 1/27). Hence $E(|X_i|) = 4/5$, and $E(|X_1 + X_2 + X_3|) = 48/27 = 16/9 = 4/3E(|X_i|)$, which is not $\geq 3/2E(|X_i|)$.

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[1] Z. W. BIRNBAUM AND HERBERT S. ZUCKERMAN, "An inequality due to H. Hornich," *Annals of Math. Stat.*, Vol. 15 (1944), pp. 328–329.

ON THE INDEPENDENCE OF THE EXTREMES IN A SAMPLE[1]

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In a previous article [1] the assumption was used that the mth observation in ascending order (from the bottom) and the mth observation in descending order (from the top) are independent variates, provided that the rank m is small compared to the sample size n. In the following it will be shown that this assumption holds for the usual distributions.

Let x be a continuous, unlimited variate, let $\Phi(x)$ be the probability of a value equal to, or less than, x; let $\varphi(x)$ be the density of probability, henceforth called the initial distribution. The mth observation from the bottom is written ${}_mx$ and the kth observation from the top is written x_k. Thus, the bivariate distribution $\mathfrak{w}_n({}_mx, x_k)$ of ${}_mx$ and x_k, is such that there are $m - 1$ observations less than ${}_mx$, $k - 1$ observations greater than x_k and $n - m - k$ observations between ${}_mx$ and x_k.

For simplicity's sake write

$$\Phi({}_mx) = {}_m\Phi; \qquad \Phi(x_k) = \Phi_k.$$
$$\varphi({}_mx) = {}_m\varphi; \qquad \varphi(x_k) = \varphi_k.$$

Then

$$\mathfrak{w}_n({}_mx, x_k) = C\,{}_m\Phi^{m-1}\,{}_m\varphi(\Phi_k - {}_m\Phi)^{n-m-k}\varphi_k(1 - \Phi_k)^{k-1}, \tag{1}$$

where

$$C = \frac{n!}{(m-1)!(k-1)!(n-m-k)!}. \tag{1'}$$

In the expression (1) no assumption about dependence or independence of ${}_mx$ and x_k is implied except that these values are taken from the same population.

The distribution (1) is now modified by introducing three conditions. First,

[1] Research done with the support of a grant from the American Philosophical Society.

that the two variates are extreme, namely that the ranks m and k are of the same order of magnitude and small compared to the sample size n.

(2) $$n >> m \simeq k = 0(1).$$

Furthermore it is assumed that the initial distribution $\varphi(x)$ is, for small and for large values of the variate, subject to L'Hospital's rules

(3) $$\lim_{x=-\infty} \frac{\varphi'(x)}{\varphi(x)} = \lim_{x=-\infty} \frac{\varphi(x)}{\Phi(x)}; \qquad \lim_{x=\infty} \frac{\varphi'(x)}{\varphi(x)} = -\lim_{x=\infty} \frac{\varphi(x)}{1-\Phi(x)}.$$

Finally it is assumed that n is so large that the equality of the limits may be replaced by the equality of the quotients. Then it is legitimate to write

(3′) $$\frac{{}_m\varphi'}{{}_m\varphi} = \frac{{}_m\varphi}{{}_m\Phi}; \qquad \frac{\varphi_k'}{\varphi_k} = -\frac{\varphi_k}{1-\Phi_k}.$$

Clearly, the three conditions do not imply any assumption about dependence or independence of the two extremes.

From (1) the most probable mth value from the bottom, ${}_mu$, and the most probable kth value from the top, u_k, are the solutions of

$$\frac{m-1}{{}_m\Phi}\,{}_m\varphi + \frac{{}_m\varphi'}{{}_m\varphi} = \frac{n-m-k}{\Phi_k - {}_m\Phi}\,{}_m\varphi.$$

$$\frac{n-m-k}{\Phi_k - {}_m\Phi}\,\varphi_k + \frac{\varphi_k'}{\varphi_k} = \frac{k-1}{1-\Phi_k}\,\varphi_k.$$

These two equations may be written by virtue of (3′)

$$\frac{m}{{}_m\Phi} = \frac{n-m-k}{\Phi_k - {}_m\Phi} = \frac{k}{1-\Phi_k}.$$

Consequently the probabilities of the most probable mth and kth values ${}_mu$ and u_k are

(4) $$\Phi({}_mu) = \frac{m}{n}; \qquad \Phi(u_k) = 1 - \frac{k}{n}.$$

The expansion of the probabilities ${}_m\Phi$ and Φ_k around the modes ${}_mu$ and u_k leads [2, 3] by virtue of (2), (3), (4), to

(5) $$ {}_m\Phi = \frac{m}{n}\,e^{{}_my}; \qquad \Phi_k = 1 - \frac{k}{n}\,e^{-y_k}.$$

where

(6) $$ {}_my = \frac{n}{m}\,\varphi({}_mu)({}_mx - {}_mu); \qquad y_k = \frac{n}{k}\,\varphi(u_k)(x_k - u_k).$$

Therefore, distributions, subject to L'Hospital's rules (3), may be said to be of the exponential type. Since the derivatives ${}_m\varphi$ and φ_k are

$$(7) \qquad {}_m\varphi = {}_m\alpha_m\Phi; \qquad \varphi_k = \alpha_k(1 - \Phi_k),$$

where

$$(7') \qquad {}_m\alpha = \frac{m}{n}\varphi({}_mu); \qquad \alpha_k = \frac{n}{k}\varphi(u_k),$$

the product of the first two and the last two functions in formula (1) may be written as a product of two functions

$$(8) \qquad {}_m\Phi^{m-1}{}_m\varphi\,\varphi_k(1 - \Phi_k)^{k-1} = \left({}_m\alpha\frac{m^m}{n^m}e^{m_my}\right)\left(\alpha_k\frac{k^k}{n^k}e^{-ky_k}\right)$$

Clearly, each factor in (8) depends only on one variable.

In the same way the function of ${}_mx$ and x_k in the middle of (1) can be split up into a product of two independent functions, each depending only on one variate. By virtue of (5)

$$\Phi_k - {}_m\Phi = 1 - \frac{1}{n}(me^{my} + ke^{-y_k})$$

and by virtue of (2)

$$(9) \qquad (\Phi_k - {}_m\Phi)^{n-m-k} = \exp(-me^{my})\ \exp(-ke^{-y_k}),$$

where

$$\exp(x) = e^x.$$

From (2) the constant factor (1′) may also be split into a product

$$(10) \qquad \frac{n!}{(m-1)!(k-1)!(n-m-k)!} = \frac{n^m}{(m-1)!}\cdot\frac{n^k}{(k-1)!}.$$

Introducing (10), (9) and (8) into (1), the bivariate distribution of the mth extreme value from the bottom and the kth extreme value from the top is obtained as a product of two independent distributions

$$(11) \qquad \mathfrak{w}_n({}_mx, x_k) = {}_mf({}_mx)\cdot f_k(x_k)$$

where

$$(12) \qquad {}_mf({}_mx) = \frac{{}_m\alpha m^m}{(m-1)!}\exp(m\,{}_my - me^{my})$$

and

$$(12') \qquad f_k(x_k) = \frac{\alpha_k k^k}{(k-1)!}\exp(-ky_k - ke^{-y_k})$$

are the distributions of the mth extreme values from the bottom, alone, and of the kth extreme values from the top, alone.

In the special case $m = k$ and for a symmetrical initial distribution with mean zero, the following equations hold

$$(13) \qquad {}_m\alpha = \alpha_k = \alpha_m, \qquad {}_mu = -u_k = -u_m.$$
$$(13') \qquad {}_m\Phi = 1 - \Phi_k = 1 - \Phi_m; \qquad {}_m\varphi = \varphi_k = \varphi_m.$$

and the bivariate distribution of the mth values from the bottom ${}_mx$, and from the top x_m, is

$$(14) \qquad \mathfrak{w}_n({}_mx, x_m) = {}_mf({}_mx)\cdot f_m(x_m),$$

where

$$(14') \qquad {}_mf({}_mx) = f_m(-x_m)$$

is the expression used in the beginning of article [1]

It follows from (11) that the mth observation in ascending order, and the kth observation in descending order, may be dealt with as independent variates provided that n is large, the ranks m and k are small, and that the initial continuous unlimited distribution is of the exponential type as defined by equations (3).

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A NOTE ON SAMPLING INSPECTION

By Paul Peach and S B. Littauer

North Carolina State College and *Newark College of Engineering*

In designing an industrial sampling plan conformable to the Pearson-Neyman approach, the operating characteristic is made to pass as nearly as possible through two predetermined points Wald [1] has used this method for setting up sequential sampling plans.

A similar type of single sampling plan can be designed by using tables of the incomplete Beta function. Unfortunately, tables of this function are not generally available, and the existing tables do not cover the range for large sample sizes.

An approximate solution of the problem for single sampling can be based on the widely available tables of percentage points of the chi-square distribution. This is equivalent to assuming a Poisson distribution of defectives in the sample, utilizing the well known fact that for even degrees of freedom the chi-square distribution gives the summation of a Poisson series.

We use the following well established notation:

n = sample size

c = acceptance number

p_1 = acceptable fraction defective

p_2 = objectionable fraction defective

α = risk of rejecting a lot if $p = p_1$.

β = risk of accepting a lot if $p = p_2$.

There seems little to be gained by using a large assortment of possible risk values, since the necessary adjustment to secure a desired effect can be made on the p's. We suggest the adoption of .05 as a standard value for both α and β. This convention conforms to much existing statistical practice, in particular to some existing inspection tables.

We propose also the use of

$$R_0 = p_2/p_1,$$

which we call the "operating ratio," as a measure of the power of discrimination of an inspection scheme. Dodge and Romig [2] used what is essentially the reciprocal of R_0 as a basis for the construction of sampling plans. Now, assume a binomial distribution of defectives in samples and a series of single sampling plans with the same c but different n. As n increases, the effective values of p_1 and p_2 clearly decrease. Their ratio R_0 is not constant, but it does not change very much after n has got beyond the range of very small samples—say $5(c + 1)$. The value obtained from the chi-square table is the upper limit of R_0 for a fixed c and increasing n. Since R_0 is to a first approximation a function of c alone, provided n is not very small, it is a useful index for the construction of tables, and gives great compactness.

Using the chi-square approach, we note that

$$D.F. = 2c + 2$$

$$np_1 = \tfrac{1}{2}\chi^2_{2c+2,1-\alpha}$$

$$np_2 = \tfrac{1}{2}\chi^2_{2c+2,\beta}$$

$$R_0 = \frac{\chi^2_{2c+2,\beta}}{\chi^2_{2c+2,1-\alpha}}.$$

Table I gives R_0, c, and np_1 over a considerable range, with $\alpha = \beta = .05$. Given p_1 and p_2, we calculate R_0 and use it to enter the table; c is read off directly, and the sample size is $n = np_1/p_1$.

Sample sizes obtained in this way will be too large when the true distribution of defectives follows the binomial or hypergeometric laws. There is, however, a gain in protection due to the extra inspection. For the binomial case the exact

TABLE I

Single sample inspection plans

$\alpha = \beta = .05$

R_0	c	np_1
58.	0	.051
13.	1	.355
7.5	2	.818
5.7	3	1.366
4.6	4	1.970
4.0	5	2.61
3.6	6	3.29
3.3	7	3.98
3.1	8	4.70
2.9	9	5.43
2.7	10	6.17
2.63	11	6.92
2.53	12	7.69
2.44	13	8.46
2.37	14	9.25
2.30	15	10.04
2.24	16	10.83
2.19	17	11.63
2.14	18	12.44
2 10	19	13.25
2.07	20	14.07
2.03	21	14.89
2.00	22	15.72
1.92	25	18.22
1.81	30	22.44
1.71	37	28.46
1.61	47	37.20
1.51	63	51.43
1.335	129	111.83
1.251	215	192.41

In view of the approximate nature of this table due to the Poisson distribution, it is suggested that when the calculated value of R_0 does not appear, the table be entered with the next larger value. This rule will result in partial compensation for the approximation.

values p_1 and p_2 for a given n and c can be calculated, using a table of the 5 per cent points of the F (variance ratio) distribution. We may take

$$n_1 = 2(n - c)$$

$$n_2 = 2(c + 1)$$

$$F_1 = F(n_1, n_2)$$

$$F_2 = F(n_2, n_1)$$

Then
$$p_1 = \frac{n_2}{n_2 + n_1 F_1}$$

and
$$p_2 = \frac{n_2 F_2}{n_1 + n_2 F_2},$$

utilizing a property of the F distribution pointed out in [3], page 2.

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ON AN EQUATION OF WALD

By David Blackwell

Howard University

Let $X_1, X_2, \cdots$ be a sequence of independent chance variables with a common expected value a, and let $S_1, S_2, \cdots$ be a sequence of mutually exclusive events, S_k depending only on $X_1, \cdots, X_k$, such that $\sum_{k=1}^{\infty} P(S_k) = 1$. Define the chance variables $n = n(X_1, X_2, \cdots) = k$ when S_k occurs and $W = X_1 + \cdots + X_n$. We shall consider conditions under which the equation

$$E(W) = aE(n), \tag{1}$$

due to Wald [3, p. 142], holds.

This equation has various interpretations:

A. n may be considered as defining a sequential test on the X_i. If a and $E(W)$ are known, (1) may be used to determine $E(n)$, the expected number of observations required by the sequential test, [3, p. 142 et seq].

B. n may be considered as representing a gambling system, i.e. it represents the point at which a player decides to stop. W then represents his winnings,

and (1), in the special case $a = 0$, says that, if each play is a fair game, then the system leads to a fair game.

C. n may be considered as the duration of a random walk. The meaning of W and (1) is obvious.

More exactly, we shall investigate conditions on X_i under which (1) holds for every test n of finite expected value. Our results, Theorems 1 and 2, are that (1) holds if the X_i have identical distributions, or if they are uniformly bounded Theorem 1 is a generalization of a result of Wald [3, p. 142].

The test n may be considered as a test on the variables $Y_i = X_i - a$. Then $W' = Y_1 + \cdots + Y_n = W - na$, so that $E(W') = 0$ is equivalent to (1) for tests of finite expected value. Thus it is no loss of generality to assume $a = 0$ and to seek conditions under which $E(W) = 0$. We remark that if $E(n)$ does not exist, then $E(W)$ need not be zero. For example define $X_i = \pm 1$ with probability $\frac{1}{2}$, and n as the smallest integer k for which $X_1 + \cdots + X_k = 1$. Then $E(W) = 1$. (It follows from Theorem 1 or 2 that $E(n)$ cannot exist, which can also be shown directly.)

THEOREM 1. *If* $X_1, X_2, \cdots$ *have identical distributions,* $E(X_i) = 0$, $E(n) < \infty$, *then* $E(W) = 0$.

PROOF: Define chance variables n_k inductively as follows: $n_1 = n$. Supposing $n_1, \cdots, n_k$ to be defined, define $n_{k+1} = n(X_{n_1+\cdots+n_k+1}, X_{n_1+\cdots+n_k+2}, \cdots)$ i.e. $n_1, n_2, \cdots$ are the successive values of n obtained by iterating the test. Then

$$P(n_1, \cdots, n_k; n_{k+1} = j) = P(S_j). \tag{2}$$

For the event $\{n_1 = a_1, \cdots, n_k = a_k\} = R$ depends only on $X_1, \cdots, X_{a_1+\cdots+a_k}$, while under the hypothesis R the event $\{n_{k+1} = j\}$ coincides with the event $S = \{n(X_{a_1+\cdots+a_k+1}, \cdots) = j\}$. Thus $P_R(S) = P(S)$. Finally $P(S) = P(S_j)$ since S is defined by imposing the same conditions on $X_{a_1+\cdots+a_k+1}, \cdots$ that S_j imposes on $X_1, \cdots, X_j$. (2) shows inductively that $n_1, n_2, \cdots$ are defined everywhere and are mutually independent with identical distributions. Now define $W_k = X_{n_1+\cdots+n_{k-1}+1} + \cdots + X_{n_1+\cdots+n_k}$. A similar argument shows that $W_1(= W), W_2, \cdots$ are also independent variables with identical distributions. The strong law of large numbers [2, p. 488] asserts that, with probability one,

$$\frac{X_1 + \cdots + X_N}{N} \to 0 \qquad \text{as } N \to \infty. \tag{3}$$

It follows that, with probability one,

$$\frac{W_1 + \cdots + W_k}{n_1 + \cdots + n_k} \to 0 \qquad \text{as } k > 0.$$

$$\text{For if } \left|\frac{W_1 + \cdots + W_k}{n_1 + \cdots + n_k}\right| > \epsilon \qquad \text{for an infinite number of } k,$$

$$\text{then } \left|\frac{X_1 + \cdots + X_N}{N}\right| > \epsilon \qquad \text{for an infinite number of } N,$$

which by (3) is an event of probability zero. Also from the strong law of large numbers $\frac{n_1 + \cdots + n_k}{k} \to E(n)$ with probability one. Then

$$\frac{W_1 + \cdots + W_k}{k} = \left(\frac{W_1 + \cdots + W_k}{n_1 + \cdots + n_k}\right)\left(\frac{n_1 + \cdots + n_k}{k}\right) \to 0$$

with probability one. It follows from the converse of the strong law of large numbers [2, p. 488] that $E(W_i) = E(W) = 0$.

Write $S_1 + \cdots + S_k = U_k$, $C(U_k) = V_k$ so that $V_k = \{n > k\}$. Then (a) V_k depends only on $X_1, \cdots, X_k$, (b) $V_1 \supset V_2 \supset \cdots$, $P(V_k) \to 0$. Conversely any sequence of sets V_k satisfying (a) and (b) defines a sequential test on X_i; define $n = k$ on $V_{k-1}C(V_k)$. Moreover $E(n) < \infty$ if and only if (c) $\sum_{k=1}^{\infty} P(V_k)$ converges [1, p 297]. Now

$$E(W) = \lim_{N\to\infty} \sum_{k=1}^{N} \int_{S_k} (X_1 + \cdots + X_k)\, dP = \lim_{N\to\infty} \sum_{k=1}^{N} \int_{S_k} (X_1 + \cdots + X_N)\, dP$$

$$= \lim_{N\to\infty} \int_{U_N} (X_1 + \cdots + X_N)\, dP = -\lim_{N\to\infty} \int_{V_N} (X_1 + \cdots X_N)\, dP.$$

This establishes the following

LEMMA: *If $E(X_i) = 0$, then $E(W) = 0$ for every test of finite expected value if and only if for every sequence of sets V_N satisfying* (a), (b), (c),

$$\int_{V_N} (X_1 + \cdots + X_N)\, dP \to 0.$$

From this condition we obtain easily

THEOREM 2. *If $E(X_i) = 0$, $|X_i| < M$, $E(n) < \infty$, then $E(W) = 0$.*

PROOF: If V_N is a sequence of sets satisfying (a), (b), (c), then

$$\left|\int_{V_N} (X_1 + \cdots + X_N)\, dP\right| < MNP(V_N).$$

Now the series $\Sigma P(V_N)$ is a convergent series with decreasing positive terms. It is well known that under these conditions $NP(V_N) \to 0$. It follows from the lemma that $E(W) = 0$

The question of finding sufficient conditions for $E(W) = 0$ more general than those given in Theorems 1 and 2 is of interest. The bare condition $E(X_i) = 0$ is not sufficient, as the following example (which is simply the system of doubling the stake) shows: $X_i \pm 2^i$ with probability $\frac{1}{2}$, n is the smallest integer k for which $X_k > 0$. A simple computation shows $E(n) = E(W) = 2$. It is well known that the expected amount of capital required for the above system is infinite. That this is generally true for such systems is shown by the following theorem, in which no hypothesis is made concerning the existence of $E(n)$.

THEOREM 3. *If $E(X_i) = 0$, $E(W) > 0$, then $E(Z) = -\infty$, where*

$$Z = \min_{k \leq n} (X_1 + \cdots + X_k).$$

PROOF: It follows from the proof of the lemma that

$$\int_{V_N} (X_1 + \cdots + X_N)\, dP \to -E(W).$$

Now on V_N, $Z \leq (X_1 + \cdots + X_N)$. Hence

$$\lim_{N\to\infty} \int_{V_N} Z\, dP \leq -E(W).$$

Thus $E(Z)$ cannot exist if $E(W) > 0$, since $P(V_N) \to 0$. Since $Z \leq X_1$, $\int_{Z\geq 0} Z\, dP$ exists; consequently $E(Z) = -\infty$.

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CORRECTION TO THE PAPER "ON A PROBLEM OF ESTIMATION OCCURING IN PUBLIC OPINION POLLS"

BY H. B. MANN

Ohio State University

In the paper "On a problem of estimation occurring in public opinion polls" (*Annals of Math. Stat.*, Vol. 16 (1945), pp. 85–90) the author made the assertion that, in the notation of the paper, $E[(\epsilon_i - r_i)^2]$ is always smaller than $E[(\epsilon_i - e_i)^2]$. This statement is incorrect and its supposed proof contains a numerical error in the fourth line from above on p. 90.

We have

$$E(r_i^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{1/2}^{\infty} \int_{1/2}^{\infty} \frac{1}{2\pi\sigma_i^3} \exp\left[-\frac{1}{2\sigma_i^2} Q(x, y, p_i)\right] dx\, dy\, dp_i$$

$$= \frac{1}{2\pi} \frac{2}{\sqrt{3}} \int_{c/\sqrt{2}}^{\infty} \int_{c/\sqrt{2}}^{\infty} \exp\left[-\frac{1}{2}\frac{4}{3}(x^2 + y^2 - xy)\right] dx\, dy$$

$$c = \frac{\frac{1}{2} - \pi_i}{\sigma_i}.$$

The last integral is tabulated in Karl Pearson's *Tables for Statisticians and Biometricians*, Vol. 2, p. 93. Comparing this table with a table of the normal probability integral it may be seen that there exists a value $\bar{c}$ such that

$$E(e_i^2) \geq E(r_i^2) \text{ for } c \leq \bar{c},$$
$$E(e_i^2) < E(r_i^2) \text{ for } c > \bar{c}.$$

The quantity $\bar{c}$ lies in the neighborhood of 2.

I am indebted to Professor J. W. Tukey for bringing the error to my attention.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

The following members of the Institute are teaching in Army University Centers in Shrivenham, England; Biarritz, France; and Florence, Italy: T. A. Bancroft, Alonzo Cohen, E. E. Blanche, P. R. Rider.

Dean Walter Bartky of the University of Chicago has been appointed as the representative of the Institute of Mathematical Statistics to the Division of Physical Sciences of the National Research Council.

Mr. Clyde A. Bridger represented the Institute at the Inauguration of Dr. F. S. Harris as President of Utah State Agricultural College on November 16.

Dr. C. West Churchman has resigned his position at Frankfort Arsenal and has accepted the appointment of Assistant Professor of Philosophy at the University of Pennsylvania.

Assistant Professor D. B. DeLury of the University of Toronto has been appointed to an associate professorship at Virginia Polytechnic Institute.

Mr. George Eldredge, formerly with the Aluminum Research Laboratories at New Kensington, Pennsylvania is now corrosion chemist with the Shell Development Company at Emeryville, California.

Dr Will Feller of Cornell University has been appointed as the representative of the Institute of Mathematical Statistics on the Policy Committee of the Mathematical Organizations.

M. Bernard Hecht has joined the International Resistance Company, Philadelphia, as head of the Quality Control Department.

Lt. Col. Paul Horst has returned to his previous position at Proctor and Gamble at Cincinnati.

Professor Harold Hotelling of Columbia University has been made a part time consultant on statistical problems to the Division of Statistical Standards of the Bureau of the Budget.

Dr. S. B. Littauer is now chairman of the Mathematics Department of Newark College of Engineering at Newark, N. J.

Lieutenant Commander A. L. O'Toole has been decorated with a Bronze Star Medal for his outstanding service in the South Pacific during the past two years.

Associate Professor H. H. Pixley of Wayne University has been appointed Assistant Dean of the College of Liberal Arts.

Dr. H. B. Mann has been appointed to an associate professorship at Ohio State University.

Miss Dorthy J. Morrow has been appointed to an assistant professorship at George Washington University.

Professor C. J. Rees of the University of Delaware has received a citation for his work in a civilian capacity with the 14th Air Force Headquarters.

Dr. L. V. Toralballa is a special instructor in the Mathematics Department at the University of Michigan.

Associate Professor Abraham Wald of Columbia University has been promoted to a professorship.

Mr Grover C. Wirick, Jr. is doing graduate work at the University of Michigan.

Henry Goldberg of the Columbia University Statistical Research Group died April 19, 1945.

During the last quarter of 1945, many members of the Institute engaged in statistical quality control were favored by visits from Messrs. W. A. Bennett and M Milbourn, the successful candidates in a scholarship competition organized by the Quality Control Panel associated with the Midland Region of the British Ministry of Production. In addition to the competition, for which with a three months' trip to the United States as a prize, 92 papers on industrial applications of statistical methods were submitted. This Panel has been active in organizing regular discussion groups and in arranging courses of lectures at the Birmingham Technical College, later published by the Birmingham District Committee as a "Symposium of Papers on Quality Control", copies of which are still available.

Mr. Bennett is Works Manager of the English Needle and Fishing Tackle Co., Ltd., of Redditch, and Mr. Milbourn is a physicist who has worked mainly in the field of spectrographic analysis and physical metallurgy in the Research Department of Imperial Chemical Industries, Metals Division, Birmingham. It is natural, therefore, that Mr. Bennett's paper dealt with the management problem of organizing a Statistical Quality Control Bureau and defining its duties, whereas Mr. Milbourn's paper considered the operation of quality control techniques as a means for detecting and identifying causes in production research.

Toward the close of their visit in this country they indicated that the future of Quality Control, both here and abroad, will depend on establishing an adequate theory of control that includes statistical along with all other necessary factors. This provides a challenge that must be answered by the statistical societies and the colleges, as well as by the quality control people.

New Members

The following persons have been elected to membership in the Institute:

Bal, Kenan Y. (Columbia) Statistical Control, Hq. AFPDC, 830 West Broadway, Louisville 3, Kentucky

Coles, James Stacy, Ph D. (Columbia) Research Supervisor, Underwater Explosives Research Laboratory, Woods Hole, Oceanographic Institution, *Box 631, Woods Hole, Mass.*

Frank, David H. Administrative Ass't, Long Island City H.S., 411 W. 114th St., New York 25, N. Y.

Greider, C. Edwin, Jr., B.A. (Michigan) Actuarial Clerk, *1066 Glenwood Blvd., Schenectady 8, N. Y.*

Gulliksen, Prof. Harold, Ph.D. (Chicago) Psychology Dept., Princeton University, Princeton, N. J.

Harrison, Joseph O., Jr., B.S (George Washington) *2605 Kingsbridge Ave., Apt 3F, New York, N Y*

Hodges, Joseph Lawson, Jr., A B (California) Operations Analyst, Army Air Forces, *1857 Park Road, N.W , Washington 10, D. C.*

Hoskins, Robert Heywood, A B. (Harvard) Radio Technician, Third Class, U. S Navy Teaching Fellow in Mathematics, Harvard University, *Separation 3, Separation Center, Shoemaker, California*

Lowry, Edward D. Statistician (Western Cartridge Co., E Alton) *692 5th St., East Alton, Ill.*

Rees, Prof. Carl J., Ph.D (Pennsylvania) Head of Math Dept., Univ. of Delaware, Newark, Delaware

Seth, Gobind Ram, M.A (Delhi) Lecturer in Math. Hindu College, Delhi (On Leave) *1345, John Jay Hall, 116th Street, Columbia University, New York 27, N Y.*

Silber, Jack, B.S. (Chicago) *4908 N. Springfield Ave , Chicago 25, Ill*

Stone, Goldie F., A M. (New York) *678 Dawson St , Bronx, New York, N. Y.*

Szatrowski, Zenon, Ph.D. (Northwestern) Instructor in Economics Department, Northwestern University, Evanston, Ill.

Wadley, Francis Marion, Ph D. (Minnesota) Statistical Consultant, Bur. of Entomology and Pl. Duar., USDA, *3215 N. Albemarle, Arlington, Virginia*

Waugh, Frederick V., Ph D (Columbia) Agricultural Economist (Office of War Mobil. and Recon.) *1006-26 Street, South, Arlington, Virginia*

REPORT ON THE CLEVELAND MEETING OF THE INSTITUTE

A meeting of the Institute of Mathematical Statistics was held in Cleveland, Ohio, Thursday to Sunday, January 24–27, 1946 in conjunction with the Annual Meetings of the American Statistical Association and the Econometric Society. The following 115 members of the Institute attended the meeting:

Beatrice Aitchison, Armen A. Alchian, Franz L. Alt, Richard L. Anderson, Kenneth J. Arnold, Max Astrachan, George J Auner, Kenan Y. Bal, Walter Bartky, William D Baten, Harold R Bellison, Archie Blake, Chester I. Bliss, Albert H. Bowker, T. H. Brown, Robert W Burgess, Oscar K Buros, Irving W Burr, Burton H. Camp, C. West Churchman, William G Cochran, Edward P. Coleman, Francis G Cornell, Jerome Cornfield, Donald R. G. Cowan, Dudley J Cowden, Gertrude M Cox, John H Curtiss, Joseph F. Daly, Cuthbert Daniel, Besse B Day, Walter L. Deemer, Jr , Daniel B. DeLury, W. Edwards Deming, Bernard Dempsey, Paul S Dwyer, Churchill Eisenhart, Mary L. Elveback, Benjamin Epstein, Wilmoth D. Evans, Carl H Fischer, Irving Fisher, T. N. E. Greville, Trygve Haavelmo, Clausin D Hadley, Margaret J. Hagood, K. W. Halbert, Morris H. Hansen, Boyd Harshbarger, Byron R. Hayden, Harold Hotelling, Earl E. Houseman, Leonid Hurwicz, William Hurwitz, Calvin J Kirchen, Lila F. Knudsen, Hendrik S. Konijn, Tjalling Koopmans, Morton Kramer, Anita R. Kury, Robert Ladd, Dickson H. Leavens, Roy Leipnik, E Vernon Lewis, Eugene Lukacs, Henry B Mann, George F. T. Mayer, Edward C. Molina, Alexander M Mood, Margaret Moore, Joseph E. Morton, Frederick C. Mosteller, Charles McC. Mottley, Paul M Neurath, Horace W. Norton, Edwin G Olds, Paul S. Olmstead, Guy H Orcutt, James G. Osborne, Russell F. Passano, Paul Peach, Alice E. Andrews Priestley, James Rafferty, Sophie Rakesky, Charles F. Roos, A C. Rosander, Herman Rubin, Phillip J. Rulon, Marion M Sandomire, Franklin E. Satterthwaite, Esther Schaeffer, Edward M. Schrock, David H. Schwartz, G. R. Seth, Lawrence W. Shaw, Jack Sherman, Walter A. Shewhart, Walt R. Simmons, Leslie E. Simon, John H. Smith, J. R. Steen, Joseph Steinberg, Henry W. Steinhaus, J. W. Sullivan, Zenon Szatrowski, Benjamin Tepping, John W. Tukey, Helen M. Walker, W. Allen Wallis, A. E. R. Westman, S. S Wilks, Elizabeth W. Wilson, Charles P. Winsor, Gerald N. Winston, Theodore O. Yntema

The first session of the meeting was held jointly with the American Statistical Association on Thursday afternoon on *Numerical Solution of Regression Equations*, under the chairmanship of Dr. W. E. Deming of the Bureau of the Budget. The following papers were presented:

1. *A Machine for Determination of Correlation and Regression Coefficients.* Dr Guy Orcutt, Massachusetts Institute of Technology
2. *A Square Root Method for the Solution of Regression Equations.* Mr D B Duncan, Royal Australian Air Force
3. *Error Control in Matrix Calculation.* Dr F E Satterthwaite, Aetna Life Insurance Company
4. *The Compact Computation of Canonical Correlations.* Professor P S. Dwyer, University of Michigan.

On Friday, a symposium, consisting of a morning and an afternoon section, was held jointly with the Econometric Society and the American Statistical Association on *Estimating Relations from Nonexperimental Observations.* Dr.

Mordecai Ezekiel acted as chairman of the morning session and Dr. R. L. Anderson was chairman of the afternoon session. Of the following four papers, the first two, were presented in the morning and the last two in the afternoon:

1 *The Economist's Problem of Statistical Inference*
Professor J Marschak, Cowles Commission
2. *Prediction and Structural Estimation.*
Mr. Leonid Hurwicz, Cowles Commission
3 *Iterative Computation Methods in Estimating Simultaneous Relations*
Dr T Koopmans, and Mr. Roy B. Leipnik, Cowles Commission
4 *Multivariate Analysis in Economies*
Professor Gerhard Tintner, Iowa State College

On Friday afternoon a session on *Experimental Designs and their Analysis* was held jointly with the Biometrics Section of the American Statistical Association under the chairmanship of Professor Gertrude Cox of North Carolina State College. The following papers were presented:

1 *On the Uses of Orthogonal Functions in the Analysis of Incomplete Latin Squares*
Professor D B DeLury, Virginia Polytechnic Institute
2 *Use of Adjusting Factors in the Analysis of Data with Disproportionate Subclass Numbers.*
Professor R. E. Patterson, Texas A. and M. College
3. *Selection of Sample Size for Detecting Treatment Differences*
Professor A M. Mood, Iowa State College
4 *Rectangular Lattices*
Professor Boyd Harshbarger, Virginia Agricultural Experiment Station

On Saturday, a two-session symposium was held jointly with the Econometric Society and the American Statistical Association on *Sampling in the Social Sciences*. Professor Arnold J. King of Iowa State College acted as chairman for the morning session and Professor S. S. Wilks of Princeton University presided in the afternoon. The following seven papers were presented, of which the first three were presented in the morning and the remainder in the afternoon:

1. *Problems and Methods of a Sample Survey of Business.*
Mr. M. H. Hansen, Bureau of the Census
2. *Problems of Area Sampling in Agriculture.*
Mr J. R. Goodman, Bureau of the Census, and Mr. E. E. Houseman, Bureau of Agricultural Economics
3 *Problems of Area Sampling in Population.*
Mr. B. J. Tepping and Mr. J. S Steinberg, Bureau of the Census
4 *The Problems of Non-Response.*
Mr W. N Hurwitz, Bureau of the Census
5. *Systematic Sampling and its Relation to Other Sampling Designs.* (Read by Title.)
Mrs. Lillian H Madow, Washington
6. *Relative Accuracies of Systematic and Stratified Random Sampling for a Specified Class of Populations.*
Professor W. G Cochran, Iowa State College

7 *On the Design of a Sample of Dealers' Inventories.*
Dr. W E Deming, Bureau of the Budget and Dr. Willard Simmons, Office of Price Administration

On Sunday, a symposium was held jointly with the American Statistical Association on *Acceptance Sampling* under the chairmanship of Professor John W. Tukey of Princeton University. The morning session of the symposium was devoted to acceptance sampling by attributes and the afternoon session to acceptance sampling by variables. The following program was presented at the morning session:

Papers:
1 *Prewar Developments.*
Mr Paul Peach, North Carolina State College
2. *Wartime Developments.*
Professor E G. Olds, Carnegie Institute of Technology

Prepared Discussion by.
Mr. H R. Bellinson, Army Ordnance Department
Mr D H Schwartz, Quartermaster Corps
Professor Walter Bartky, University of Chicago

In the afternoon session the following program was presented:

Papers
1 *Lot Quality Measured by Average or Variability.*
Lt Commander J. H. Curtiss, Bureau of ships
2. *Lot Quality Measured by Proportion Defective.*
Mr. W. A Wallis, Columbia University

Prepared Discussion:
Mr E M. Schrock, Army Ordnance
Professor A. M. Mood, Iowa State College
Professor K. J. Arnold, University of Wisconsin
Lt Commander J. F. Daly, Bureau of ships
Dr. A. E. R. Westman, Ontario Research Foundation

A business meeting of the Institute was held at 5 p.m. on Saturday afternoon at which time reports were made by the President, Secretary-Treasurer, Editor and Chairman of the Committee on Development. These reports are all printed in the current issue of the *Annals*.

PAUL S. DWYER,
Secretary.

ANNUAL REPORT OF THE PRESIDENT OF THE INSTITUTE

(For 1945)

I. Development of Public Appreciation for Mathematical Statistics

The aims of the Institute, as stated in the constitution, are to promote the interests of mathematical statistics. First and foremost, research must go on. The *Annals* must be published and its position maintained as the world's leading journal in mathematical statistics. Meetings must be held to provide for further dissemination and discussion of research. But this is not all. We should fall short of our opportunities for promoting the interests of mathematical statistics if we were to lose sight of the need for creating an environment in which mathematical statistics and statisticians can thrive and take their proper place for rendering the service that they are capable of rendering in the political, industrial, and scientific life of the nation.

A fair share of the efforts of the officers and committees of the Institute this past year has been devoted to the creation of this environment. The Institute has assumed leadership in several movements of importance in this direction and has lost no opportunity to cooperate with other organizations toward the same ends. Momentum has thus been given to important developments which are bound to affect the scientific advancement and employment opportunities of all people engaged in statistical work of any kind, whether it be mathematical research, consulting, teaching, major or minor roles in large-scale statistical projects, preparing questionnaires, designing experiments, analyzing results, formulating conclusions and recommendations, or taking part in any other way in the collection or use of statistical data. Briefly, these developments fall under three main headings.

(*i*) *Setting standards of professional competence.*. The Description of the Profession of Statistics, put out by the National Roster this year, has gone a long way as a first step toward setting standards of professional competence. The officers and many members of the Institute assisted the Roster, particularly Professor Harold Hotelling and his Committee on the Teaching of Statistics, together with Dr. C. I. Bliss representing the American Statistical Association. Although the Roster Description is not intended to represent the official attitude of the Institute, it does represent cooperative effort toward cultivation of public understanding of statistical work.

(*ii*) *Raising the standards of teaching.* Standards of teaching go hand in hand with standards of professional competence. The Institute can proudly point to the accomplishments of its Committee on the Teaching of Statistics, which under the chairmanship of Professor Hotelling, has persistently set forth standards of teaching which are bound to bring about important changes in the arrangement of statistical courses and organization of statistical teaching. An inevitable result will be greater competence in statistical theory, better research, and expanding avenues for more effective application of theory.

(*iii*) *Promoting public understanding and appreciation for the statistician.* More adequate public appreciation of statistical theory can be brought about in several ways. The first two of these are being actively pursued by the officers of the Institute The third constitutes a proposal; and the fourth, an obligation incumbent on every member of the Institute.

First, through joint meetings with other professions such as sociologists, economists, psychologists, engineers, biometricians, etc. The Cleveland meeting is an example, the St. Louis meeting of the AAAS to be held in March is another. These joint sessions give opportunity for other groups to become aware of the impact of mathematical statistics on their own work, and for mathematical statisticians to hear of the statistical problems in other fields. Opportunities for such diffusion of knowledge exist in local chapters as well as in national meetings, and every member of the Institute should be on the lookout for opportunities to explain how problems in administration, management, economics, and manufacturing, are going to require modification in the future owing to new work in sampling techniques, acceptance procedures, quality control, and other developments of mathematical statistics.

The federation of statistical societies (see Part III) will afford better means than existed heretofore for an admixture of mathematical statistics with fields of application, both in national and local meetings.

Second, through the work of committees whose responsibility is to advise professional groups, and government and private research agencies, concerning the use of mathematical statistics. A notable example is the Joint Committee for the Development of Statistical Applications in Engineering and Manufacturing, of which Dr. W. A. Shewhart is chairman. The Institute has two representatives on it. Much of the recent advancement of statistics in industry is traceable to the work of this committee.

Third, through the establishment and publication of colloquium lectures as recommended by Dr Shewhart in his report for the preceding year, or of an annual Rietz lecture of broad interest as recommended by this years' Committee on Development (cf. Appendix A, Part V).

Fourth, information through expository nonmathematical articles and lectures delivered by leading mathematical statisticians before gatherings of nonstatistical groups of professional and business men. Such activity is of course informal and without record, carried on by individuals as opportunity permits and not by official announcement from the office of the Institute.

II. Long-range Planning

Through the work of several of the Institute's committees, each tackling specific areas of enquiry, the Institute is being provided with long-range policies and planning. In particular, the reports of the following committees should be cited in this connection:

The Committee on Development (Appendix A)

The Committee on the Teaching of Statistics (Appendix B)

The Committee on Finance (Appendix C)
The Committee on Policy in Regard to Local Chapters (Appendix D)
These committees are obviously alive to the recent rapid expansion of mathematical statistics in industry and government, and to the opportunities that lie ahead for developing proper environment for greater expansion and service of mathematical statistics.

III. Federation of Statistical Societies

A movement of extreme importance to all statistical workers is the proposed reorganization of the American Statistical Association as the central organization for all statistical societies. This movement owes its impetus largely to the recommendation made by our Committee on Development a year ago, and to the active part that our officers and representatives played in organizing and assisting the Inter-Society Committee. This movement is centripetal and replaces the centrifugal forces that were splitting statistical organizations. Under the new arrangement, statistics will possess a united front on matters of common interest, yet each organization will maintain its autonomy. Nothing is to be sacrificed in the way of standards of membership, meetings, or publications Economies will be effected through combined office operations. Much will be gained through coordinated effort; wide distribution of a journal of general methodology and applications; development of public appreciation for statistical work through dissemination of reliable information concerning statistical science and its contributions; cooperation with local and international statistical groups; promotion and development of professional standards of statistical work; and through cooperation with other professional groups in fields of application.

This federation is not yet accomplished; it is still in process of formulation, but it is probably safe to say that agreement on general aims has been reached, as well as on many items of detail. The proposition will in time be put up to each statistical organization for acceptance.

IV. Growth and Expansion

During the year the membership increased from 606 to 777. The work of the Institute, vitally affecting many thousands of statistical workers through its efforts to enhance public confidence and appreciation for theoretical statistics as well as to improve the quality of statistical work, extends far beyond the environment of its nearly 800 members. Concerted drives for membership should continue, but should not be expected to take the place of personal invitation in the form of explanation, one man to another, of what the Institute stands for. The outlook is encouraging. Year by year as the work and influence of the Institute receive wider success and recognition, more and more people will be found ready and desirous of joining.

V. Administrative Affairs

As with any active organization, there are certain chores to be done and internal affairs to be administered. The chief burden falls on the executive officer, our Secretary-Treasurer, Paul S. Dwyer, who is expected

i. To keep the list of members up to date with addresses and titles. Furnish information to the Board regarding increases and decreases in membership, and issue the Directory.

ii. To send out notices, to keep the membership informed concerning meetings and other items of interest.

iii. To send out bills, and keep the books showing payment of dues and subscriptions.

iv. To fill orders for back numbers of the *Annals*.

v. To estimate the probable demand for copies of the *Annals*, current and past, and to place orders with the printer to be able to supply the demand.

vi. With the Committee on Finance, to keep the Board posted on the expected expenditures and income for the year ahead.

vii. To answer correspondence from other organizations and individuals who desire information concerning the Institute.

viii. To keep a record of proceedings of the Board and business meetings of the Institute.

ix. To work with the various committees of the Institute, keeping them informed and in line on policy, constitution, by-laws, and other commitments.

x. With the Committee on Programs, to arrange sessions of contributed papers, and to find space in hotels or elsewhere for holding meetings and housing members.

xi. To keep the Board informed concerning recommendations and reports of committees, and other matters brought to his attention requiring action by the Board.

xii To conduct continuous membership and subscription drives with or without the aid of committees.

It is obvious that when an organization reaches the size and activity of the Institute, these duties are too onerous to carry on without proper assistance. Our Secretary-Treasurer should be freed for proper performance of important functions which only he can render toward the growth and vitalization of the Institute Consideration is being given to two possible plans, either of which will call for some increase in expenditure. One plan is to provide competent and sufficient assistance in the office of the Secretary-Treasurer, and the other is to transfer some of his duties (e.g., Items i, ii, iii, iv, x, and xii) to the American Statistical Association on a cost basis. A cooperative arrangement of this kind between the A.S.A. and the Institute has been discussed informally with Mr. Lester Kellogg, Secretary of the A.S.A., who will be able to provide us with cost estimates a little later. This kind of arrangement would be a first step and serves

as a pilot study in cost-accounting for the ultimate federation of statistical societies (Part III).

The constitution must be revised, and a committee has been formed to undertake the task. The one we have has served well, with minor revisions, over the first ten years in the life of the Institute, but conditions are now different and thorough reconsideration is needed. Among other things, it needs to be revised to permit federation with other statistical societies. As it stands it is totally deficient in specifying responsibilities between local chapters and the parent society. It should embody the recommendations of the Committee on Policy in Regard to Local Chapters, or modifications of these recommendations. Also, there are ambiguities in the present constitution that need to be cleared up, and there is no provision for carrying out the business of the Institute by correspondence when a Board meeting or Committee meeting can not be held.

The Committee on Meetings must not only seek out suitable papers for meetings, carrying out the wishes of the Board in regard to the subject-matter to be covered, but must also be concerned with the geographic location of meetings, cooperation with other professional societies, and choice of dates. During the past few years, in addition, this committee has had to contend with restrictions on transportation and hotel space. The Committee on Finance must decide what expenditures are wise and allowable; they must make decisions on investments and surety bonds. They have calculated the price of life-memberships for purchase at various ages. Committees on Membership and on Subscriptions must be active. The services rendered by these committees deserve the grateful thanks of the members of the Institute.

Undoubtedly the most lasting contribution that is being made by the Institute to research in mathematical statistics is the publication of the *Annals of Mathematical Statistics*. Without some first-hand knowledge of the problems that are encountered in publishing a professional journal of high standing it is hardly possible to be conscious of the depth of the debt owed by the Institute to Dr. Samuel S. Wilks, Editor. During the past few years, in addition to the normal editor's problems of maintaining standards of excellence in the articles published, there have been additional difficulties and delays arising from paper and manpower shortages in printing.

In closing this section it is a pleasure to record our appreciation of the assistance and advice received at various times during the year from Mr. Lester Kellogg, Secretary of the A.S.A.; also from Mr. E. A. Stephens of the Ohio Bell Telephone Company in Cleveland in regard to the difficult problems of hotel space which arose in connection with the Cleveland meeting in January 1946.

VI. Election of Fellows

Acting in consideration of the advice of the Committee on Membership, the Board advanced the following members to the grade of Fellow:

M. S. Bartlett, Cambridge University
Trygve Haavelmo, The Norwegian Embassy

William N. Hurwitz, Bureau of the Census
John von Neumann, Institute for Advanced Study

VII. Election of Officers

The following officers were duly nominated and elected for 1946:

President, William G. Cochran
Vice Presidents, Will Feller
Edwin G. Olds

VIII. Committees and Reports of Committees

Our committees and representatives on joint committees for the year 1945 are shown below. The reports of these committees are appended for the information of members. It should be borne in mind that committee reports are for consideration of the Board, they do not commit the Board to any specific action one way or another. As already intimated, every member of the Institute may take pride in the splendid work of these committees. Like the deliberations of the Board, most of the deliberations of the committees were necessarily carried out by correspondence because no large meetings were held at which the members of any committee or the Board could all be brought together.

During the year we have been asked by Dean L. P. Eisenhart, Chairman of the Division of Physical Sciences of the National Research Council, to name a representative. The Board duly appointed Dean Walter Bartky. The invitation from Dean Eisenhart to be so represented is a distinct honor and a recognition of the importance of the Institute in pure and applied research.

We have also been invited to name a representative to the Policy Committee for Mathematicians, to which the Board has named Professor Will Feller. On the committee are four representatives from the American Mathematical Society, one from the Society for Symbolic Logic, and one from the Institute of Mathematical Statistics. The Mathematical Association of America has been invited to name two representatives. The constitution and purposes of this committee are explained in the following paragraphs which are taken from a statement that was approved by the A.M.S. Council on November 23, 1945:

Representatives of each organization shall be selected in accordance with a plan approved by the governing body of that organization.

The Secretary of the American Mathematical Society shall be a non-voting, ex officio member of the committee and shall act as secretary for the committee.

The Policy Committee shall study those problems affecting the mathematical profession which are the common concern of the constituent organizations. It shall be empowered to speak for the constituent organizations on matters which concern the position of mathematics in such matters as proposed or enacted legislation concerned with science, problems concerning the effective use of mathematicians or potential members of our profession, and other questions which tend to affect the dignity and the effective position of mathematics among related sciences, both nationally and internationally.

Nothing in the powers of this committee shall be construed to affect any commitments already made on a national or international basis by any of the constituent organizations

(i e., among these is the International Congress of Mathematicians for which an invitation was issued by the American Mathematical Society in 1936).

This Policy Committee shall be appointed for a period of five years. At the end of that time the work of the committee shall be reviewed and a decision made concerning the continuation of the committee

A supplemental motion passed by the A.M.S. Council asks the Policy Committee to concern itself primarily with the profession of mathematics and only secondarily with the teaching of mathematics.

W. EDWARDS DEMING,
President, 1945.

Committees of the Institute

Committee	*Personnel*	*Report in Appendix*
Development	William G. Cochran, Chairman Paul S. Olmstead, Acting Chairman Chester I. Bliss Henry Scheffé C. C. Craig Frederick Mosteller	A
The Teaching of Statistics	Harold Hotelling, Chairman Walter Bartky Milton Friedman W. Edwards Deming	B
Finance	Paul S. Dwyer, Chairman Charles F. Roos Carl Fischer A. C. Olshen	C
Policy in Regard to Local Chapters	Morris H. Hansen, Chairman Gertrude Cox Samuel S. Wilks	D
Meetings	John H. Curtiss, Chairman T. Koopmans William G. Madow	E
Membership	Joseph L. Doob, Chairman Paul S. Dwyer T. Koopmans Will Feller	F
Increasing Subscriptions to Libraries and Laboratories	W. D. Baten, Chairman Harold F. Dodge Irving W. Burr L. Aroian	G
Tabulation	Paul S. Dwyer, Chairman Will Feller Churchill Eisenhart	

Nominations	C. C. Craig, Chairman Frederick F. Stephan Gertrude Cox
Revising the Constitution and By-Laws	Morris H. Hansen Allen T. Craig Chester I. Bliss John Curtiss
Representatives to the Inter-Society Committee on Federation	John H Curtiss Paul S. Olmstead
Representative to the Division of Physical Sciences, National Research Council	Walter Bartky
Representative to the Policy Committee for Mathematicians	Will Feller
Representative to Explain the Need of Mathematical Statistics in Research for Defense	W. Edwards Deming
Representatives to the Joint Committee for the Development of Statistical Applications in Engineering and Manufacturing	Samuel S. Wilks Paul R. Rider

Appendix A

Report from the Committee on Development

I. General

Continuing the work of the 1944 Committee on Post-War Development, this Committee has analyzed the purpose and policy of the Institute to see what additional activities the Institute should undertake in order to provide further stimulus to the development of the field of mathematical statistics. The following existing and proposed activities were considered:

1. Maintenance of professional standards

2. Publications program
3. Meetings program
4. Rietz Lecture
5. Chapter policy
6. Cooperation in determining educational standards
7. Maintaining relationships with other technical societies
8. Increasing membership of the Institute

In general, each of these activities is placed in the hands of a committee. Except in a few instances, reports of these committees have not been published in the *Annals*. This committee recommends that each of the committees of the Institute together with the representatives of the Institute on joint committees be requested by the Board of Directors to submit a yearly report for possible publication in the March issue of the *Annals* so that the members of the Institute may be kept informed concerning the Institute's affairs.

II. Professional Standards

This committee believes that the Report of the Membership Committee published in the March 1945 issue of the *Annals* is typical of the kind of report desired, providing, as it does, an outline of present standards for membership in the Institute.

III. Publications

The publication program has been discussed with the Editor and we find that we are in agreement with the present editorial policy. We recommend that the Editor submit a yearly report.

Although an increased membership among those engaged primarily in the application of statistics is desirable, it is not considered advisable to alter radically the character of the *Annals* in order to attract such membership. However, writers on theoretical topics in the *Annals* should be encouraged to include illustrations of applications whenever feasible. A desirable goal at which to aim would be for every issue of the *Annals* to contain an expository paper reviewing progress in a broad field of theory or devoted to new fields of existing theory (these functions are not mutually exclusive). It seems more difficult to obtain good papers of this kind than research papers. Now that statisticians are leaving war work the prospect for obtaining such papers should improve. The committee has been informed that the Editor has invited certain writers to contribute expository papers on assigned topics and it is recommended that this policy be continued It is believed that the members of the Institute would like to be informed in the Editor's report concerning progress in receiving such papers.

Last year this committee considered the possibility that the Institute sponsor the publication of a series of books and monographs. In view of recent developments in the commercial publishing field it seems that there is ample opportunity for the publication of such works as the Institute might otherwise undertake to

publish, and the committee therefore recommends against such Institute action at this time.

IV. Meetings

Under normal conditions of transportation, the Institute has held at least two meetings each year, one with the mathematical societies in the summer and one with the social science societies in the winter. This committee favors the continuation of this system Occasionally, meetings have been held with an engineering society. This program does not provide specifically for joint meetings with societies devoted to (a) standardization, (b) engineering, or (c) natural sciences Arrangements for meetings under (a) and (b) could be made through our representatives on the Joint Committee for the Development of Statistical Applications in Engineering and Manufacturing, which has representation from each of these groups. This committee recommends that the Program Committee have on its membership one of the Institute's representatives on the Joint Committee and one who is active in the natural sciences. Important duties of these members are to give advice on the type of program desired for joint meetings in these applied fields and to make arrangements for the meetings. It is also recommended that the Program Committee include Institute members who are active in the mathematical societies and in the social science societies so that our participation in meetings with these groups will be integral to their programs. Other members of the Program Committee may be chosen with similar aims in mind. The yearly report of the Program Committee should discuss among other matters the progress made in arranging joint meetings.

V Rietz Lecture

To direct attention to the work of the Institute, it is recommended that the Institute sponsor an annual lecture of broad interest, to be named after its first president, the late Professor Henry L Rietz. It is suggested that the lecturer be appointed by the Board of Directors, that he be given a year's notice, and that the lecture be arranged for a meeting with an appropriate society.

VI. Chapters

In establishing chapters, the Institute has undertaken obligations that to date have not been fulfilled. Two courses are open. Either the Institute should abolish its existing chapters or it should formulate a policy that will provide for a vigorous chapter program. Some requirements for chapters have been set down by the Committee on Policy with Regard to Local Chapters (Appendix D). It is proposed that this be submitted to the secretaries of our chapters for their comments. Further, certain broader aspects of the problem require additional consideration. Discussion with various members of the Institute indicates that some believe that the interests of the Institute because of its relatively small membership might be better served by organizing geographical sections rather than chapters. Pending final agreement on these points, this committee

recommends that the Board of Directors hold in abeyance any requests for the formation of new chapters.

VII. Educational Standards

The matter of educational standards for college courses is now in the hands of the Committee on the Teaching of Statistics. Such a committee should be a permanent committee of the Institute.

It is our further recommendation that one member of this committee be one of the representatives of the Institute on the Joint Committee for the Development of Statistical Applications in Engineering and Manufacturing. It should be his duty to assess needs for statistics courses, particularly in relation to standardization and engineering.

VIII. Relationships with Other Technical Societies

In 1929, the Joint Committee for the Development of Statistical Applications in Engineering and Manufacturing was formed. The Institute has had two representatives since 1937. The other sponsor societies for the Joint Committee are:

American Society of Mechanical Engineers
American Society for Testing Materials
American Statistical Association
American Mathematical Society
American Institute of Electrical Engineers

Much of the use of statistical method in the war effort is traceable directly to the activity of this committee. In particular, this committee is working continuously to see that statistical methods and statistical concepts are introduced in connection with work on standardization, engineering, and the natural and social sciences. In a report published in the December 1940 issue of the *Annals*, the Institute's War Preparedness Committee made the following recommendations:

> The Institute should "cooperate to the fullest in matters pertaining to quality control and specification with the 'Joint Committee for the Development of Statistical Applications in Engineering and Manufacturing,' of which the Institute is a sponsor."

Six specific steps for a cooperative program with the Joint Committee were outlined. However, although this report was accepted by the Board, no action was taken on these recommendations. In view of the above, we make the following recommendations to the Board:

1. That the Institute's representatives be requested to make a report on the activities of the Joint Committee. (This should be the first of a series of yearly reports.)
2. That the Board request a report from the Joint Committee on the status of statistics and statisticians in engineering and manufacturing including forecasts of future needs and opportunities.
3. That the Board request a report from the Joint Committee on the status of statistics in the training of engineers including recommendations for such training in the future.
4. That at least one of the Institute's representatives be from the engineering or manufacturing field.

IX. Growth of the Institute

The Committee on Development has examined the record of growth of the Institute and finds that the largest increase in recent years has been among people from industry, a group that is still less than a quarter of the total membership. It is believed that the program outlined above will stimulate growth in membership among all users and potential users of mathematical statistics.

X. Publicizing Mathematical Statistics

This Committee recommends that the Institute make available to appropriate channels of public information reliable communications concerning mathematical statistics. As a specific recommendation, the case for the science of statistics should be presented at the hearings of the National Research (Science) Foundation Acts pending in Congress, preferably by representatives acting jointly for the Institute and the American Statistical Association.

XI. The Intersociety Committee

A second meeting of the Intersociety Committee mentioned in last year's report is to be held on December 8th. This Committee feels that consideration of proposals for reorganization of the Institute should not be undertaken prior to advice concerning the action of that Committee.

W. G. Cochran, *Chairman* P. S. Olmstead, *Acting Chairman*
C. I. Bliss C. C Craig
F. C. Mosteller H. Scheffé

November 5, 1945

Appendix B

Report from the Committee on the Teaching of Statistics

A preliminary draft of recommendations in the teaching of statistics was read by the chairman of this committee at the Rutgers Meeting at the Institute in September 1945. These recommendations are at present being re-drafted by members of the Committee and it is hoped that they would be ready to present to the Board in the near future for possible publication in the *Annals.*

Assistance was rendered during the first part of the year to the National Roster of Scientific and Specialized Personnel, toward the development of a formal description of the profession of statistics (mentioned in Part I of the Annual Report of the President). This assistance was carried out jointly with Dr. Chester I. Bliss who was appointed by the American Statistics Association to assist with this project. It is believed by this Committee that the description put forth by the Roster will help bring about recognition of standards of professional competence in statistics and in the teaching of statistics.

Harold Hotelling, *Chairman*
Walter Bartky
Milton Friedman
W. Edwards Deming

Appendix C

Report from the Committee on Finance

The Committee on Finance met in the office of Dr. C. F. Roos in New York City on September 14, 1945. Present were Messrs. Roos, C. H. Fischer, P. S. Dwyer; absent, A. C. Olshen.

The Treasurer presented a summary of income and expenses during the third quarter of 1945 through September 13. This information was considered along with the first half year reports which were prepared some months ago. The Treasurer also presented a graph showing balance on hand at the end of each month (1939–1945) and one showing income during each month (1939–1945). These facts, as well as other pertinent information, were used in formulating the recommendations which follow.

The Finance Committee proposes to the Board of Directors that the following recommendations be approved by the Board as policy for the Institute of Mathematical Statistics.

1. That no revision be made with reference to the adoption of the expected budget for 1945. It appears now that the income will be somewhat higher than the amount indicated on the expected budget ($6450) and that the amount of expense should be somewhat lower the amount there estimated ($6050).

2. That the Secretary-Treasurer be instructed to prepare an Annual statement for 1945 on the general plan of previous annual statements with the addition of an analysis of assets and liabilities. The main assets are cash, bonds, and back issues of the *Annals* It is recommended that the back issues be valued at 75 cents per copy (for inventory purpose)—a fair estimate of cost. It is further recommended that no value be placed on exchanges and office equipment.

3. That the Secretary-Treasurer prepare the annual statement prior to the winter meeting, which means presumably that the books will be closed about December 10th.

4. That, in consideration of the nature of the graph of the income of the Institute, the Institute adopt the policy of having its yearly report run from July 1 to July 1 and that the Secretary-Treasurer be instructed to draw up an additional annual report as of June 30, 1946.

5. That the Secretary-Treasurer be instructed to draw up a budget for 1946 and to submit it to the Finance Committee in sufficient time so that action may be taken on it by the Board at its winter meeting.

6 That the U. S Government G Bonds now owned by the Institute ($3000) be listed on the books at their face values even though the market values of these bonds are slightly lower.

7 That the total amounts of all life membership payments be placed in a special life membership fund and that these funds, at least twice a year, be used in the purchase of U S. Government F Bonds. The market value of these bonds shall be used in determining the amount of this fund at any accounting period.

8. That the Secretary-Treasurer be authorized to take whatever steps are

necessary to obtain adequate interest on our liquid assets. That he maintain sufficient cash position to carry on the business transactions of the Institute and that he invest the remainder (a) either in U. S. Government G bonds or (b) in short term bonds.

9. That the purchase from Professor Carver of all back issues jointly owned by Professor Carver and the Institute be made an item of the budget for 1946.

10. That the Secretary-Treasurer be instructed to purchase a $2,000 fidelity Bond Form B (a form which covers negligence as well as dishonesty) for 3 years for the office of Secretary-Treasurer.

11. That a policy be adopted of allowing a straight 10% discount to all agencies and booksellers who send us subscriptions or orders for back issues.

12 That the Institute set up a permanent Committee on Finance with the Secretary-Treasurer as ex-officio member and chairman. There shall be three additional members with terms of three years with a new member each year. At the formation of the Committee one member shall be appointed for one year, one for two years, and one for three years. A resignation from the Committee shall be followed by an appointment for the unexpired term.

13. That the Board notify any committee working on revision of the Constitution and By-Laws that it is supporting a permanent committee on Finance and believes it appropriate that a statement of the organization and duties of this committee should appear in the By-Laws.

Paul S. Dwyer, *Chairman*
Carl H. Fischer
Abraham C. Olshen
Charles F. Roos

September 15, 1945

Appendix D

Report from the Committee on Policy with Regard to Local Chapters

Attached to this report is a summary of provisions for organizing and working with local chapters; it might be cast into appropriate form and incorporated into the Constitution of the Institute From these recommended provisions it will be clear that this committee does not favor the organization of weak inactive chapters. Unless the membership of the Institute grows substantially it will be possible to have only a very limited number of local chapters under these provisions

It is the opinion of the Committee that it is desirable for members of the Institute to amalgamate with members of other statistical organizations in the same area to form local statistical societies. We believe this will build stronger local statistical organizations and will effect greater advances in the application and development of effective statistical methods. Such amalgamation in the formulation of local societies can best be stimulated, and national leadership provided,

after the national statistical organizations have accomplished a federation or amalgamation. We therefore urge the Institute to use its influence in stimulating discussion and action concerning national federation or amalgamation.

The following further comments are made in addition to or supplementing those provisions recommended for incorporation into the Constitution of the Institute:

1. Do not accept or reject the petition from any group until a plan of organization is formulated. There should be clearance on the following questions:
   a. What are the reciprocal responsibilities of chapters and the parent organization? What type of chapter activity should the Institute seek to promote? What kind of things can chapters do that will advance the purposes for which the Institute exists?

      We have indicated in the recommended provisions that the President of the Institute should personally undertake or designate someone to work with the chapters in answering these and similar questions.
   b If local chapters are not active will they hinder the efforts of the parent organization? We believe that the existence of an inactive organization is a detriment to development of an active statistical group in a community. Activity can be measured in various ways:
      a. Meetings for research in mathematical statistics
      b. Joint meetings with other professions
      c. Bringing in new members to the parent organization
      d. Annual election of officers
1. If members of a chapter must be members of the parent organization, the Secretary-Treasurer of the Institute should notify the secretary of a local chapter whenever a new member joins within his area.
3. It is recommended that if a local chapter desires it, bills for Institute dues contain provision for collection of local dues.
4. The Institute should not allow any local group to use its name unless the group contributes to the accomplishment of the aims of the Institute.

MORRIS H. HANSEN, *Chairman*
GERTRUDE COX
SAMUEL S. WILKS

Suggested Article on Local Chapters for addition to the Constitution

1. Local chapters of the Institute of Mathematical Statistics may be organized to promote the work of the Institute by a local organization of members who are resident within a given limited territory.

2. The members of the local chapter shall be members of the Institute.

3. A local chapter may be established upon acceptance by the Board of Directors of a petition signed by at least twenty-five members of the Institute residing in the area the chapter is to serve.

4. Local chapters shall elect their own officers, designate committees, assess dues, and make any rules for their government not inconsistent with the Constitution of the Institute of Mathematical Statistics.

5. The affairs of local chapters shall be in general charge of the President of the Institute or a representative assigned by him to be responsible for local chapters, under the Direction of the Board of Directors.

6. Any local chapter will be dissolved by:

(a) failing for two successive years to maintain a paid membership of at least 25 members or to hold at least one meeting per year which shall include election of officers; or

(b) by vote of the Board of Directors of the Institute

7. Each local chapter shall transmit a report to the Secretary-Treasurer of the Institute within 30 days of the annual business meeting, reporting among other things, on its officers, the number of members, and on the meetings held during the year.

Appendix E

Report from the Committee on Meetings

A meeting was held at Rutgers University on Sunday Sept. 16, which was attended by 115 members of the Institute. Simultaneously a meeting was held by the American Mathematical Society. The first session, which commenced at 10 a.m was a symposium on sequential analysis. The chairman was Professor W. Allen Wallis of Stanford University and Director of the Statistical Research Group at Columbia University. The speakers and their titles are listed below.

1 *Theory of sequential analysis.*
Professor A. Wald, Columbia University

2. *Construction of multiple sampling inspection plans for attributes from sequential principles*
Dr. Milton Friedman, National Bureau of Economic Research and the Statistical Research Group

3 *Applications of sequential analysis to the ranking of two populations with respect to a single parameter.*
Mr. Meyer A. Girshick, Bureau of Agricultural Economics and the Statistical Research Group

The afternoon session was a series of contributed papers, followed by a preliminary report from the Institute's Committee on the Teaching of Statistics, which was delivered by Professor Harold Hotelling. Dr. W. Edwards Deming, President of the Institute, was chairman of this meeting. The list of contributed papers follows hereunder.

1. *On the variance of a random set in n dimensions.*
Dr. Herbert E. Robbins, The Post Graduate School, Annapolis

2. *The non-central Wishart distribution and its application to problems in multivariate analysis.*
Dr T. W Anderson, Jr., Princeton University

3. *The effect on a distribution function of small changes in the population function.*
Professor Burton H. Camp, Wesleyan University

4. *On composite distributions.*
Dr. Casper Goffman and Dr Benjamin Epstein, Westinghouse Electric and Manufacturing Company

5. *Population, expected values, and sample*
   Professor Emil J. Gumbel, New School for Social Research
6. *On the selection of a sample in repeated steps*
   Dr. William G. Madow, Bureau of the Census
7. *On optimum estimates for stratified samples* (Presented by Margaret Gurney, Bureau of the Census)
   Mr. Morris H. Hansen and Mr. William N. Hurwitz, Bureau of the Census
8. *Pearsonian correlation coefficients associated with least squares theory.* (Presented by title)
   Professor Paul S. Dwyer, University of Michigan

At this writing preparations are being made for a meeting to be held in Cleveland, January 24–27, 1946, and for a meeting with the A.A.A.S. to be held in St. Louis, March 27–30.

JOHN H. CURTISS, *Chairman*
T. KOOPMANS
WILLIAM G. MADOW

Appendix F

Report from the Committee on Membership

The Committee, after study and consideration, recommended to the Board of Directors that Messrs. M. S. Bartlett, T. Haavelmo, William N. Hurwitz, and John von Neumann be advanced to the grade of Fellow. This recommendation was approved by the Board.

The Committee, with the advice and approval of the Board is preparing a letter to be sent to groups of people who are not members of the Institute to call their attention to the work of the Institute. This letter will be accompanied by reprints of a recent paper by Wald and Wolfowitz on *Sampling inspection-plans for continuous production*, with a brief explanation of the field covered by the Wald-Wolfowitz paper, and the statement that it and others that have appeared in recent issues of the *Annals* have already modified statistical practice in important ways.

JOSEPH L. DOOB, *Chairman*
PAUL S. DWYER
T. KOOPMANS
WILL FELLER

Appendix G

Report from the Committee for Increasing Subscriptions to Libraries and Laboratories

This committee prepared suitable literature to send to prospective subscribers. This literature contained a concise description of the nature of the *Annals*, a table of contents for a year, and a subscription blank.

Alphabetical lists of public, college, university and industrial libraries were prepared. These lists contained the name, the librarian, and the address of each library They were checked for duplicates for present subscribers and sent to Professor Dwyer, Secretary-Treasurer. Altogether, the list contained about 1500 libraries.

Professor Dwyer took care of printing the literature, further checking for duplicates, addressing the envelopes, and mailing.

William Dowell Baten, *Chairman*
Harold F. Dodge
Irving W. Burr
L. Aroian

ANNUAL REPORT OF THE SECRETARY-TREASURER OF THE INSTITUTE

(For 1945)

Accounts of the Rutgers meeting of the Institute appeared in the September issue of the *Annals*. Notices of meetings of the Washington Chapter have been sent out from the office of the Secretary-Treasurer.

Due to a large extent to activity of the members, the Institute has enjoyed a large increase in memberhip during the year. The 606 members of a year ago have increased to 777. This is an increase of over 28%.

The Secretary-Treasurer wishes to acknowledge the continued assistance of Professor Lloyd Knowler in looking after the back issues of the *Annals* which are stored at Iowa City.

The following financial statement is drawn up along lines specified by the Finance Committee and the Board of Directors. It covers the period December 31, 1944 to December 31, 1945.

FINANCIAL STATEMENT

December 31, 1944, to December 31, 1945

A. RECEIPTS

BALANCE ON HAND, DECEMBER 13, 1944	$6,790.65
DUES	4,108.40
LIFE MEMBERSHIP PAYMENTS	885.00
SUBSCRIPTIONS	1,515.73
SALE OF BACK NUMBERS	1,737.46
INCOME FROM INVESTMENTS	75.00
MISCELLANEOUS	.20
TOTAL	$15,112.44

B. EXPENDITURES

ANNALS—CURRENT		
Office of Editor.	$400.00	
Waverly Press.	4,056.42	
		$4,456.42
ANNALS—BACK NUMBERS		
Purchase from H. C. Carver	280.51	
Reprinted 300 copies Vol. I No 2, Vol. II No. 1, Vol. IX No. 1.	727.50	
Iowa City Office	45.00	
		1,053.01
OFFICE OF PRESIDENT.		130.25
MATHEMATICAL REVIEWS.		100.00
OFFICE OF THE SECRETARY-TREASURER		
Printing, Mimeographing, programs, etc. (including stamped envelopes)	754.58	

Postage and supplies	166 25	
Clerical help...	852 95	
		1,773.78
MISCELLANEOUS		50 76
BALANCE ON HAND, DECEMBER 31, 1945 (Cash and Bonds)		7,548.22
		$15,112.44

C. SUMMARY OF RECEIPTS AND EXPENDITURES

BALANCE ON HAND,* DECEMBER 31, 1944	$6,790.65
RECEIPTS DURING 1945	8,321 79
EXPENDITURES DURING 1945	7,564.22
BALANCE ON HAND,* DECEMBER 31, 1945	7,548.22
NET EXCESS OF RECEIPTS OVER EXPENDITURES, 1945	757 57

D COMPARISON OF ASSETS ON DECEMBER 31, 1944 AND DECEMBER 31, 1945

	1944	1945
US Government G Bonds	$3,000.00	$6,000 00
Life Membership Funds	330 00 Bank	888.00 F Bonds 327.00 Bank Dep.
Additional Bank Deposits	3,460.65	333 22
Current Accounts Receivable	303.73	255.35
Estimated Value (Cost)**		
Of back issues of *Annals*		
At Iowa City	4,219.25	3,825 75
At Ann Arbor	567.00	1,242.80
Deduct Estimated Value of issues owned by H. C. Carver	879 60	570.60
Total	$11,001.03	12,301.52
Net Gain 1945		1,300.49

E. LIABILITIES OF INSTITUTE OF MATHEMATICAL STATISTICS AS OF DECEMBER 31, 1945

All bills which have been presented have been paid and there are no outstanding accounts against the Institute of appreciable size. The $1215 in Life Membership payments require the Institute to provide the privileges of membership for life for the 17 members who have made payments. About $2500 should be credited to 1946 dues and subscriptions

PAUL S. DWYER
Secretary-Treasurer.

December 31, 1945

* In form of bank deposit and government bonds.

** Value of *Annals* calculated at 75 cents per copy. All 1944 figures and 1945 Ann Arbor figures based on physical inventory. 1945 Iowa City figures based on book inventory.

ANNUAL REPORT OF THE EDITOR

(For 1945)

In spite of the war, enough papers in mathematical statistics have been proposed for publication in the *Annals* in 1945 to keep the total volume of material at approximately 450 pages, the level which has been maintained during the last few years. A total of 40 papers were published in the 1945 volume of the *Annals* of which 14 were short notes published in the "notes" section. The outlook for a sufficient number of acceptable papers to maintain the usual volume of publication during 1946 looks quite favorable. Many mathematical statisticians who were engaged in war work are now free to resume their research. In some cases statistical theory developed in connection with classified war research projects can be expected to be declassified in the near future and made available for open publication.

Most of the material which has been published in the *Annals* consists of original research or extensions of work already published in mathematical statistics as contrasted with material of an expository character. In view of the considerable number of newcomers into the Institute, as well as a general increase of interest in probability and statistics during recent years, it would be highly desirable to publish more expository or survey material. Invitations have been accepted by several individuals to prepare expository articles, but they have been so heavily burdened with extra work during the war that they have been unable to complete their tasks. It is hoped that circumstances will now permit the preparation of expository articles.

On behalf of the Editorial Committee for the *Annals*, the Editor takes this opportunity to acknowledge with thanks the refereeing assistance which has been received from the following individuals during 1945: R. L. Anderson, T. W. Anderson, George W. Brown, A. H. Copeland, W. J. Dixon, J. L. Doob, Milton Friedman, M. A. Girshick, M. Kac, T. Koopmans, Carl Kossack, D. H. Lehmer, H. B. Mann, P. J. McCarthy, F. C. Mosteller, H. E. Robbins, J. W. Tukey, W. A. Wallis, J. D. Williams, and C. P. Winsor. The Editor is also indebted to the following individuals at Princeton University for preparation of manuscripts for the printer, and other editorial assistance from time to time in connection with the *Annals*: Mrs. Gladys B. Huling, Luis F. Nanni, Mrs. Euthie Ross, Mrs. Eleanor C. Schoenly, and John E. Walsh.

S. S. Wilks
Editor

December 31, 1945

CONSTITUTION

OF THE

INSTITUTE OF MATHEMATICAL STATISTICS

ARTICLE I

Name and Purpose

1. This organization shall be known as the Institute of Mathematical Statistics.
2. Its object shall be to promote the interests of mathematical statistics.

ARTICLE II

Membership

1. The membership of the Institute shall consist of Members, Fellows, Honorary Members, and Sustaining Members.

2. Voting members of the Institute shall be (a) the Fellows, and (b) all others, Junior members excepted, who have been members for twenty-three months prior to the date of voting.

3. No person shall be a Junior Member of the Institute for more than a limited term as determined by the Committee on Membership and approved by the Board of Directors.

ARTICLE III

Officers, Board of Directors, and Committee on Membership

1. The Officers of the Institute shall be a President, two Vice-Presidents, and a Secretary-Treasurer. The terms of office of the President and Vice-Presidents shall be one year and that of the Secretary-Treasurer three years. Elections shall be by majority ballots at Annual Meetings of the Institute. Voting may be in person or by mail.

(a) Exception. The first group of Officers shall be elected by a majority vote of the individuals present at the organization meeting, and shall serve until December 31, 1936.

2. The Board of Directors of the Institute shall consist of the Officers, the two previous Presidents, and the Editor of the Official Journal of the Institute.

3 The Institute shall have a Committee on Membership composed of a Chairman and three Fellows. At their first meeting subsequent to the adoption of this Constitution, the Board of Directors shall elect three members as Fellows to serve as the Committee on Membership, one member of the Committee for a term of one year, another for a term of two years, and another for a term of three years. Thereafter the Board of Directors shall elect from among the Fellows one member annually at their first meeting after their election for a term of three years. The president shall designate one of the Vice-Presidents as Chairman of this Committee.

ARTICLE IV

Meetings

1. A meeting for the presentation and discussion of papers, for the election of Officers, and for the transaction of other business of the Institute shall be held annually at such

time as the Board of Directors may designate. Additional meetings may be called from time to time by the Board of Directors and shall be called at any time by the President upon written request from ten Fellows. Notice of the time and place of meeting shall be given to the membership by the Secretary-Treasurer at least thirty days prior to the date set for the meeting. All meetings except executive sessions shall be open to the public. Only papers accepted by a Program Committee appointed by the President may be presented to the Institute.

2. The Board of Directors shall hold a meeting immediately after their election and again immediately before the expiration of their term. Other meetings of the Board may be held from time to time at the call of the President or any two members of the Board. Notice of each meeting of the Board, other than the two regular meetings, together with a statement of the business to be brought before the meeting, must be given to the members of the Board by the Secretary-Treasurer at least five days prior to the date set therefor. Should other business be passed upon, any member of the Board shall have the right to reopen the question at the next meeting.

3. Meetings of the Committee on Membership may be held from time to time at the call of the Chairman or any member of the Committee provided notice of such call and the purpose of the meeting is given to the members of the Committee by the Secretary-Treasurer at least five days before the date set therefor. Should other business be passed upon, any member of the Committee shall have the right to reopen the question at the next meeting. Committee business may also be transacted by correspondence if that seems preferable.

4. At a regularly convened meeting of the Board of Directors, four members shall constitute a quorum. At a regularly convened meeting of the Committee on Membership, two members shall constitute a quorum.

ARTICLE V

Publications

1. The *Annals of Mathematical Statistics* shall be the Official Journal for the Institute. The Editor of the *Annals of Mathematical Statistics* shall be a Fellow appointed by the Board of Directors of the Institute. The term of office of the Editor may be terminated at the discretion of the Board of Directors.

2. Other publications may be originated by the Board of Directors as occasion arises.

ARTICLE VI

Expulsion or Suspension

1. Except for non-payment of dues, no one shall be expelled or suspended except by action of the Board of Directors with not more than one negative vote.

ARTICLE VII

Amendments

1. This constitution may be amended by an affirmative two-thirds vote at any regularly convened meeting of the Institute provided notice of such proposed amendment shall have been sent to each voting member by the Secretary-Treasurer at least thirty days before the date of the meeting at which the proposal is to be acted upon. *Voting* may be in person or by mail.

BY-LAWS

ARTICLE I

DUTIES OF THE OFFICERS, THE EDITOR, BOARD OF DIRECTORS, AND COMMITTEE ON MEMBERSHIP

1. The President, or in his absence, one of the Vice-Presidents, or in the absence of the President and both Vice-Presidents, a Fellow selected by vote of the Fellows present, shall preside at the meetings of the Institute and of the Board of Directors. At meetings of the Institute, the presiding officer shall vote only in the case of a tie, but at meetings of the Board of Directors he may vote in all cases. At least three months before the date of the annual meeting, the President shall appoint a Nominating Committee of three members. It shall be the duty of the Nominating Committee to make nominations for Officers to be elected at the annual meeting and the Secretary-Treasurer shall notify all voting members at least thirty days before the annual meeting. Additional nominations may be submitted in writing, if signed by at least ten Fellows of the Institute, up to the time of the meeting.

2. The Secretary-Treasurer shall keep a full and accurate record of the proceedings at the meetings of the Institute and of the Board of Directors, send out calls for said meetings and, with the approval of the President and the Board, carry on the correspondence of the Institute. Subject to the direction of the Board, he shall have charge of the archives and other tangible and intangible property of the Institute and once a year he shall publish in the *Annals of Mathematical Statistics* a classified list of all Members and Fellows of the Institute. He shall send out calls for annual dues and acknowledge receipt of same; pay all bills approved by the President for expenditures authorized by the Board or the Institute; keep a detailed account of all receipts and expenditures, prepare a financial statement at the end of each year and present an abstract of the same at the annual meeting of the Institute after it has been audited by a Member or Fellow of the Institute appointed by the President as Auditor. The Auditor shall report to the President.

3. Subject to the direction of the Board, the Editor shall be charged with the responsibility for all editorial matters concerning the editing of the *Annals of Mathematical Statistics* He shall, with the advice and consent of the Board, appoint an Editorial Committee of not less than twelve members to co-operate with him; four for a period of five years, four for a period of three years, and the remaining members for a period of two years, appointments to be made annually as needed. All appointments to the Editorial Committee shall terminate with the appointment of a new Editor. The Editor shall serve as editorial adviser in the publication of all scientific monographs and pamphlets authorized by the Board.

4. The Board of Directors shall have charge of the funds and of the affairs of the Institute, with the exception of those affairs specifically assigned to the President or to the Committee on Membership. The Board shall have authority to fill all vacancies ad interim, occurring among the Officers, Board of Directors, or in any of the Committees. The Board may appoint such other committees as may be required from time to time to carry on the affairs of the Institute. The power of election to the different grades of Membership, except the grades of Member and Junior Member, shall reside in the Board.

5 The Committee on Membership shall prepare and make available through the Secretary-Treasurer an announcement indicating the qualifications requisite for the

different grades of membership. The Committee shall review these qualifications periodically and shall make such changes in these qualifications and make such recommendations with reference to the number of grades of membership as it deems advisable. The power to elect worthy applicants to the grades of Member and Junior Member shall reside in the Committee, which may delegate this power to the Secretary-Treasurer, subject to such reservations as the Committee considers appropriate. The Committee shall make recommendations to the Board of Directors with reference to placing members in other grades of membership. The Committee shall give its attention to the question of increasing the number of applicants for membership and shall advise the Secretary-Treasurer on plans for that purpose.

ARTICLE II

Dues

1. Members shall pay five dollars at the time of admission to membership and shall receive the full current volume of the Official Journal. Thereafter, Members shall pay five dollars annual dues. The annual dues of Junior Members shall be two dollars and fifty cents

The annual dues of Fellows shall be five dollars. The annual dues of Sustaining Members shall be fifty dollars Honorary Members shall be exempt from all dues.

(a) Exception. In the case that two Members of the Intitute are husband and wife and they elect to receive between them only one copy of the Official Journal, the annual dues of each shall be three dollars and seventy-five cents.

(b) Exception Any Member or Fellow may make a single payment which will be accepted by the Institute in place of all succeeding yearly dues and which will not otherwise alter his status as a Member or Fellow. The amount of this payment will depend upon the age of this Member or Fellow and will be based upon a suitable table and rate of interest, to be specified by the Board of Directors.

(c) Exception. Any Member or Junior Member of the Institute serving, except as a commissioned officer, in the Armed Forces of the United States or of one of its allies, may upon notification to the Secretary-Treasurer be excused from the payment of dues until the January first following his discharge from the Service He shall have all privileges of membership except that he shall not receive the Official Journal. However during the first year of his resumed regular membership he may have the right to purchase, at $2.50 per volume, one copy of each volume of the Official Journal published during the period of his service membership.

2 Annual dues shall be payable on the first day of January of each year.

3 The annual dues of a Fellow, Member, or Junior Member include a subscription to the Official Journal. The annual dues of a Sustaining Member include two subscriptions to the Official Journal.

4. It shall be the duty of the Secretary-Treasurer to notify by mail anyone who[illegible] may be six months in arrears, and to accompany such notice by a copy of this [illegible] If such person fail to pay such dues within three months from the date of mailin[illegible] notice, the Secretary-Treasurer shall report the delinquent one to the Board o[illegible] by whom the person's name may be stricken from the rolls and all privileges [illegible]ship withdrawn. Such person may, however, be re-instated by the Board [illegible] upon payment of the arrears of dues.

ARTICLE III

Salaries

1. The Institute shall not pay a salary to any Officer, Director, or member of any committee.

ARTICLE IV

Amendments

1. These By-Laws may be amended in the same manner as the Constitution or by a majority vote at any regularly convened meeting of the Institute, if the proposed amendment has been previously approved by the Board of Directors.

CONTRIBUTIONS TO THE THEORY OF SEQUENTIAL ANALYSIS. I

By M. A. Girshick

United States Department of Agriculture

PART I Applications of Sequential Analysis to the Ranking of Two Populations With Respect to a Single Parameter.

1. Summary. Given two populations π_1 and π_2 each characterized by a distribution density $f(x, \theta)$ which is assumed to be known, except for the value of the parameter θ. It is desired to test the composite hypothesis $\theta_1 < \theta_2$ against the alternative hypothesis $\theta_1 > \theta_2$ where θ_i is the value of the parameter in the distribution density of π_i, $(i = 1, 2)$.

The criterion proposed for testing this hypothesis is based on the sequential probability ratio and consists of the following:

Choose two positive constants a and b and two values of θ, say θ_1^0 and θ_2^0. Take pairs of observations $x_{1\alpha}$ from π_1 and $x_{2\alpha}$ from π_2, $(\alpha = 1, 2, \ldots)$, in sequence and compute $Z_j = \sum_{\alpha=1}^{j} z_\alpha$ where

$$z_\alpha = \log\left[\frac{f(x_{2\alpha}, \theta_1^0) f(x_{1\alpha}, \theta_2^0)}{f(x_{2\alpha}, \theta_2^0) f(x_{1\alpha}, \theta_1^0)}\right].$$

The hypothesis tested is accepted or rejected depending on whether $Z_n \geq a$ or $Z_n \leq -b$ where n is the smallest integer j for which either one of these relationships is satisfied.

The boundaries a and b are partly given in terms of the desired risks of making an erroneous decision. The values θ_1^0 and θ_2^0 define the magnitude of the difference between the values of θ in π_1 and in π_2 which is considered worth detecting. It is shown that the power of this test is constant on a curve $h(\theta_1, \theta_2) = \text{constant}$. If $E\left(\log \frac{f(x, \theta_2^0)}{f(x, \theta_1^0)}\right)$ is a monotonic function of θ, then the test is unbiased in the sense that all points (θ_1, θ_2) which lie on the curve $h(\theta_1, \theta_2) = \text{constant}$ are such that either every $\theta_1 < \theta_2$ or every $\theta_1 > \theta_2$. For a large class of known distributions the quantity h is shown to be an appropriate measure of the difference between θ_1 and θ_2 and the test procedure for this class of distributions is simple and intuitively sensible.

For the case of the binomial, the exact power of this test as well as the distribution of n is given.

1.1 General discussion. Consider two processes (populations) π_1 and π_2 each yielding a measurable quantity x whose distribution density $f(x, \theta)$ is assumed to be known except for the value of the parameter θ. On the basis of a random sample obtained from each, it is desired to choose that process which yields the smaller (or larger) θ. That is, it is desired to devise a test which will

result in a high probability of accepting π_1 if the θ characterizing its distribution density is smaller (or larger) than the θ in π_2, a high probability of rejecting π_1 (i.e. accepting π_2) when the opposite is true, and approximately equal probability of making one or the other decision if the value of θ in π_1 is the same as in π_2.

As an illustration of the type of problem here considered, let us assume that a manufacturer is faced with a choice between two competing processes of production, each process yielding an unknown fraction defective p and each entailing about the same operating cost. Based on the evidence of a random sample selected from each, the manufacturer wishes to choose that process which yields the smaller fraction defective. If the fractions defective in the two processes differ by a significant amount, he will want a test which guarantees a high probability of making a correct decision. If, however, the fraction defective in the two processes are of approximately the same magnitude, it will be a matter of indifference to him which decision is reached.

The solution given in this paper to the above problem is based on Wald's sequential probability ratio test [1]. The resulting procedure not only requires on the average, fewer observations for the same protection than any other test (which is always the case with sequential tests of this type) but is also direct and simple when applied to a large class of distributions commonly met in practice.

1.2 Derivation of the sequential test when the existence of a priori probabilities is assumed. The choice of the probability ratio as a method of discriminating between the two processes is suggested by considerations of a priori probabilities. Let us assume that each process may have either θ_1^0 or θ_2^0 as the value of a parameter θ in its distribution density and that the value θ_1^0 is more desirable than θ_2^0. Let us further assume that there exists an a priori probability g_1 that a process will have θ_1^0 as a parameter and an a priori probability $g_2 = 1 - g_1$ that it will have θ_2^0 as a parameter. Let the likelihood for n observations $x_{11}, x_{12}, \cdots, x_{1n}$ drawn from π_1 be designated by $p(x_{11}, x_{12}, \cdots, x_{1n}, \theta_1^0)$ when θ_1^0 is the parameter in π_1, and by $p(x_{11}, x_{12}, \cdots, x_{1n}, \theta_2^0)$ when θ_2^0 is the parameter in π_1. Let the likelihoods $p(x_{21}, x_{22}, \cdots, x_{2n}, \theta_1^0)$ and $p(x_{21}, x_{22}, \cdots, x_{2n}, \theta_2^0)$ be similarly defined for n observations $x_{21}, x_{22}, \cdots, x_{2n}$ drawn from π_2. Then

$$p(x_{i1}, x_{i2}, \cdots, x_{in}, \theta_j^0) = \prod_{\alpha=1}^{n} f(x_{i\alpha}, \theta_j^0), \qquad i, j = 1, 2. \tag{1.201}$$

Let β_{ij}, $(i, j = 1, 2)$, be the a posteriori probability that having obtained $x_{i\alpha}$, $(\alpha = 1, 2, \cdots, n)$, that process π_i has θ_j^0 as a parameter in its distribution density. Then

$$\beta_{i1} = \frac{g_1 p(x_{i1}, x_{i2}, \cdots, x_{in}, \theta_1^0)}{g_1 p(x_{i1}, \cdots, x_{in}, \theta_1^0) + g_2 p(x_{i1}, \cdots, x_{in}, \theta_2^0)} \tag{1.202}$$

and

$$\beta_{i2} = \frac{g_2 p(x_{i1}, \cdots, x_{in}, \theta_2^0)}{g_1 p(x_{i1}, \cdots, x_{in}, \theta_1^0) + g_2 p(x_{i1}, \cdots, x_{in}, \theta_2^0)} \tag{1.203}$$

for $i = 1, 2$.

In order to decide whether the hypothesis that θ_1^0 belongs to the distribution density of π_1 is more tenable than the hypothesis that it belongs to the distribution of π_2, it is only necessary to compare β_{11} with β_{21}. But if β_{11} is equal to or greater than β_{21}, the ratio β_{11}/β_{12} must be equal to or greater than β_{21}/β_{22} and conversely. For assume that $\beta_{11} \geq \beta_{21}$. Subtracting $\beta_{11}\beta_{21}$ from each side of the inequality we get $\beta_{11}(1 - \beta_{21}) \geq \beta_{21}(1 - \beta_{11})$. But since $1 - \beta_{21} = \beta_{22}$ and $1 - \beta_{11} = \beta_{21}$, we see that $\beta_{11}/\beta_{12} \geq \beta_{21}/\beta_{22}$. Conversely, let $\beta_{11}/\beta_{12} \geq \beta_{21}/\beta_{22}$. Then $\beta_{11}(1 - \beta_{21}) \geq \beta_{21}(1 - \beta_{11})$, or $\beta_{11} \geq \beta_{21}$.

From the above it would appear that a sensible sequential procedure for deciding whether θ_1^0 is more likely to belong to π_1 than to π_2 is as follows: Select two positive quantities A and B with $A > 1$ and $B < 1$. Take a pair of observations $(x_{1\alpha}, x_{2\alpha})$, $(\alpha = 1, 2, \cdots)$, at a time, one from each process. At each step (i.e., for each sample size n) compute the ratio $\lambda = \dfrac{\beta_{21}}{\beta_{22}} \Big/ \dfrac{\beta_{11}}{\beta_{12}}$. If at any stage $\lambda \leq B$, terminate the sampling and accept the hypothesis that θ_1^0 is a parameter in the distribution density of π_1. On the other hand, if at any stage $\lambda \geq A$, terminate sampling and accept the hypothesis that θ_1^0 is a parameter of the distribution density in π_2. If neither holds, that is if $B < \lambda < A$, then take another pair of observations, consisting of one from each process. Continue this procedure until one or the other decision is reached.[1]

The interesting point here is that the decision function λ is independent of g_1 and g_2. In fact, it is easily seen from equations (1.202) and (1.203) that

$$\lambda = \frac{p(x_{21}, x_{22}, \cdots, x_{2n}, \theta_1^0)p(x_{11}, x_{12}, \cdots, x_{1n}, \theta_2^0)}{p(x_{21}, x_{22}, \cdots, x_{2n}, \theta_2^0)p(x_{11}, x_{12}, \cdots, x_{1n}, \theta_1^0)}. \tag{1.204}$$

1.3 The proposed sequential test as a special case of a sequential probability ratio test. If we examine the expression given in (1.204) we see that it is a ratio of two likelihoods. The numerator of the ratio is the likelihood of the $2n$ observations under the hypothesis that θ_2^0 is a parameter in π_1 and θ_1^0 is a praameter in π_2; the denominator is the likelihood of the $2n$ observations under the hypothesis that θ_1^0 is a parameter in π_1 and θ_2^0 is a parameter in π_2. Thus, the proposed sequential test is equivalent to a sequential probability ratio test (see [1]) for testing the simple hypothesis that θ_1^0 belongs to π_1 and θ_2^0 belongs to π_2 against the alternative hypothesis that θ_2^0 belongs to π_1 and θ_1^0 belongs to π_2. We can, therefore, apply the theory of sequential analysis developed by A. Wald ([1] and [2]) to this problem.

While the test is posed in terms of a simple hypothesis, the solution, as will be shown later, is in fact a solution to a composite hypothesis. In order to bring this out more clearly we shall rederive a few of the results which have already been obtained by A. Wald. This will be done in sections 1.4, 1.5, and 1.6.

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[1] That a decision will be reached eventually can be asserted with probability one if the variance of the variate z_α (defined by (1.301)) below is different from zero (or if it is zero, the value of z_α is different from zero). See [2], Lemma 1 As we shall see later, if, in fact, both processes have either θ_1^0 or θ_2^0 as parameters, then the above sequential procedure will result in the acceptance of either process with approximately equal probability

In what follows we shall speak of the hypothesis (θ_1, θ_2) to mean the hypothesis that θ_1 is the value of the parameter in the distribution density of π_1 and θ_2 is the value of the parameter in the distribution density of π_2. The hypothesis (θ_1^0, θ_2^0) will represent a specific hypothesis which we may wish to test and will be used to define the decision function (the probability ratio) of the sequential test.

Let us fix $A > 1$ and $B < 1$ and set

$$z_\alpha = \log\left[\frac{f(x_{2\alpha}, \theta_1^0)f(x_{1\alpha}, \theta_2^0)}{f(x_{2\alpha}, \theta_2^0)f(x_{1\alpha}, \theta_1^0)}\right] \tag{1.301}$$

where $x_{1\alpha}$ is the αth observation from π_1, $x_{2\alpha}$ is the αth observation from π_2 and (θ_1^0, θ_2^0) is the particular hypothesis to be tested against the alternative hypothesis (θ_2^0, θ_1^0) Let $a = \log A$ and $-b = \log B$. Then a and b are positive. Since the observations from π_1 and π_2 are assumed to be independent, $\log \lambda = \sum_{\alpha=1}^{n} z_\alpha$. Hence the proposed sequential test can be carried out in the following manner. Draw one pair of observations at a time, one from π_1 and one from π_2. Let $z_1, z_2, \cdots$ be the values of z_α obtained from the first, second, etc. trial. Let $Z_n = z_1 + z_2 + \cdots + z_n$, $(n = 1, 2, \cdots)$. Continue sampling as long as $-b < Z_n < a$. Whenever $Z_n \geq a$, $(n = 1, 2, 3, \cdots)$, terminate sampling and accept π_2 (or π_1). Whenever $Z_n \leq -b$, $(n = 1, 2, 3, \cdots)$, terminate sampling and accept π_1 (or π_2).

1.3a *Basic assumptions.* In this section and throughout this paper, we shall be dealing with sequential tests involving, as above, a decision function $Z_n = z_1 + z_2 + \cdots + z_n$, $(n = 1, 2, \cdots$, ad inf.), where the z_α's are independently distributed random variables having a common distribution function. Let z denote a random variable whose distribution is the same as the common distribution of z_α, $(\alpha = 1, 2, \cdots$, ad inf.). It will be assumed, even if not explicitly stated, that the distribution of z satisfies the following conditions.

CONDITION I. *Both the expected value Ez of z and the variance of z exist and are unequal to zero.*

CONDITION II. *There exists a positive δ such that $P(e^z > 1 + \delta) > 0$ and $P(e^z < 1 - \delta) > 0$.*

CONDITION III. *For any real value h, the expected value $Ee^{hz} = g(h)$ exists.*

CONDITION IV. *The first two derivatives of the function $g(h)$ exist and may be obtained by differentiating under the integral sign.*

1.3b. *Fundamental properties of sequential tests.* Let z be defined as in 1.3a. Then under the assumption that the distribution of z satisfies the conditions specified, Wald [2] has proved the following:

LEMMA I. *The probability that a decision is reached in a finite number of steps is unity.*

LEMMA II. *There exists one and only one real value $h \neq 0$ such that the expected value $Ee^{hz} = 1$.*

FUNDAMENTAL IDENTITY: *The fundamental identity $Ee^{Z_n t}[\phi(t)]^{-n} = 1$ holds for all points in the complex plane for which $|\phi(t)| \geq 1$ where $\phi(t) = Ee^{tz}$.*

Let $w = \log \frac{f(x, \theta_1^0)}{f(x, \theta_2^0)}$ and let the distribution density of x be $f(x, \theta)$. Let θ_1 and θ_2 be any two values of θ which may be distinct from θ_1^0 and θ_2^0. Then it can easily be verified that if w satisfies the conditions specified in section 1.3a under the hypothesis $\theta = \theta_1$ as well as the hypothesis $\theta = \theta_2$, and if moreover the expected values of w under these two hypotheses are not equal, then $z = \log \frac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)}$ will also satisfy these conditions when the joint distribution density of x_1 and x_2 (x_1 representing the measurable characteristic in π_1 and x_2 in π_2) is either $f(x_1, \theta_1)\, f(x_2, \theta_2)$ or $f(x_1, \theta_2)\, f(x_2, \theta_1)$.

In what follows, we shall assume that the distribution of w satisfies the required restrictions for the θ_1 and θ_2 under consideration and that the expectation of w under the hypothesis $\theta = \theta_1$ is unequal to the expectation of w under the hypothesis $\theta = \theta_2$ Consequently, we shall assume that Lemmas I and II and the Fundamental Identity hold for all the sequential tests we shall consider.

1.4 The power of the proposed test. Let x_1 be an observation from π_1 and x_2 an observation from π_2. Let

$$z = \log \frac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)} \tag{1.401}$$

where θ_1^0 and θ_2^0 are specified parameters in the probability density of π_1 and π_2 respectively. Furthermore, let $\phi(t \mid \theta_1, \theta_2) = E(e^{tz} \mid \theta_1, \theta_2)$ be the moment generating function of z under the hypothesis (θ_1, θ_2). Then

$$E(e^{tz} \mid \theta_1, \theta_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)}\right]^t f(x_1, \theta_1)f(x_2, \theta_2)\, dx_1\, dx_2. \tag{1.402}$$

By Lemma II there exists one and only one real number $h \neq 0$ such that $E(e^{hz} \mid \theta_1, \theta_2) = 1$ Let $L_h = P(Z_n \leq -b \mid \theta_1, \theta_2)$ be the probability that the sequential test terminates and $Z_n \leq -b$ under the hypothesis (θ_1, θ_2). Then by Lemma I, $1 - L_h = P(Z_n \geq a \mid \theta_1, \theta_2)$. For any random variable u considered under the hypothesis (θ_1, θ_2), let the symbol $E_b(u)$ stand for the expected value of u under the restriction that $Z_n \leq -b$ and $E_a(u)$ stand for the expected value of u under the restriction that $Z_n \geq a$. In terms of the above definitions, the Fundamental Identity can be expressed as follows:

$$L_h E_b e^{tZ_n}[\phi(t \mid \theta_1, \theta_2)]^{-n} + (1 - L_h)E_a e^{tZ_n}[\phi(t \mid \theta_1, \theta_2)]^{-n} = 1. \tag{1.403}$$

Setting $t = h$ in (1.403) we get

$$L_h E_b e^{hZ_n} + (1 - L_h)E_a e^{hZ_n} = 1. \tag{1.404}$$

Following Wald [2], we define a two valued random variable $\bar{Z}_n$ in this manner: $\bar{Z}_n = a$ if $Z_n \geq a$ and $\bar{Z}_n = -b$ if $Z_n \leq -b$. Let $\bar{Z}_n - Z_n = \epsilon$. Then ϵ is also a random variable. In what follows, we shall substitute 0 for ϵ. The error committed in neglecting ϵ is small when θ_1^0 is close to θ_2^0. As we shall indicate later,

the quantity ϵ can, in fact, be neglected without error in the special case where $f(x, \theta)$ is the binomial distribution.

Substituting $\bar{Z}_n$ for Z_n in (1.404) we get

$$L_h e^{-hb} + (1 - L_h)e^{ha} = 1. \tag{1.405}$$

Solving for L_h we get[2]

$$L_h = \frac{1 - e^{ha}}{e^{-hb} - e^{ha}} = \frac{e^{h(a+b)} - e^{hb}}{e^{h(a+b)} - 1}. \tag{1.406}$$

As we shall see later, $h = 0$ when $\theta_1 = \theta_2$. But when $h = 0$, L_h in (1.406) is indeterminate. However, it can be easily seen that

$$\lim_{h\to 0} L_h = \frac{a}{a + b}. \tag{1.407}$$

It follows from (1.406) that the power of the test is constant for all θ_1 and θ_2 which give the same root $t = h$. The quantity h is thus fundamental in this test, and as we shall see later, is an appropriate measure of the difference between θ_1 and θ_2 for a large class of distributions.

1.5 Method of determining the sequential test. Let z be defined as in (1.401) and let $\phi_1(t) = E(e^{tz} \mid \theta_1^0, \theta_2^0)$ be the moment generating function of z under the hypothesis (θ_1^0, θ_2^0), and let $\phi_2(t) = E(e^{tz} \mid \theta_2^0, \theta_1^0)$ be the moment generating function of z under the hypothesis (θ_2^0, θ_1^0). Furthermore, let $\alpha = P(\bar{Z}_n = a \mid \theta_1^0, \theta_2^0)$ and $\beta = P(\bar{Z}_n = -b \mid \theta_2^0, \theta_1^0)$. Then by Lemma I, $1 - \alpha = P(\bar{Z}_n = -b \mid \theta_1^0, \theta_2^0)$ and $1 - \beta = P(\bar{Z}_n = a \mid \theta_2^0, \theta_1^0)$. Now, applying Wald's Fundamental Identity we have,

$$(1 - \alpha)e^{-tb}E_{1b}[\phi_1(t)]^{-n} + \alpha e^{ta}E_{1a}[\phi_1(t)]^{-n} = 1, \tag{1.501}$$

$$\beta e^{-tb}E_{2b}[\phi_2(t)]^{-n} + (1 - \beta)e^{ta}E_{2a}[\phi_2(t)]^{-n} = 1, \tag{1.502}$$

where the symbol E_{1a} stands for the conditional expectation knowing that $\bar{Z}_n = a$ and E_{1b} stand for the conditional expectation knowing that $\bar{Z}_n = -b$; with both expectations taken under the hypothesis (θ_1^0, θ_2^0). The symbols E_{2a} and E_{2b} are similarly defined but under the hypothesis (θ_2^0, θ_1^0). Setting $t = 1$ in (1.501) and $t = -1$ in (1.502), we get, in view of Corollary 2, Theorem 2 below,

$$(1 - \alpha)e^{-b} + \alpha e^{a} = 1, \tag{1.503}$$

$$\beta e^{b} + (1 - \beta)e^{-a} = 1. \tag{1.504}$$

[2] In what follows, L_h will always stand for the probability that a sequential test will terminate with $Z_n \leq -b$. In any given problem, the interpretation of the event $Z_n \leq -b$ will be clear from the context.

Now $a = \log A$ and $-b = \log B$. Hence, equations (1.503) and (1.504) become

$$(1.505) \qquad (1 - \alpha) B + \alpha A = 1,$$

$$(1.506) \qquad \frac{\beta}{B} + \frac{1 - \beta}{A} = 1,$$

or

$$(1.507) \qquad A = \frac{1 - \beta}{\alpha} \quad \text{and} \quad a = \log \frac{1 - \beta}{\alpha},$$

$$(1.508) \qquad B = \frac{\beta}{1 - \alpha} \quad \text{and} \quad b = \log \frac{1 - \alpha}{\beta}.$$

From (1.507) and (1.508) we see that the sequential test is completely determined by the function z, which, in turn, is defined by θ_1^0 and θ_2^0, and by the probabilities of making a decision for the two hypotheses (θ_1^0, θ_2^0) and (θ_2^0, θ_1^0).

Once z is defined in terms of a specific (θ_1^0, θ_2^0), the probability that $Z_n \leq -b$ will be equal to $1 - \alpha$ and the probability that $Z_n \geq a$ will be α (if we neglect the fact that $|Z_n|$, at a decision point, might exceed a or b) for the totality of hypotheses (θ_1, θ_2) for which the moment generating function $\phi(t \mid \theta_1, \theta_2) = 1$ when $t = 1$. A similar statement can be made for the corresponding hypotheses (θ_2, θ_1) for which the moment generating function will equal unity when $t = -1$. Hence, we see that while the test is defined by specifying two points (θ_1^0, θ_2^0) and (θ_2^0, θ_1^0) in the parameter space, the pre-assigned risks α and β of making the correct decision will be approximately constant on the set of points for which the moment generating function equals unity when $t = 1$ and when $t = -1$, respectively. This set of points usually will constitute a smooth curve.

If $\theta_1 = \theta_2$, $L_0 = \dfrac{a}{a + b}$ (by 1.407). Hence, the probability of accepting π_1 will be close to $\frac{1}{2}$ if a is close to b, and will equal $\frac{1}{2}$ if $a = b$. But from (1.507) and (1.508) we see that $a = b$ if $\alpha = \beta$. Thus, if we construct a test which will give a probability of rejecting π_1 when (θ_1^0, θ_2^0) is true equal to the probability of accepting π_1 when (θ_2^0, θ_1^0) is true, we shall be accepting π_1 and π_2 with equal frequency when in fact $\theta_1 = \theta_2$.

1.6 The average number of pairs of observations required to reach a decision. Let $E(n \mid \theta_1, \theta_2)$ be the expected number of pairs of observations required to reach a decision under the hypothesis (θ_1, θ_2). We shall show that

$$(1.601) \qquad E(n \mid \theta_1, \theta_2) = \frac{a(1 - L_h) - bL_h}{Ez}.$$

PROOF: Differentiating the Fundamental Identity,

$$(1.602) \qquad Ee^{tZ_n}[\phi(t)]^{-n} = 1,$$

with respect to t, we get[3]

$$E\{Z_n e^{tZ_n}[\phi(t)]^{-n} - ne^{tZ_n}\phi'(t)[\phi(t)]^{-n-1}\} = 0. \tag{1.603}$$

Setting $t = 0$, we get

$$EZ_n - \phi'(0)E(n \mid \theta_1, \theta_2) = 0. \tag{1.604}$$

But

$$E\bar{Z}_n = a(1 - L_h) - bL_h \tag{1.605}$$

and

$$\phi'(0) = Ez. \tag{1.606}$$

Hence, solving for $E(n \mid \theta_1, \theta_2)$ in (1.604) and substituting from (1.605) and (1.606) we get

$$E(n \mid \theta_1, \theta_2) = \frac{a(1 - L_h) - bL_h}{Ez}. \tag{1.607}$$

While L_h is approximately constant for all values of (θ_1, θ_2) for which the moment generating function equals unity for $t = h$ the expected value of n given by (1.607) will depend on the particular hypothesis (θ_1, θ_2). This follows from the fact that Ez is not necessarily constant for the same set of points (θ_1, θ_2) for which L_h is constant.

1.7 Some general properties of the proposed test.

THEOREM 1. *Let* $z = \log \dfrac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)}$ *where x_1 is an observation from π_1 and x_2 from π_2. Then if $F(z)$ is the distribution density of z under the hypothesis (θ_1, θ_2), $F(-z)$ is the distribution density of z under the hypothesis (θ_2, θ_1).*

PROOF: Let t be a real number and let $\psi_1(t) = E(e^{itz} \mid \theta_1, \theta_2)$ be the characteristic function of z under the hypothesis (θ_1, θ_2). Then

$$\psi_1(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)}\right]^{it} f(x_1, \theta_1)f(x_2, \theta_2)\, dx_1\, dx_2. \tag{1.701}$$

Now let $\psi_2(t) = E(e^{-itz} \mid \theta_2, \theta_1)$ be the characteristic function of $-z$ under the hypothesis (θ_2, θ_1). Then

$$\psi_2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)}\right]^{-it} f(x_1, \theta_2)f(x_2, \theta_1)\, dx_1\, dx_2. \tag{1.702}$$

Interchanging the variables of integration in (1.702) we see that $\psi_1(t) = \psi_2(t)$. Consequently, the distribution of z under the hypothesis (θ_1, θ_2) is the same as

[3] This assumes that the Fundamental Identity can be differentiated with respect to t. The results that follow can be derived without any reference to the Fundamental Identity. See Wald [1], page 142.

the distribution of $-z$ under the hypothesis (θ_2, θ_1). This theorem in conjunction with the fact that $E(z \mid \theta_1, \theta_2) \neq 0$ when $\theta_1 \neq \theta_2$ shows that the decision function z discriminates in a real sense between the two alternative hypotheses (θ_1, θ_2) and (θ_2, θ_1).

THEOREM 2. *Let $E(e^{tz} \mid \theta_1, \theta_2)$ be the moment generating function of z under the hypothesis (θ_1, θ_2) and let $E(e^{tz} \mid \theta_2, \theta_1)$ be the moment generating function of z under the hypothesis (θ_2, θ_1) Then, if $t = h$ is a root of the equation $E(e^{tz} \mid \theta_1, \theta_2) = 1$, then $t = -h$ is a root of the equation $E(e^{tz} \mid \theta_2, \theta_1) = 1$.*

PROOF: The same as Theorem 1. As we have seen in Section 1.4, the power of the proposed sequential test (neglecting ϵ) depends only on h This theorem shows that if the probability of accepting π_1 is large under the hypothesis (θ_1, θ_2), it will be small under the hypothesis (θ_2, θ_1), and conversely.

COROLLARY 1. *The only value of t for which $E(e^{tz} \mid \theta, \theta) = 1$ is $t = 0$* This follows from Theorem 2.

COROLLARY 2. *The values of t for which $E(e^{tz} \mid \theta_1^0, \theta_2^0) = 1$ and $E(e^{tz} \mid \theta_2^0, \theta_1^0) = 1$ are $t = 1$ and $t = -1$ respectively* This can be seen by expressing $E(e^{tz} \mid \theta_1^0, \theta_2^0)$ as a double integral and setting $t = 1$

THEOREM 3 *Let ω be the totality of points (θ_1, θ_2) in the parameter space for which $\theta_1 < \theta_2$. Then a necessary and sufficient condition that the values of h (for which $E(e^{hz} \mid \theta_1, \theta_2) = 1$) be of the same sign for all points in ω is that*

$$Ew \mid \theta = \int_{-\infty}^{\infty} \log \frac{f(x, \theta_2^0)}{f(x, \theta_1^0)} f(x, \theta)\, dx \tag{1.703}$$

be a monotonic function of θ.

To prove this theorem we need the following lemma.

LEMMA 1. *Let $g(x, \theta)$ be the distribution density of x and $\psi(t)$ its moment generating function. Let h be the real non-zero value of t for which $\psi(t) = 1$. Then the sign of h is opposite in sign to Ex (the expected value of x) if $Ex \neq 0$*

PROOF: For any random variable u, Wald [1] has shown that the inequality

$$Eu \leq \log Ee^u \tag{1.704}$$

holds

Setting $u = tx$, where t is a constant, we get

$$tEx \leq \log Ee^{tx} = \log \psi(t). \tag{1.705}$$

Setting $t = h$ in (1.705) we get $hEx \leq 0$. This proves the lemma.

Now let $E(z \mid \theta_1, \theta_2)$ be the expected value of z under the hypotheses (θ_1, θ_2) where (θ_1, θ_2) belongs to ω. Then

$$\begin{aligned} E(z \mid \theta_1, \theta_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \log \frac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)} f(x_1, \theta_1)f(x_2, \theta_2)\, dx_1\, dx_2 \\ &= \int_{-\infty}^{\infty} \log \frac{f(x, \theta_2^0)}{f(x, \theta_1^0)} f(x, \theta_1)\, dx \qquad (1.706) \\ &\quad - \int_{-\infty}^{\infty} \log \frac{f(x, \theta_2^0)}{f(x, \theta_1^0)} f(x, \theta_2)\, dx = Ew \mid \theta_1 - Ew \mid \theta_2 . \end{aligned}$$

From (1.706) we see that if $Ew \mid \theta$ is monotonic in θ, $E(z \mid \theta_1, \theta_2)$ will have a constant sign for all points (θ_1, θ_2) in ω and hence by Lemma 1, h will have a constant sign. Conversely, if h is of constant sign for all (θ_1, θ_2) in ω, so will $E(z \mid \theta_1, \theta_2)$ be. Consequently, by (1.706) $Ew \mid \theta$ must be monotonic.

COROLLARY 1. *Let $Ew \mid \theta$ be a monotonic function of θ and let ω_h, $(h \neq 0)$, be the totality of points (θ_1, θ_2) in the parameter space for which the power of the sequential test is constant. Then the coordinates of the points (θ_1, θ_2) in ω_h are such that either every $\theta_1 < \theta_2$ or every $\theta_1 > \theta_2$.*

PROOF: By assumption all points in ω_h have the same power. Since L_h in (1.406) is a strictly increasing function of h, the points in ω_h must yield the same h. However, if we assume that ω_h contains a point (θ_1', θ_2') with $\theta_1' < \theta_2'$ and a point (θ_1'', θ_2'') with $\theta_1'' > \theta_2''$, the sign of $E(z \mid \theta_1', \theta_2')$ by (1.706) will be opposite to the sign of $E(z \mid \theta_1'', \theta_2'')$. Hence, the value of h yielded by (θ_1', θ_2') is opposite in sign to that yielded by (θ_1'', θ_2''), which contradicts the assumption that both points yield the same h

Theorem 3 and Corollary 1 show that if $Ew \mid \theta$ is monotonic in θ, the proposed sequential test is unbiased in the sense that all points (θ_1, θ_2) that lie on the curve h = constant (and hence have the same power) will have the property that either the inequality $\theta_1 < \theta_2$ holds or the inequality $\theta_1 > \theta_2$ holds. The equality sign will hold if and only if $h = 0$.

1.8 The proposed test applied to distributions which admit sufficient statistics. Let $f(x, \theta)$ admit a sufficient estimate of θ. Then it is well known that $f(x, \theta)$ can be written in the form[4]

(1.801)
$$f(x, \theta) = e^{u(x)v(\theta)+r(x)+w(\theta)}.$$

Setting $z = \log \dfrac{f(x_2, \theta_1^0)f(x_1, \theta_2^0)}{f(x_2, \theta_2^0)f(x_1, \theta_1^0)}$, we see that for this class of distributions the decision function assumes the simple form:

(1.802)
$$z = [u(x_2) - u(x_1)][v(\theta_1^0) - v(\theta_2^0)].$$

Let $a^* = \dfrac{a}{v(\theta_1^0) - v(\theta_2^0)}$ and $b^* = \dfrac{b}{v(\theta_1^0) - v(\theta_2^0)}$. Then the decision function becomes

(1.803)
$$z^* = u(x_2) - u(x_1).$$

We shall now show that, for this class of distributions, the power of the sequential test is a function of $v(\theta_1) - v(\theta_2)$. To prove this, it is only necessary to show that $E(e^{tz^*} \mid \theta_1, \theta_2)$ equals unity for $t = v(\theta_1) - v(\theta_2)$. Now

(1.804)
$$\begin{aligned} E(e^{tz^*} \mid \theta_1, \theta_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t[u(x_2)-u(x_1)]} f(x_1, \theta_1)f(x_2, \theta_2)\, dx_1\, dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{u(x_1)[v(\theta_1)-t]+u(x_2)[t+v(\theta_2)]+r(x_1)+r(x_2)+w(\theta_1)+w(\theta_2)}\, dx_1\, dx_2. \end{aligned}$$

If we set $t = v(\theta_1) - v(\theta_2)$ in (1 804), we see that the statement is proved.

[4] See, for example, [3].

Let $En \mid h$ be the average number of pairs of observations required to reach a decision when $v(\theta_1) - v(\theta_2) = h$ Then by formula (1.607) we have

$$(1.805) \qquad E(n \mid h) = \frac{a^*(1 - L_h) - b^* L_h}{E[u(x_2) - u(x_1)]} = \frac{(1 - L_h)\log A + L_h \log B}{h_0 E[u(x_2) - u(x_1)]}.$$

Since the expected value of $u(x)$ will not necessarily equal $v(\theta)$, the average number of pairs of observations required to reach a decision will depend not only on $v(\theta_1) - v(\theta_2)$ but also on the particular hypothesis (θ_1, θ_2) considered.

Since the power of the test for this class of distributions depends on $v(\theta_1) - v(\theta_2)$, it will be constant for all θ_1 and θ_2 which lie on the curve defined by $v(\theta_1) - v(\theta_2) =$ constant. In particular, if the sequential test is defined with risks α and β, the probability of accepting π_1 (or π_2) will be approximately α for all hypotheses (θ_1, θ_2) which lie on the curve defined by $v(\theta_1) - v(\theta_2) = v(\theta_1^0) - v(\theta_2^0) = h_0$ and the probability of accepting π_2 (or π_1) will be approximately β for all hypotheses (θ_2, θ_1) which lie on the curve defined by $v(\theta_2) - v(\theta_1) = h_0$. Now, the decision function z as well as the boundaries a^* and b^* will be identical for all sequential tests provided they are defined by the same risks α and β and the parameters θ_1 and θ_2 which determine the decision function all lie on the curve $v(\theta_1) - v(\theta_2) = h_0$. Since Wald [1] has proved that the sequential probability ratio test minimizes $E(n)$, the expected number of observations required to reach a decision, when the hypothesis tested is true as well as when the alternative hypothesis is true, it must follow that in the case under consideration $E(n)$ is minimized for *all* hypotheses (θ_1, θ_2) which lie either on the curve defined by $v(\theta_1) - v(\theta_2) = h_0$ or on the curve defined by $v(\theta_2) - v(\theta_1) = h_0$. If $v(\theta)$ is a monotonic function of θ, then the test is unbiased (i.e. all points (θ_1, θ_2) which lie on the curve $v(\theta_1) - v(\theta_2) =$ constant will have the property that either every $\theta_1 < \theta_2$ or every $\theta_1 > \theta_2$).

For this type of distribution, the importance of the difference between θ_1 and θ_2 may be measured by $v(\theta_1) - v(\theta_2)$. We shall now show that the function $v(\theta_1) - v(\theta_2)$ is an appropriate measure of the difference between these parameters for a wide class of distributions which often occur in practice.

1.9 The proposed test applied to known distributions.

1.9a. *The problem of discriminating between means when the variances are known.* Let $f(x, \mu)$ be a normal distribution function with unknown mean μ and known variance σ^2 which we shall assume, without loss of generality, to be unity. Let x_1 be an observation from π_1 and x_2 an observation from π_2. Let the distribution density of x_1 be designated by $f(x_1, \mu_1)$ and that of x_2 by $f(x_2 \cdot \mu_2)$. The problem is to decide which process has the larger μ.

Since $f(x, \mu)$ is a normal distribution, it is given by

$$(1.901) \qquad f(x, \mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$$

Hence $f(x, \mu)$ is of the form considered in Section 1.8 with $u(x) = x$ and $v(\mu) = \mu$. Therefore, the decision function is given by

$$(1.902) \qquad z^* = x_2 - x_1$$

and the power of the test depends on $h = \mu_1 - \mu_2$ and is given by (1.406) with a and b replaced by a^* and b^*, respectively.

The sequential test is performed in the following manner: We take a pair of observations, one from π_1 and one from π_2, in sequence. If at any stage $\sum_{\alpha=1}^{n} (x_{2\alpha} - x_{1\alpha}) \leq -b^*$, we accept the hypothesis that π_1 has the larger mean. If, however, at any stage $\sum_{\alpha=1}^{m} (x_{2\alpha} - x_{1\alpha}) \geq a^*$, we accept the hypothesis that π_2 has the larger mean. If neither holds, we continue sampling. According to section 1.8 $a^* = \dfrac{\log A}{\mu_1 - \mu_2}$ and $-b^* = \dfrac{\log B}{\mu_1 - \mu_2}$, where $\mu_1 - \mu_2$ is assumed to be positive.

In order to determine a sequential test, we must fix a^* and b^*. That is, we must fix the quantities $\mu_1 - \mu_2$, A, and B. This can be accomplished by deciding, (1) the smallest difference between the means of the two processes which is considered worth detecting. This determines $h_0 = \mu_1^0 - \mu_2^0$, which we shall assume to be positive: (2) the maximum probability α of rejecting the hypothesis that π_1 has the larger mean when in fact μ_1 in π_1 differs from μ_2 in π_2 by as much as h_0, and (3) the maximum probability β of accepting the hypothesis that π_1 has the larger mean when in fact the difference between μ_1 and μ_2 is as large as h_0 negatively.[5] When α and β are fixed, A and B are determined by equations (1.507) and (1.508).

1.9b. *The problem of discriminating between variances when the means are known.* Let us assume that the distribution of x_1 in π_1 and x_2 in π_2 are normal with known means but unknown variances. We are required to choose that process which has the smaller variance. Without any loss of generality we shall suppose that the means of x_1 and x_2 are zero. Since $f(x, \sigma)$ is normal, it is given by

$$\text{(1.903)} \qquad \frac{1}{\sqrt{2\pi}\sigma} e^{-(x^2/2\sigma^2)} = e^{-(x^2/2\sigma^2) - \log \sigma \sqrt{2\pi}}$$

which is of the form considered in Section 1.8 with $u(x) = x^2$ and $V(\sigma) = \dfrac{1}{2\sigma^2}$. Hence the decision function z^* is given by

$$\text{(1.904)} \qquad z^* = x_2^2 - x_1^2$$

and the power of the test depends on $h = \frac{1}{2}(\sigma_2^{-2} - \sigma_1^{-2})$ and is given by (1.406) with a and b replaced by a^* and b^*, respectively. The sequential test is performed in the following manner: We take one pair of observations at a time, one from π_1 and one from π_2. We continue sampling as long as $\sum_{\alpha=1}^{n} (x_{2\alpha}^2 - x_{1\alpha}^2)$ lies between $-b^*$ and a^*. Whenever $\sum_{\alpha=1}^{n}(x_{2\alpha}^2 - x_{1\alpha}^2) \geq a^*$, we conclude that $\sigma_2^2 > \sigma_1^2$.

[5] The power curve defined by (1.406) is a monotonic function of $h = \mu_1 - \mu_2$. Hence the probability of rejecting the hypothesis that π_1 has the larger mean is $\leq \alpha$ whenever $\mu_1 - \mu_2 \geq h_0$. Thus α is in fact the *maximum* risk of making an erroneous decision. A similar statement can be made concerning the risk β.

Whenever $\sum_{\alpha=1}^{n} (x_{2\alpha}^2 - x_{1\alpha}^2) \leq -b^*$, we conclude that $\sigma_2^2 < \sigma_1^2$. The quantities a^* and b^* are defined by

$$a^* = \frac{\log A}{\frac{1}{2}[(\sigma_2^0)^{-2} - (\sigma_1^0)^{-2}}$$

and

$$-b^* = \frac{\log B}{\frac{1}{2}[(\sigma_2^0)^{-2} - (\sigma_1^0)^{-2}]}.$$

Thus a^* and b^* are defined by a specific value of $\sigma_2^{-2} - \sigma_1^{-2}$ and A and B. If we take $(\sigma_2^0)^{-2} - (\sigma_1^0)^{-2}$ as negative, then $A = \dfrac{\beta}{1-\alpha}$ and $B = \dfrac{1-\beta}{\alpha}$ where α = probability of concluding that $\sigma_1^2 < \sigma_2^2$ when in fact $\sigma_2^{-2} - \sigma_1^{-2} = -[(\sigma_2^0)^{-2} - (\sigma_1^0)^{-2}]$ and β is the probability of concluding $\sigma_1^2 < \sigma_2^2$ when in fact $\sigma_2^{-2} - \sigma_1^{-2} = [(\sigma_2^0)^{-2} - (\sigma_1^0)^{-2}]$.

1 9c. *The problem of discriminating between variances when the means are unknown.* Let the measured characteristics in π_1 and π_2 be assumed to be normally distributed with unknown means and unknown variances. We desire to choose, on the basis of a sequential test, that process which has the smaller variance no matter what the means are. This will be accomplished by reducing the problem to that treated in Section 1.9b.

Let $x_{11}, x_{12}, x_{13}, \cdots$ be the successive observations from π_1 and $x_{21}, x_{22}, x_{23}, \cdots$ the successive observations from π_2. Consider the transformation

$$\begin{aligned}
y_{11} &= \frac{1}{\sqrt{2}} x_{11} - \frac{1}{\sqrt{2}} x_{12}, \\
y_{12} &= \frac{1}{\sqrt{2.3}} x_{11} + \frac{1}{\sqrt{2.3}} x_{12} - \frac{2}{\sqrt{2.3}} x_{13}, \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
y_{1(n-1)} &= \frac{1}{\sqrt{n(n-1)}} x_{11} + \frac{1}{\sqrt{n(n-1)}} x_{12} \cdots - \frac{n-1}{\sqrt{n(n-1)}} x_{1n}, \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

with $y_{21}, y_{22}, \cdots y_{2(n-1)} \cdots$ similarly defined in terms of $x_{21}, x_{22}, \cdots x_{2n} \cdots$. It is obvious that this transformation can be applied sequentially. Moreover, it is easy to show that

(1) The expected values of the y's are zero.

(2) The variances of the y's are the same as the variances of the x's.

(3) The y's are normally and independently distributed.

Hence we can apply the sequential test developed in Section 1.9b to the y's without any alterations. The decision function Z_n^* will be given by

$$Z_n^* = \sum_{\alpha=1}^{n} (y_{2\alpha}^2 - y_{1\alpha}^2). \tag{1.905}$$

But it can be easily shown that

$$\sum_{\alpha=1}^{n} y_{2\alpha}^2 = \sum_{\alpha=1}^{n+1} (x_{2\alpha} - \bar{x}_2)^2$$

and

$$\sum_{\alpha=1}^{n} y_{1\alpha}^2 = \sum_{\alpha=1}^{n+1} (x_{1\alpha} - \bar{x}_1)^2$$

where $\bar{x}_1$ and $\bar{x}_2$ are the arithmetic means of the observations in π_1 and π_2 respectively. Hence (1.905) is equivalent to

$$Z_n^* = \sum_{\alpha=1}^{n+1} (x_{2\alpha} - \bar{x}_2)^2 - \sum_{\alpha=1}^{n+1} (x_{1\alpha} - \bar{x}_1)^2 . \tag{1.906}$$

Thus, to perform this sequential test, the population means need not be known. The only difference between the tests considered in 1.9b and 1.9c is that 1.9c requires one additional pair of observations.[6]

1.9d. *The problem of discriminating between means when the variates have a Poisson distribution.* Let the distribution of x_1 in π_1 be given by $\dfrac{e^{-m_1} m_1^{x_1}}{x_1!}$ and the distribution of x_2 in π_2 be given by $\dfrac{e^{-m_2} m_2^{x_2}}{x_2!}$ where x_1 and x_2 each take on the values $0, 1, 2, \cdots$. It is desired to test the hypothesis that the mean in π_1 is smaller than the mean in π_2 against the alternative that the reverse is true. Since the Poisson distribution can be written as

$$f(x, m) = \frac{1}{x!} e^{x \log m - m}, \tag{1.907}$$

it is of the form considered in Section 1.8 with $u(x) = x$ and $v(m) = \log m$. Hence the decision function z^* is given by

$$z^* = x_2 - x_1$$

and the power of the test depends on $h = \log \dfrac{m_1}{m_2}$. The sequential test is performed in the following manner: We take one observation from π_1 and one from π_2 in succession. If at any stage $\sum_{\alpha=1}^{n} (x_{2\alpha} - x_{1\alpha}) \leq -b^*$, we conclude that m_2 is smaller than m_1. If $\sum_{\alpha=1}^{n} (x_{2\alpha} - x_{1\alpha}) \geq a^*$, we conclude that m_1 is smaller than m_2. If neither holds, we take another pair of observations. This process is

[6] The method employed here was discovered independently by Charles Stein and the author as a solution to a different sequential problem

continued until one or the other decision is reached. The quantities a^* and b^* are given by

$$a^* = \frac{\log \frac{\beta}{1-\alpha}}{\log u_0} \tag{1.908}$$

$$b^* = \frac{\log \frac{1-\beta}{\alpha}}{\log u_0} \tag{1.909}$$

where $u_0 = m_1^0/m_2^0$ which is assumed to be less than one, α is the desired probability of concluding that m_2 is smaller than m_1 when in fact $m_1^0/m_2^0 = u_0 < 1$, and β is the probability of concluding that m_1 is smaller than m_2 when in fact $m_1^0/m_2^0 = 1/u_0$. The power curve is given by

$$L_u = \frac{u^{a^*+b^*} - u^{b^*}}{u^{a^*+b^*} - 1}, \tag{1.910}$$

where $u = m_1/m_2$.

1 9c. *Double dichotomies.*[7] We are given two processes π_1 and π_2, one yielding a fraction defective p_1 and the other p_2. We shall assume that p_1 and p_2 are unknown. We desire to choose on the basis of a sample that process which gives the smaller fraction defective. That is, we wish to devise a test which gives a high probability of accepting π_1 if $p_1 < p_2$ and a high probability of accepting π_2 if $p_2 < p_1$. If $p_1 = p_2$, we might be more or less indifferent as to which process we select,

Before we can answer this question, we must decide: (a) the minimum difference between the two processes which we consider worth detecting; and (b) if the two processes differ at least by the amount specified in (a), the minimum probability with which we desire to make the correct decision.

In the proposed test, the decision function is given by $z^* = x_2 - x_1$ where x_i, $(i = 1, 2)$, takes on the values 0 or 1, depending on whether the ith process yields a nondefective or defective item. The difference between the two processes is measured by[8] $u = \frac{p_1}{1-p_1} / \frac{p_2}{1-p_2}$ (the ratio of the odds). It can easily be seen that when $u < 1$, $p_1 < p_2$ and when $u > 1$, $p_1 > p_2$. If $u = 1$, $p_1 = p_2$. Let u_0 represent a quantity less than 1. Furthermore, let α be the probability of accepting π_2 when in fact the point (p_1, p_2) lies on the curve $\frac{p_1q_2}{q_1p_2} = u_0$; and β be the probability of accepting π_1 when in fact the true point (p_1, p_2) lies on the

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[7] For a solution of a more general problem in double dichotomies using a different approach, see [1], section 5 32 and [4] section 3.

[8] This follows from the fact that the binomial distribution can be written as $f(x, p) = e^{x \log(p/q) + \log q}$ where x takes on the values 0 or 1. Hence the distribution is of the form considered in section 1.8 with $v(p) = \log p/q$, $w(p) = \log q$, and $z^* = x_2 - x_1$.

curve $\frac{p_2 q_1}{q_2 p_1} = u_0$. Once u_0, α and β are chosen, we compute

$$(1)\quad a^* = \frac{\log \frac{\beta}{1-\alpha}}{\log u_0}$$

and

$$(2)\quad -b^* = \frac{\log \frac{1-\beta}{\alpha}}{\log u_0}.$$

We then proceed as follows: We take one item from each process in sequence and cumulate the number of defective d_1 in process π_1 and d_2 in process π_2. Whenever $d_2 - d_1 \leq -b^*$, we choose process π_2. Whenever $d_2 - d_1 \geq a^*$, we choose process π_1. Whenever $d_2 - d_1$ lies between a^* and $-b^*$, we take another pair of observations, one from each process. This procedure is continued until one or the other decision is reached.

1.9e1. *The exact value of the power function for double dichotomies.* Since $d_2 - d_1$ changes at most in steps of one unit, it must follow that whenever a decision is reached at a^*, the difference between a^* and $d_2 - d_1$ is either zero (if a^* is an integer), or the difference between a^* and $d_2 - d_1$ is constant for all values of n. A similar argument holds for b^*. This permits us to compute the power function without any approximations. Let $\bar{a}$ be the next positive integer larger than a^* if a^* is not an integer, and $\bar{a} = a^*$ if a^* is an integer. Let $\bar{b}$ be the next positive integer larger than b^* if b^* is not an integer, and $\bar{b} = b^*$ if b^* is an integer. Then we see that the equation (1.406) for the power curve can be given without any approximations by the formula

$$(1.9101)\qquad L_u = (u^{\bar{a}+\bar{b}} - u^{\bar{b}})/(u^{\bar{a}+\bar{b}} - 1)$$

1.9e2. *The exact average sample number for double dichotomies.* Let $Z_n = d_2 - d_1$ and let the point (p_1, p_2) be on some curve $\frac{p_1 q_2}{p_2 q_1} = u$. Let $E(n \mid p_1, p_2)$ be the expected number of pairs of observations required before a decision is reached. Let L_u = probability of reaching $-\bar{b}$ (i.e., L_u is the probability that π_2 is accepted). Then $1 - L_u$ is the probability of reaching $\bar{a}$ (i.e., $1 - L_u$ is the probability that π_1 is accepted). Then by Wald's Fundamental Identity we have[9]

$$(1.911)\qquad EZ_n = EzE(n \mid p_1, p_2).$$

Now, $Ez = p_2 - p_1$, and $EZ_n = -L_u\bar{b} + (1 - L_u)\bar{a}$. Hence

$$(1.912)\qquad E(n \mid p_1, p_2) = \frac{L_u(\bar{a} + \bar{b}) - \bar{a}}{p_2 - p_1}.$$

[9] For a derivation of formula (1.911) which does not depend on the Fundamental Identity, see Wald [1], page 142.

It will be noted that while L_u depends only on $u = \frac{p_1 q_2}{p_2 q_1}$, $E(n \mid p_1, p_2)$ depends not only on the ratio of the odds but also on the difference between the two fraction defectives.

1 9e3. *The distribution of n for double dichotomies.* In this section we shall be concerned with the probability of reaching a decision with exactly n pairs of observations.

Let a and b be two positive integers and let the sequential test be defined by the decision function $Z_n^* = \sum_{\alpha=1}^{n} z_\alpha$ where z_α takes on the values -1, 0, and 1 with probabilities P_1, P_2, and P_3, respectively. In terms of double dichotomies, $Z_n^* = d_2 - d_1$ where d_2 and d_1 are the cumulative number of defectives obtained sequentially from π_1 and π_2, respectively, and $P_1 = p_1 q_2$, $P_2 = p_1 p_2 + q_1 q_2$, $P_3 = p_2 q_1$, where p_1 is the fraction defective yielded by π_1 and p_2 the fraction defective yielded by π_2.

By the Fundamental Identity we have for any t in the complex plane for which $|\phi(t)| \geq 1$,

$$L_u e^{-tb} E_1[\phi(t)]^{-n} + (1 - L_u) e^{ta} E_2[\phi(t)]^{-n} = 1 \tag{1.913}$$

where L_u is the probability that $Z_n^* = -b$ when p_1 and p_2 are such that $\frac{P_1}{P_3} = u$, E_1 and E_2 are the appropriate conditional expectations, and

$$\phi(t) = P_1 e^{-t} + P_2 + P_3 e^{t}. \tag{1.914}$$

If we examine Wald's proof of Lemma II [2], we see that $\phi(t) \geq 1$ for all real values of t which lie outside the open interval $(0, h)$ where h is the root of the equation $\phi(t) = 1$. Hence, it must follow that the Fundamental Identity (1 913) must also hold for all real values of t with the possible exception of the open interval $(0, h)$. This fact will be used in the subsequent discussion.

We shall first obtain the distribution of n when $a = \infty$. From equation (1.910) we see that when a approaches ∞, L_u approaches 1 for $u \geq 1$ and u^b for $u < 1$. We shall assume that $u \geq 1$. Then for t negative and $a = \infty$, the Fundamental Identity (1.913) becomes

$$e^{-tb} E[\phi(t)]^{-n} = 1 \tag{1.915}$$

or

$$E[\phi(t)]^{-n} = e^{tb}. \tag{1.916}$$

Now for all $u > 1$, $P_1 > P_3$, and hence $Ez = P_3 - P_1$ is negative. Since the real roots of $\phi(t) = 1$ are opposite in sign to Ez, it must follow that (1.916) holds for all t in the interval $(-\infty, 0)$. Now set $e^t = x$. Then (1.916) can be written as

$$E\left(P_1 \frac{1}{x} + P_2 + P_3 x\right)^{-n} = x^b \tag{1.917}$$

and (1.917) is valid for all x in the interval $0 \leq x \leq 1$.

Now set

$$P_1 \frac{1}{x} + P_2 + P_3 x = \frac{1}{\tau}. \tag{1.918}$$

Then for any specified value of τ there will be two values of x, say $x_1(\tau)$ and $x_2(\tau)$. As τ approaches 0, one of these values of x will approach zero and the other infinity. Let $x_1(\tau)$ be the value of x in (1.918) which approaches zero as τ approaches zero. Substituting (1.918) in (1.917) we get

$$E\tau^n = [x(\tau)]^b. \tag{1.919}$$

But $E\tau^n$ is the generating function of n. Hence if we could expand $E\tau^n$ as a power series in τ, then the probability $Z_n^* = -b$ in exactly n steps would be given by the coefficient of τ^n. We are thus led to consider the expansion of $[x(\tau)]^b$ in a power series in τ.

We multiply (1.918) by τx and get

$$x = \tau(P_3 x^2 + P_2 x + P_1). \tag{1.920}$$

Then since $x_1(\tau)$ approaches 0 as τ approaches 0, we can expand $[x_1(\tau)]^b$ by Lagrange formula,[10] and get

$$[x_1(\tau)]^b = \sum \frac{b^m}{m!} \frac{d^{m-1}}{d\xi^{n-1}} [\xi^{b-1}(P_1 + P_2\xi + P_3\xi^2)^m]_{\xi=0} \tag{1.921}$$

where the expansion is valid for $x_1(\tau)$ sufficiently close to zero. Hence, if $P_n(b)$ is the probability that exactly n pairs of observations are required to reach a decision, then

$$P_n(b) = \frac{b}{n!} \frac{d^{n-1}}{d\xi^{n-1}} [\xi^{b-1}(P_1 + P_2\xi + P_3\xi^2)^n]_{\xi=0}. \tag{1.922}$$

Now

$$\begin{aligned} &\frac{d^{n-1}}{d\xi^{n-1}} [\xi^{b-1}(P_1 + P_2\xi + P_3\xi^2)^n]_{\xi=0} \\ &= \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} P_3^i \sum_{j=0}^{n-1} \frac{(n-i)!}{j!(n-i-j)!} P_1^j P_2^{n-i-j} \frac{d^{n-1}}{d\xi^{n-1}} \xi^{n+i-j+b-1}]_{\xi=0}. \end{aligned} \tag{1.923}$$

But

$$\frac{d^{n-1}}{d\xi^{n-1}} \xi^{n+i-j+b-1}\Big]_{\xi=0} = 0 \tag{1.924}$$

unless $n = n + i - j + b$, i.e., $j = i + b$, in which case

$$\frac{d^{n-1}}{d\xi^{n-1}} \xi^{n+i-j+b-1}\Big|_{\xi=0} = (n-1)! \tag{1.925}$$

[10] See, for example, *Mathematical Analysis*, Vol. 1 (paragraph 189), by Goursat-Hedrick.

Also, since the subscript j ranges from 0 to $n - i$, it must follow that $j \leq n - i$. Hence, $i + b \leq n - i$, or $i \leq \frac{n - b}{2}$. Substituting (1.924) and (1.925) into (1.923) and simplifying, we get for $P_n(b)$

$$P_n(b) = b \sum_{i=0}^{m} \frac{(n-1)!\, P_1^{i+b} P_2^{n-2i-b} P_3^{i}}{i!(i+b)!\,(n-2i-b)!} \tag{1.926}$$

where $m = \frac{n-b}{2}$ when $n - b$ is even and $m = \frac{n-b-1}{2}$ when $n - b$ is odd.

We shall now obtain the distribution of n when a is finite.

As before, let $x_1(\tau)$ and $x_2(\tau)$ be the roots of the equation (1.918). Then from (1.913) we have

$$L_u[x_1(\tau)]^{-b}E_1\tau^n + (1 - L_u)\,[x_1(\tau)]^a E_2\tau^n = 1, \tag{1.927}$$

$$L_u[x_2(\tau)]^{-b}E_1\tau^n + (1 - L_u)\,[x_2(\tau)]^a E_2\tau^n = 1. \tag{1.928}$$

Solving for $E_1\tau^n$ and $E_2\tau^n$ from (1.927) and (1.928) we get

$$L_u\, E_1\,\tau^n = \frac{[x_1(\tau)x_2(\tau)]^b[x_2(\tau)^a - x_1(\tau)^a]}{x_2(\tau)^{a+b} - x_1(\tau)^{a+b}} \tag{1.929}$$

$$(1 - L_u)E_2\,\tau^n = \frac{x_2(\tau)^b - x_1(\tau)^b}{x_2(\tau)^{a+b} - x_1(\tau)^{a+b}}. \tag{1.930}$$

We shall first obtain the probability $Q_n(b)$ that $Z_n^* = -b$. This is given by the coefficient of τ^n in the expansion of $L_uE_1\tau^n$ in a power series in τ. From (1.918) we see that $x_1(\tau)x_2(\tau) = \frac{P_1}{P_3}$. Hence we can write (1.929) as

$$L_u\, E_1\,\tau^n = \frac{x_1(\tau)^b - \left(\frac{P_3}{P_1}\right)^a x_1(\tau)^{b+2a}}{1 - \left(\frac{P_3}{P_1}\right)^{b+a} x_1(\tau)^{2b+2a}}. \tag{1.931}$$

Applying Lagrange formula, we get for $Q_n(b)$

$$Q_n(b) = \frac{1}{n!}\frac{d^{n-1}}{d\xi^{n-1}}\,[(P_1 + P_2\,\xi + P_3\,\xi^2)^n f'(\xi)]_{\xi=0} \tag{1.932}$$

where

$$f(\xi) = \frac{\xi^b - \left(\frac{P_3}{P_1}\right)^a \xi^{b+2a}}{1 - \left(\frac{P_3}{P_1}\right)^{b+a} \xi^{2b+2a}}. \tag{1.933}$$

But $f(\xi)$ can be expanded in a power series in ξ,

$$(1.934) \qquad f(\xi) = \sum_{k=0}^{\infty} \left(\frac{P_3}{P_1}\right)^{kb+ka} \xi^{(2k+1)b+2ka} - \left(\frac{P_3}{P_1}\right)^{kb+(k+1)a} \xi^{(2k+1)b+(2k+2)a}$$

Hence

$$(1.935) \qquad \begin{aligned} Q_n(b) = {} & \frac{1}{n!} \sum_{k=0}^{\infty} [(2k+1)b + 2ka] \left(\frac{P_3}{P_1}\right)^{kb+ka} \\ & \cdot \frac{d^{n-1}}{d\xi^{n-1}} [\xi^{(2k+1)b+2ka-1} (P_1 + P_2\xi + P_3\xi^2)^n]_{\xi=0} \\ & - \frac{1}{n!} \sum_{k=0}^{\infty} [(2k+1)b + (2k+2)a] \left(\frac{P_3}{P_1}\right)^{kb+(k+1)a} \\ & \cdot \frac{d^{n-1}}{d\xi^{n-1}} [\xi^{(2k+1)b+(2k+2)a-1} (P_1 + P_2\xi + P_3\xi^2)^n]_{\xi=0} . \end{aligned}$$

Comparing (1.935) with (1.922) we see that

$$(1.936) \qquad Q_n(b) = P_n(b) - \left(\frac{P_3}{P_1}\right)^{a} P_n(b+2a) + \left(\frac{P_3}{P_1}\right)^{b+a} P_n(3b+2a) - \cdots ,$$

the terms in the series being alternately of the form

$$\left(\frac{P_3}{P_1}\right)^{kb+ka} P_n[(2k+1)b + 2ka] \quad \text{and}$$

$$- \left(\frac{P_3}{P_1}\right)^{kb+(k+1)b} P_n[(2k+1)b + (2k+2)a], \quad \textit{for } k = 0, 1, \cdots .$$

The series stops by itself as soon as the argument of P_n becomes greater than n

If we compare (1.930) with (1.929), we see that the probability that $Z_n^* = a$ with exactly n pairs of observations is given by (1.936) with a and b interchanged and the result multiplied by $(P_3/P_1)^a$.

It is to be noted that the problem of double dichotomies is similar to the following problem in games of chance. Two players A and B, possessing a and b dollars, respectively, are playing a game of chance which admits a draw. The stake is one dollar per game. The probability that A will win one dollar is P_1, the probability that B will win one dollar is P_3 and the probability of a draw is P_2. In terms of this game, L_u given by (1.910) is the probability that B will be ruined in the long run, and $Q_n(b)$ in (1.936) is the probability that B will be ruined in exactly n games.

For a discussion of games of chance which do not permit a draw, see *Introduction to Mathematical Probability*, Chapter VIII, by J. V. Uspensky. The development presented above is in some respects similar to that given in Uspensky's book In Part II, we shall give a different and more general approach to the problem of deriving the distribution of n for sequential tests in which the variate takes on a finite number of integral values.

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AN APPROACH FOR QUANTIFYING PAIRED COMPARISONS AND RANK ORDER[1]

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1. Summary. Research for the Army demobilization point system evolved a new approach to paired comparisons and rank order. Each of N individuals compares or ranks n things, the problem is to determine a numerical value for each of the n things that will best represent the comparisons in some sense. The new criterion adopted is that the numerical values be determined so as best to distinguish between those things judged higher and those judged lower *for each individual.* Least-squares is employed in the analysis, and the solution appears in the form of the latent vector associated with the largest root of a matrix obtained from the comparisons or rankings.

This approach applies to the conventional problem of ordinary comparisons, the numerical solution being easily obtainable by simple iterations; the conventional use of hypothetical variables and unverified hypotheses is avoided. The Army point system is an example of a new and more complicated class of problems; the same principle for the solution applies here, only more details occur in the derivations and computations.

2. Introduction. The problem of paired comparisons arises when it is desired to obtain numerical values for a set of n things, with respect to one characteristic, such that these values will represent the judgments of a population of N individuals.

One procedure for obtaining the judgments is to have the individuals compare the things two at a time and to judge for each comparison which of the two things should be given the higher rank. An alternative procedure is to have each individual rank all the n things simultaneously. Such a ranking implies judging all the $n(n - 1)/2$ comparisons at once; hence, the two procedures are substantially equivalent. Two noteworthy differences between the procedures are: (a) comparing two things at a time allows inconsistencies to appear within judgments of an individual, and (b) it is sometimes harder in practice for people to judge n things simultaneously than to compare them two at a time.

The problem of quantification, of course, is identical for both procedures, so we do not distinguish between them in this paper. The judgments vary from person to person (and possibly within a person), and the problem is to determine a set of numerical values for the things being compared that will in some sense best represent or average the judgments of the whole population.

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[1] Adapted from Report D-3, "An approach for quantifying paired comparisons," Research Branch, Information and Education Division, Headquarters Army Service Forces, Washington, D C , 1945.

In some situations, the things being compared may be single items or objects, this we shall call the case of *ordinary* comparisons. In other situations, the things may be *combinations* of items or objects.

This paper is devoted to the presentation of a general approach to quantifying comparisons or rank orders, with particular application to ordinary comparisons and to the comparison of combinations of two things It seems to differ from previous approaches in at least two important respects: (a) it is based on but one simple principle, namely, that the quantification shall be the one best able to *reproduce the judgment of each person in the population on each comparison;* and, as a consequence, (b) the approach yields solutions not only to the traditional case of ordinary comparisons, but also to more complex cases that do not seem to have been discussed previously.

An example of a major practical use of this approach is with respect to the demobilization score card of the United States Army. The problem was to determine the number of points to assign each of the variables on the score card according to the opinions of the soldiers themselves. The research on this was based on a form of paired comparisons more complicated than the ordinary one, and had additional complications of curvilinearities of various sorts in the data. Our approach handles such problems as well as the problem of ordinary comparisons.

Let us describe the score card problem in somewhat more detail. In a survey of enlisted men throughout the world by means of a questionnaire administered by field teams of the Research Branch, it was found that there were five variables that the men thought should receive consideration on the score card to determine order of demobilization: length of time in the Army, length of time overseas, amount of combat, age, and number of children.

The problem now was to determine how much weight to give each of these variables in obtaining total scores. According to ordinary paired comparisons, one would ask, for example, "Who should get out first after the war: a man who has two children or a man who has been in two battles?" But respondents refuse to judge such a comparison because the battle experience of the first man is not specified, nor is the number of progeny of the second man, so that there is insufficient basis for judgment.

Therefore, in the actual research, judgments were asked on each of ten comparisons put in the following form:

"Here are three men of the same age, all overseas the same length of time. Check the one you would want to have let out first:

—— A single man . . . . through two campaigns of combat

—— A married man with no children . . . . through one campaign of combat

—— A married man with two children . . . not in combat."

Each variable was compared with every other one in this fashion.

The equations were derived for computing the relative number of points to assign to each month in the army, each month overseas, etc., which would be most consistent according to our principle. These are essentially the equations developed in section 6 of this paper.

The results showed strong curvilinearities in the men's judgments. Amount of combat received one amount of emphasis when compared with age, and another amount of emphasis when compared with number of children. Since the score card would be too complicated in practice if curvilinear scoring were used, equations were derived for the *linear* scoring scheme that would be most consistent according to our principle. These are essentially the equations derived in section 7. The weights arising out of the research were computed from such equations.

The variable age received a slight negative weight, which justified dropping it from the score card. The weights the Army finally adopted for the remaining factors were modified from the research weights, but yield essentially the same results as the research weights. Demobilization scores obtained from the one system of weights correlate very highly with scores obtained from the other.

It can now be revealed that the Army's modification was essentially to reverse the weights for children and battles. In subsequent attitude surveys on how well the soldiers liked the point system [8], a major complaint was found to be that battles got too little weight compared with babies!

3. The basic principle. Our basic principle in deriving numerical values—let us call them "x-values"—for the things being compared requires that the x-values of things a given person judges higher than other things should be as different as possible from the x-values of the things he judges to be lower than other things. This will be achieved if we make the x-values of things judged higher as homogeneous as possible among themselves, and the x-values of things judged lower as homogeneous as possible among themselves, for each individual. In the language of analysis of variance, our principle calls for *minimizing the variation within individuals*, compared with that within the group as a whole.[2] The resulting x-values will tend to be the best for reproducing the judgment of each individual on each comparison with a minimum overall proportion of errors of reproduction [3, pp. 342–343]. The smaller this overall proportion of error, the better the quantification represents the data. Least squares is used for convenience for measuring variation in deriving the equations.

The previous literature, on ordinary paired comparisons,[3] seems to have concentrated largely on the problem of estimating the differences between means of hypothetical variables assumed to underlie the judgments. Thurstone has shown that by using assumptions of normality of distribution, equality of variances, and zero correlations among hypothetical variables, it is possible to estimate relative distances between means for some kinds of data.

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[2] This principle for quantification was suggested by previous work on scale analysis; see [3] This theory has been developed further by the definition of a perfect scale in [4] The equations for the perfect scale have interesting properties that may be related to paired comparisons, these equations are being prepared for publication. The referees have called my attention to related work on quantification by R A. Fisher in [1, p. 283].

[3] A good survey of the previous work, including that of Thurstone, is given in [2, pp. 217–243]. For more recent work, see [7]

The problem of estimating differences between means is not identical with that of reproducing individual judgments. For example, it can be shown, within the same framework of hypothetical variables conventionally used, that if variances are unequal and/or correlations are unequal then the means of the hypothetical variables are *not* in general the best quantification for reproducing individual judgments; the principal axis of certain product-moments of raw scores is the best quantification. It is in the special case where variances are equal, and where correlations are equal—not even necessarily equal to zero—that the principal axis *is* the set of means. Proof of this is given in the appendix.

The approach of this paper does not use hypothetical variables, but inquires directly as to what numerical values can be derived from the observations that will best reproduce those observations.

In the next section is treated the case of ordinary comparisons. The more complicated problem of the demobilization score card is formalized in section 5, and the equations for its unrestricted solution are derived in section 6. Since the unrestricted solution brings out curvilinearities that may be present, and since the score card in practice required a linear scoring scheme, equations for the most consistent linear quantification are derived in section 7. These are essentially the equations used in the research on the weights for the score card.

The appendix shows a distinction between the conventional principle of estimating mean differences of hypothetical variables and the present principle of representing the comparisons of each individual.

4. The case of ordinary comparisons. Paired comparisons as treated in the literature seem concerned largely with the ordinary case where separate things are compared, rather than where combinations of things are compared. Our principle covers the ordinary case as well as more complex cases, and we shall treat the ordinary case first since it involves less details.

Let $O_1, O_2, \cdots, O_n$ be the n things to be compared, where the assigning of subscripts is arbitrary. Each of N individuals is asked to make judgments of the form that O_j is higher than (or lower than) O_k. For convenience, we assume the rules of the experiment to exclude judgments of equality. We shall also assume that all people compare all the pairs. Hence, there are N sets of $n(n-1)/2$ comparisons. Considering each comparison as comprising two judgments—one of "higher than" for one object and one of "lower than" for the other—there is a total of $Nn(n-1)$ judgments in the experiment.

The judgments of all the individuals on all the comparisons can be represented compactly as follows. Let

$$(4.1)\qquad e_{ijk} \equiv \begin{cases} 1 \text{ if individual } i \text{ judges } O_j > O_k \\ 0 \text{ if individual } i \text{ judges } O_j < O_k \\ 0 \; j = k. \end{cases}$$

The ranges of subscripts, whether free or dummy, will always be:

$$i = 1, 2, \cdots, N \tag{4.2}$$
$$j, k = 1, 2, \cdots, n,$$

so that the ranges will not be explicitly stated again.

Definition (4.1) implies that if $e_{ijk} = 1$, then $e_{ikj} = 0$, and that

$$e_{ijk} + e_{ikj} \doteq 1, \qquad (j \neq k). \tag{4.3}$$

Let f_{ij} be the number of things individual i judged to be lower than O_j, and let g_{ij} be the number of things he judged to be higher than O_j. Then

$$f_{ij} \equiv \sum_k e_{ijk}, \qquad g_{ij} \equiv \sum_k e_{ikj}. \tag{4.4}$$

From (4.3) and (4.4), we have

$$f_{ij} + g_{ij} \equiv n - 1. \tag{4.5}$$

Let F be the total number of comparisons made by each person; then

$$F = n(n-1)/2 \equiv \sum_k f_{ik} \equiv \sum_k g_{ik}. \tag{4.6}$$

Let c be the number of times each O_j was judged in the whole experiment, and let C be the total number of judgments in the experiment:

$$c = N(n-1) \equiv \sum_i (f_{ij} + g_{ij}), \quad C = Nn(n-1). \tag{4.7}$$

Both c and C count each comparison as two judgments, one of "lower than" and one of "higher than."

The means and variances to be considered are defined as follows. Let x_j be the numerical value to be derived for O_j on the basis of the comparisons. Let t_i be the mean of the x-values of the things individual i ranked *higher* than the other things, weighted by the respective frequencies of the judgments, and et y_i be the sum of squares of deviations from their mean of these x-values:

$$t_i \equiv \frac{1}{F} \sum_k x_k f_{ik}, \tag{4.8}$$

$$y_i \equiv \sum_k (x_k - t_i)^2 f_{ik} \equiv \sum_k x_k^2 f_{ik} - t_i^2 F. \tag{4.9}$$

Similarly, let u_i and z_i be the mean and sum of squares respectively for the x-values of the things individual i ranked *lower* than other things:

$$u_i \equiv \frac{1}{F} \sum_k x_k g_{ik}. \tag{4.10}$$

$$z_i \equiv \sum_k (x_k - u_i)^2 g_{ik} \equiv \sum_k x_k^2 g_{ik} - u_i^2 F. \tag{4.11}$$

Let V be the mean of all the x-values in the experiment, and let W be the sum of squares of deviation from their mean of the x-values·

$$V = \frac{1}{C} \sum_k x_k c \cdot = \frac{1}{n} \sum_k x_k, \tag{4.12}$$

$$W = \sum_k (x_k - V)^2 c = c \sum_k x_k^2 - V^2 C. \tag{4.13}$$

W is the total sum of squares for the experiment. Let R be the sum of squares *between* individuals, and let S be the sum of squares *within* individuals:

$$(4.14) \quad R = \sum_i [(t_i - V)^2 + (u_i - V)^2]F = F\sum_i (t_i^2 + u_i^2) - V^2 C,$$

$$(4.15) \quad S = \sum_i (y_i + z_i) = W - R.$$

Our principle is to quantify the judgments by obtaining the x-values that will *minimize the variation within individuals* compared to that of the group as a whole. This means making S as small as possible compared with W, which is equivalent to making R as large as possible compared with W.

Therefore, if we define the correlation ratio E by

$$(4.16) \quad E^2 = 1 - S/W,$$

the problem is to determine the x_j that will maximize E^2.

A convenient formula for E^2 is, from (4.15) and (4.16),

$$(4.17) \quad E^2 = R/W.$$

Since E^2 is invariant with respect to translations of the x-values, we can without loss of generality set

$$(4.18) \quad V = 0.$$

Then we can write from (4.14) and (4.13), respectively,

$$(4.19) \quad R = F\sum_i (t_i^2 + u_i^2)$$

$$(4.20) \quad W = c\sum_k x_k^2 .$$

To find the maximizing values x_j for E^2, we differentiate the right member of (4.17) with respect to the x_j, set the derivatives equal to zero, and obtain the stationary equations

$$(4.21) \quad \frac{\partial R}{\partial x_j} = E^2 \frac{\partial W}{\partial x_j} .$$

The derivatives of R can be evaluated by differentiating the right member of (4.19) with the aid of (4.8):

$$(4.22) \quad \frac{\partial R}{\partial x_j} = \frac{2}{F} \sum_k x_k \sum_i (f_{ij} f_{ik} + g_{ij} g_{ik}).$$

From (4.20), the derivatives of W are

$$(4.23) \quad \frac{\partial W}{\partial x_j} = 2cx_j .$$

If we let

$$(4.24) \quad H_{jk} \equiv \frac{1}{cF} \sum_i (f_{ij} f_{ik} + g_{ij} g_{ik}),$$

then (4.21) can be re-written from (4.22), (4.23), and (4.24) as:

$$\sum_k x_k H_{jk} = E^2 x_j \,. \tag{4.25}$$

Equations (4.25) are the equations to be solved numerically for the maximizing x_j.

Before indicating a procedure for the numerical solution, let us first verify that a solution of (4.25) will satisfy (4.18). Summing both members of (4.25) over j, and using (4.24) and relations among the notation previously defined, we get

$$\sum_k x_k = E^2 \sum_j x_j \,,$$

or, from (4.12),

$$(1 - E^2)\, V = 0. \tag{4.26}$$

Therefore, if $E^2 \neq 1$, we must have $V = 0$. Since a perfect correlation ratio will not in general occur in practice, condition (4.18) will in general be satisfied by a solution of (4.25).

There is always a trivial solution of (4.25) for which E^2 is formally equal to unity. This is $x_j \equiv 1$. For this trivial solution, $t_i \equiv u_i \equiv 1$; $R = W = C$; $E^2 = 1$; and (4.25) is satisfied. Of course, E is not an actual correlation ratio for this trivial solution.

The non-trivial solution of (4.25) can be carried out with the aid of matrix algebra. Let x be a row vector of the n elements x_j, and let H be the $n \times n$ symmetric matrix $\| H_{jk} \|$. H is not only symmetric but Gramian, since its elements are product sums. Now (4.25) becomes the matric equation

$$xH = E^2 x \,. \tag{4.27}$$

Equation (4.27) shows that x is a latent vector of H, and E^2 is a latent root to which this vector corresponds. Since we want the largest possible correlation ratio, we seek the largest of the non-trivial roots. If the two largest non-trivial roots are not equal, which should be the general case in practice, then there is a unique vector associated with the largest root which is the solution to our problem.

The numerical solution of (4.27) can be carried out by the simple iterative technique for latent roots and vectors (see, for example [6]). The iterations converge in general to the vector associated with the largest root. To avoid convergence to the trivial solution (which formally has the largest root), the trial vectors should be adjusted to satisfy (4.18), then they will converge in general to the vector associated with the largest non-trivial root.

A good way to choose a first trial vector is first to guess what the rank order of the x-values will be. Let r_j be the guessed rank of x_j, the r_j comprising the integers from one to n. If n is odd, then as the first trial x_j use $r_j - (n + 1)/2$. If n is even, then as the first trial x_j use $2\, r_j - n - 1$.

A marginal check on the internal consistency of the judgments of the population is to compare each difference $(x_j - x_k)$ with the corresponding difference $(\sum_i e_{ijk} - \sum_i e_{ikj})$. If the population's judgments are sufficiently consistent, the signs of the two differences will be alike for all the comparisons. $\sum_i e_{ijk}$ is the frequency with which O_j is judged greater then O_k, and can be used as a basis for guessing the ranks of x_j and x_k.

5. Comparing combinations of two things. The problem of the score card is but one example of a class of problems that can be formalized as follows. Consider a set of n items, where the jth item has m_j categories. Let O_{jp} be the pth category of the jth item, $(p = 1, 2, \cdots, m_j; j = 1, 2, \cdots, n)$. The O_{jp} may be either qualitative or quantitative, and the order of subscripts assigned the categories can be arbitrary.

Each of N individuals is asked to make judgments of the form that the combination (O_{jp}, O_{kr}) is greater than (or less than) the combination (O_{jq}, O_{ks}). We shall assume that all people compare each of the pairs of combinations, and that the rules of the experiment exclude judgments of equality.

The judgments of all the individuals on all the comparisons can be represented compactly as follows. Let

$$e_{ijk/pr,qs} \equiv \begin{cases} 1 \text{ if individual } i \text{ judges } (O_{jp}, O_{kr}) > (O_{jq}, O_{ks}) \\ 0 \text{ otherwise.} \end{cases} \tag{5.1}$$

Here and throughout this paper the ranges of subscripts, whether free or dummy, will always be as follows:

$$\begin{aligned} i &= 1, 2, \cdots, N \\ j, k &= 1, 2, \cdots, n \\ p, q, r, s &= 1, 2, \cdots, m_j, \text{ (or } m_k \text{, as the case may be),} \end{aligned} \tag{5.2}$$

so that the ranges will not be explicitly stated again.

Definition (5.1) implies the symmetry

$$e_{ijk/pr,qs} \equiv e_{ikj/rp,sq}, \tag{5.3}$$

and that

$$e_{ijk/pr,qs} + e_{ijk/qs,pr} \equiv \begin{cases} 0 \text{ if individual } i \text{ omits the comparison of } (O_{jp}, \\ \quad O_{kr}) \text{ with } (O_{jq}, O_{ks}) \\ 1 \text{ if he judges these two combinations to be} \\ \quad \text{unequal.} \end{cases} \tag{5.4}$$

Additional notation is defined as follows. Let $a_{ijk/pr}$ be the number of combinations individual i judged to be lower than (O_{jp}, O_{kr}), and let $b_{ijk/pr}$ be the number of combinations he judged to be higher than (O_{jp}, O_{kr}):

$$(5.5) \qquad a_{ijk/pr} \equiv \sum_q \sum_s e_{ijk/pr,qs} \equiv a_{ikj/rp}$$

$$(5.6) \qquad b_{ijk/pr} \equiv \sum_q \sum_s e_{ijk/qs,pr} \equiv b_{ikj/rp} .$$

Let $c_{jk/pr}$ be the number of comparisons for all individuals involving (O_{jp}, O_{kr}):

$$(5.7) \qquad c_{jk/pr} \equiv \sum_i (a_{ijk/pr} + b_{ijk/pr}) \equiv c_{kj/rp} .$$

Let f_{ijp} be the number of times that O_{jp} occurred in combinations that were judged to be higher than other combinations by individual i, and let g_{ijp} be the number of times O_{jp} occurred in combinations judged lower than others:

$$(5.8) \qquad f_{ijp} \equiv \sum_k \sum_r a_{ijk/pr} \equiv \sum_k \sum_r a_{ikj/rp} ,$$

$$(5.9) \qquad g_{ijp} \equiv \sum_k \sum_r b_{ijk/pr} \equiv \sum_k \sum_r b_{ikj/rp} .$$

Let A_{jp} be the total number of times in the entire experiment that O_{jp} was judged:

$$(5.10) \qquad A_{jp} \equiv \sum_i (f_{ijp} + g_{ijp}) \equiv \sum_k \sum_r c_{jk/pr}$$

Let F be the total number of comparisons made by each person, and let C be the total number of judgments in the entire experiment (a comparison comprises two judgments, one of "higher than" and one of "lower than"):

$$(5.11) \qquad F \equiv \sum_j \sum_p f_{ijp} \equiv \sum_j \sum_p g_{ijp} ,$$

$$(5.12) \qquad C = \sum_j \sum_p A_{jp} = 2NF.$$

The means and variances required for the problem are defined as follows. Let x_{jp} be the numerical value to be derived for O_{jp} from the judgments. Let t_i be the mean of the x-values of the combinations individual i judged to be *higher* than other combinations, weighted by the respective frequencies of such judgments, and let u_i be the analogous mean of combinations judged lower than others:

$$(5.13) \qquad t_i \equiv \frac{1}{F}\sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})\, a_{ijk/pr} \equiv \frac{2}{F}\sum_k \sum_r x_{kr} f_{ikr} ,$$

$$(5.14) \qquad u_i \equiv \frac{1}{F}\sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})\, b_{ijk/pr} \equiv \frac{2}{F}\sum_k \sum_r x_{kr}\, g_{ikr} .$$

Let y_i be the sum of squares of deviations from their mean of these "higher than" x-values, and let z_i be the analogous sum of squares for the "lower than" x-values:

$$
\begin{aligned}
y_i &= \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr} - t_i)^2 a_{ijk/pr} \\
&= \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 a_{ijk/pr} - t_i^2 F,
\end{aligned} \tag{5.15}
$$

$$
\begin{aligned}
z_i &= \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr} - u_i)^2 b_{ijk/pr} \\
&= \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 b_{ijk/pr} - u_i^2 F.
\end{aligned} \tag{5.16}
$$

Let V be the mean of all x-values, weighted by their respective frequencies in the entire experiment, and let W be the sum of squares of deviations from their mean of these x-values:

$$
V = \frac{1}{C} \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr}) c_{jk/pr} = \frac{2}{C} \sum_k \sum_r x_{kr} A_{kr}, \tag{5.17}
$$

$$
\begin{aligned}
W &= \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr} - V)^2 c_{jk/pr} \\
&= \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 c_{jk/pr} - V^2 C.
\end{aligned} \tag{5.18}
$$

W is the total sum of squares for the experiment. Let R be the sum of squares *between individuals* for the experiment, and let S be the sum of squares *within individuals*:

$$
R = \sum_i [(t_i - V)^2 + (u_i - V)^2]F = F \sum_i (t_i^2 + u_i^2) - V^2 C, \tag{5.19}
$$

$$
S = \sum_i (y_i + z_i) = W - R. \tag{5.20}
$$

Our principle for quantifying the judgments is to derive the x-values that will *minimize the variation within individuals* compared with that within the group as a whole. This means making S as small as possible compared with W.

Therefore, if we define the correlation ratio E by

$$
E^2 = 1 - S/W, \tag{5.21}
$$

our problem is to determine the x_{jp} that will maximize E^2.

A convenient formula for E^2 is, from (5 20) and (5.21),

$$
E^2 = R/W. \tag{5.22}
$$

Since E^2 is invariant with respect to translations of the x-values, we can without loss of generality set

$$
V = 0. \tag{5.23}
$$

Then we can write, from (5.19) and (5.18) respectively,

$$
R = F \sum_i (t_i^2 + u_i^2) \tag{5.24}
$$

$$
W = \sum_j \sum_k \sum_p \sum_r (x_{jp} + x_{kr})^2 c_{jk/pr}. \tag{5.25}
$$

6. The unrestricted maximum. To find the maximizing x-values for E^2, we differentiate the right member of (5.22) with respect to the x_{jp} and set the derivatives equal to zero This yields the stationary equations

$$\frac{\partial R}{\partial x_{jp}} = E^2 \frac{\partial W}{\partial x_{jp}}. \tag{6.1}$$

To evaluate the partial derivatives of R, we differentiate the right member of (5.24), using (5.13) and (5.14), and obtain

$$\frac{\partial R}{\partial x_{jp}} = \frac{8}{F} \sum_k \sum_r x_{kr} \sum_i (f_{ijp} f_{ikr} + g_{ijp} g_{ikr}). \tag{6.2}$$

Similarly for W, we differentiate the right member of (5.25) and obtain

$$\frac{\partial W}{\partial x_{jp}} = 4(x_{jp} A_{jp} + \sum_k \sum_r x_{kr} c_{jk/pr}) \tag{6.3}$$

From (6.2) and (6.3), (6.1) can be written as

$$\sum_k \sum_r x_{kr} h_{jk/pr} = \tfrac{1}{2} E^2 (x_{jp} A_{jp} + \sum_k \sum_r x_{kr} c_{jk/pr}), \tag{6.4}$$

where

$$h_{jk/pr} \equiv \frac{1}{F} \sum_i (f_{ijp} f_{ikr} + g_{ijp} g_{ikr}). \tag{6.5}$$

The numerical solution of the x-values is to be obtained from (6.4).

Before showing a procedure for the numerical solution, let us verify that a solution of (6.4) will also satisfy (5.23). Summing both members of (6.4) over j and p, and using (6.5) and relations among the notation laid down in the previous section, we get

$$\sum_k \sum_r x_{kr} A_{kr} = \tfrac{1}{2} E^2 (\sum_j \sum_p x_{jp} A_{jp} + \sum_k \sum_r x_{kr} A_{kr})$$

or

$$(1 - E^2) \sum_k \sum_r x_{kr} A_{kr} = 0. \tag{6.6}$$

From (5.17), this can be written as

$$(1 - E^2)\, V = 0. \tag{6.7}$$

Therefore, if $E^2 \neq 1$, we must have $V = 0$. Hence, any solution of (6.4) which does not yield a perfect correlation ratio must have a weighted mean of zero for the x-values. Since a perfect correlation ratio will not in general occur in practice, condition (5.23) will in general be satisfied and is no restriction.

It should be noted that there is always a trivial solution for which E^2 is formally equal to unity. The trivial solution is to set $x_{jp} \equiv 1$. Then $t_i \equiv u_i \equiv 2$; $R = W = 4C$, $E^2 = 1$, and (6.4) is satisfied since it reduces to (6.7). For this trivial solution, E is of course not an actual correlation ratio.

The non-trivial numerical solution of (6 4) can be carried out in practice with the aid of matrix algebra Instead of regarding the x_{jp} as elements of a table with n rows with m_j elements in the jth row, consider the rows of such a table placed end to end to form a single row of $M = \sum_j m_j$ elements Denote this as the row vector $\boldsymbol{x}$. Correspondingly, consider the values $h_{jk/pr}$ arranged to form the elements of a symmetric matrix $\boldsymbol{H}$ of M rows and columns; consider the M values A_{jp} to be the *diagonal* elements of an $M \times M$ diagonal matrix $\boldsymbol{A}$; and consider the values of $c_{jk/pr}$ arranged to form an $M \times M$ symmetric matrix $\boldsymbol{C}$. Let $\lambda = \frac{1}{2}E^2$. Then (6 4) becomes in matric form.

$$(6.8) \qquad \boldsymbol{xH} = \lambda(\boldsymbol{xA} + \boldsymbol{xC}) = \lambda\boldsymbol{x}(\boldsymbol{A} + \boldsymbol{C}).$$

In the next paragraph it is shown that, in general, $(\boldsymbol{A} + \boldsymbol{C})$ is non-singular, so that it has an inverse by which the members of (6 8) can be postmultiplied, yielding

$$(6.9) \qquad \boldsymbol{xH}(\boldsymbol{A} + \boldsymbol{C})^{-1} = \lambda\boldsymbol{x}.$$

This shows that $\boldsymbol{x}$ is a latent vector of $\boldsymbol{H}(\boldsymbol{A} + \boldsymbol{C})^{-1}$, and λ is the latent root to which this vector corresponds. Since we want the largest possible correlation ratio, we seek the largest of the non-trivial latent roots. If the two largest non-trivial roots are not equal, which should ordinarily be the case in practice, then there will be a unique latent vector associated with the largest root.

It is of interest to show that all the latent roots of $\boldsymbol{H}(\boldsymbol{A} + \boldsymbol{C})^{-1}$ are real and non-negative, and that all the latent vectors are real. First, we notice that $\boldsymbol{H}$ is Gramian, for its elements are product sums. To see that $\boldsymbol{A} + \boldsymbol{C}$ is Gramian, we notice that from (5.18) and (5.10),

$$(6.10) \qquad W = 2\sum_j \sum_p x_{jp}^2 A_{jp} + 2\sum_j \sum_k \sum_p \sum_r x_{jp}x_{kr}c_{jk/pr} - V^2C,$$

or, in matric notation, and transposing members,

$$(6.11) \qquad 2\boldsymbol{x}(\boldsymbol{A} + \boldsymbol{C})\boldsymbol{x}' = W + V^2C.$$

Since W is a sum of squares, the right member is clearly non-negative; and hence

$$(6.12) \qquad \boldsymbol{x}(\boldsymbol{A} + \boldsymbol{C})\boldsymbol{x}' \geqq 0,$$

for all $\boldsymbol{x}$. Thus, $\boldsymbol{A} + \boldsymbol{C}$ is nonnegative-definite, or Gramian Furthermore, $\boldsymbol{A} + \boldsymbol{C}$ is in general nonsingular, because according to (5.17) and (5.18), V and W cannot vanish simultaneously unless

$$(6\ 13) \qquad (x_{jp} + x_{kr})c_{jk/pr} \equiv 0$$

If $n \geqq 3$, then (6 13) will ordinarily imply that $x_{jp} \equiv 0$, that is, the equality in (6.12) will hold if and only if $\boldsymbol{x} = 0$. In such a case, $\boldsymbol{A} + \boldsymbol{C}$ is *positive*-definite, or is nonsingular as well as Gramian, and possesses an inverse.

As is well known, the inverse of a Gramian matrix is Gramian (see [5, p 71], for example), so that $(\boldsymbol{A} + \boldsymbol{C})^{-1}$ is Gramian. That the latent roots of $\boldsymbol{H}(\boldsymbol{A} + \boldsymbol{C})^{-1}$ are all nonnegative follows from a general theorem that the latent roots of

the product of two Gramian matrices are always nonnegative [5, p. 116] The proof of this is brief, and will be repeated here in a little different variation in order to prove in addition that the latent vectors are all real. Let G be a symmetric square root of $A + C$, so that $G^2 = A + C$. If we postmultiply both members of (6.9) by G, we can write the results as:

$$(xG)(G^{-1}HG^{-1}) = \lambda(xG). \tag{6.14}$$

This shows that xG is a latent vector of $G^{-1}HG^{-1}$ corresponding to the root λ. But $G^{-1}HG^{-1}$ is symmetric, and in fact Gramian, for it can be written in the form $(G^{-1}K)(G^{-1}K)'$, where $KK' = H$. Hence, each λ is nonnegative, and each xG is real, whence each x is real

The numerical solution of (6.9) can be carried out by the simple iterative technique for latent roots and vectors (see, for example, [6]). The iterations converge in general to the vector associated with the largest root. To avoid convergence to the trivial solution (which formally has the largest root), the trial vectors should be adjusted to satisfy (5.23); then they will in general converge to the vector associated with the largest non-trivial root.

A marginal indication of the internal consistency of the judgments is the agreement in sign of

$$(x_{jp} + x_{kr}) - (x_{jq} + x_{ks})$$

with

$$\sum_i e_{ijk/pr,qs} - \sum_i e_{ijk/qs,pr} ,$$

for each of the comparisons. If one combination is judged higher by more people in comparison with another, then its x-values should exceed those of the other for marginal consistency.

7. The maximum under certain linear restrictions. In the previous section, no restrictions were placed on the x_{jp} in maximizing E^2 For some problems, the O_{jp} may be quantitative, and it may be desired within each item to keep the distances between the x_{jp} proportionate to the distances between the O_{jp}. This was the case for the score card, where a linear system of weighting had to be used to be practicable for the army. It was necessary to derive a constant multiplier for length of service, a constant multipler for time overseas, etc., even though there were curvilinearities in the judgments.

Our principle enables us to handle such restrictions just as well as the unrestricted case. We shall derive the set of multipliers which is most consistent for the judgments in the sense of least squares. The ordering of categories within an item will no longer be considered arbitrary. Instead, subscripts will be assigned in a fashion to make $(O_{jp} - O_{jq})$ proportional to $(p - q)$ *within* each item. For convenience, the subscripts can be assigned beginning from zero for each item.

The linear restriction is to determine x-values in the form

$$x_{jp} = \xi_j + p\eta_j \,, \tag{7.1}$$

where the ξ_j and the η_j are now the basic unknowns to be solved for to maximize E^2. It is the η_j that are of interest, for they will be the multipliers; but the ξ_j have to be used in the analysis to help determine the multipliers even though they are only additive constants that will not affect the order of total scores of people.

To maximize E^2 under the linear restrictions, we differentiate the right member of (5.22) with respect to the ξ_j and the η_j, set the derivatives equal to zero, and obtain the stationary equations

$$\frac{\partial R}{\partial \xi_j} = E^2 \frac{\partial W}{\partial \xi_j} \tag{7.2}$$

$$\frac{\partial R}{\partial \eta_j} = E^2 \frac{\partial W}{\partial \eta_j} \,. \tag{7.3}$$

In order to evaluate the indicated derivatives, it is helpful to introduce some more notations. Let:

$$l_{0,ik} \equiv \sum_r f_{ikr} \,, \qquad m_{0,ik} \equiv \sum_r g_{ikr} \tag{7.4}$$

$$l_{1,ik} \equiv \sum_r r f_{ikr} \,, \qquad m_{1,ik} \equiv \sum_r r g_{ikr} \tag{7.5}$$

$$d_{a,jk} \equiv \sum_p \sum_r p^a c_{jk/pr} \tag{7.6}$$

$$d_{11,jk} \equiv \sum_p \sum_r p r c_{jk/pr} \equiv d_{11,kj} \tag{7.7}$$

$$D_{a,j} \equiv \sum_k \sum_p \sum_r p^a c_{jk/pr} \equiv \sum_k d_{a,jk} \tag{7.8}$$

$$h_{0,jk} \equiv \frac{1}{F} \sum_i (l_{0,ij} l_{0,ik} + m_{0,ij} m_{0,ik}) \tag{7.9}$$

$$h_{1,jk} \equiv \frac{1}{F} \sum_i (l_{1,ij} l_{0,ik} + m_{1,ij} m_{0,ik}) \tag{7.10}$$

$$h_{2,ik} \equiv \frac{1}{F} \sum_i (l_{1,ij} l_{1,ik} + m_{1,ij} m_{1,ik}). \tag{7.11}$$

It is important to notice that $d_{0,jk} \equiv d_{0,kj}$, but that $d_{1,jk} \not\equiv d_{1,kj}$. Similarly, $h_{0,jk} \equiv h_{0,kj}$ and $h_{2,jk} \equiv h_{2,kj}$, but $h_{1,jk} \not\equiv h_{1,kj}$.

To evaluate the derivatives of R, it is helpful to re-write the right members of (5.13) and (5.14) by means of (7.1), (7.4), and (7.5):

$$t_i = \frac{2}{F} \sum_k (\xi_k l_{0,ik} + \eta_k l_{1,ik}) \tag{7.12}$$

$$u_i = \frac{2}{F} \sum_k (\xi_k m_{0,ik} + \eta_k m_{1,ik}) \tag{7.13}$$

Differentiating the right member of (5.24) with respect to the ξ_j and the η_j respectively with the aid of (7.12) and (7.13), and using (7.9), (7.10), and (7.11), yields

$$\frac{\partial R}{\partial \xi_j} = 8\sum_k (\xi_k h_{0,jk} + \eta_k h_{1,kj}) \tag{7.14}$$

$$\frac{\partial R}{\partial \eta_j} = 8\sum_k (\xi_k h_{1,jk} + \eta_k h_{2,jk}). \tag{7.15}$$

For the derivatives of W, we re-write (5.25) using (7.1):

$$W = \sum_j \sum_k \sum_p \sum_r (\xi_j + p\eta_j + \xi_k + r\eta_k)^2 c_{jk/pr} . \tag{7.16}$$

Differentiating with respect to the ξ_j and η_j respectively, we obtain, using (7.6), (7.7), and (7.8),

$$\frac{\partial W}{\partial \xi_j} = 4[\xi_j D_{0,j} + \eta_j D_{1,j} + \sum_k (\xi_k d_{0,jk} + \eta_k d_{1,kj})] \tag{7.17}$$

$$\frac{\partial W}{\partial \eta_j} = 4[\xi_j D_{1,j} + \eta_j D_{2,j} + \sum_k (\xi_k d_{1,jk} + \eta_k d_{11,jk})] . \tag{7.18}$$

The stationary equations (7.2) and (7.3) can now be re-written by means of (7.14), (7.15), (7.17), and (7.18) as:

$$\sum_k (\xi_k h_{0,jk} + \eta_k h_{1,kj}) = \tfrac{1}{2}E^2[\xi_j D_{0,j} + \eta_j D_{1,j} + \sum_k (\xi_k d_{0,jk} + \eta_k d_{1,kj})] \tag{7.19}$$

$$\sum_k (\xi_k h_{1,jk} + \eta_k h_{2,jk}) = \tfrac{1}{2}E^2[\xi_j D_{1,j} + \eta_j D_{2,j} + \sum_k (\xi_k d_{1,jk} + \eta_k d_{11,jk})]. \tag{7.20}$$

These are the equations to be solved numerically for the maximizing ξ_j and η_j.

Before showing a procedure for the numerical solution, let us verify that a solution of (7.19) and (7.20) will satisfy (5.23). From (7.1), (5.17), and (7.8),

$$V \doteq \frac{2}{C} \sum_k (\xi_k D_{0,k} + \eta_k D_{1,k}) . \tag{7.21}$$

Summing both members of (7.19) over j shows that

$$(1 - E^2) \sum_k (\xi_k D_{0,k} + \eta_k D_{1,k}) = 0,$$

or, from (7.21),

$$(1 - E^2)V = 0.$$

Hence, if $E^2 \neq 1$, the corresponding solution will satisfy the condition that $V = 0$.

As in the unrestricted case, there is always a trivial solution that will yield an E^2 formally equal to unity. This trivial solution is $\xi_j \equiv 1$, $\eta_j \equiv 0$, which makes $x_{jp} \equiv 1$ as in the previous case. These values satisfy (7.19) and (7.20), and have $E^2 = 1$. Of course, E is again not an actual correlation ratio for this trivial solution.

To obtain a non-trivial solution, it is convenient to write (7.19) and (7.20) in matric notation. Let

$$(7.22) \qquad z = \| [\xi_j] \quad [\eta_j] \|.$$

z is a row vector of $2n$ elements, the first n elements being the ξ_j and the last n elements being the η_j. Let

$$(7.23) \qquad h = \left\| \begin{matrix} [h_{0,jk}] & [h_{1,jk}] \\ [h_{1,kj}] & [h_{2,jk}] \end{matrix} \right\|.$$

h is $2n \times 2n$ and is symmetric, in fact it is also Gramian, since its elements are product sums. Let δ_{jk} be Kronecker's delta, and let

$$(7.24) \qquad c = \left\| \begin{matrix} [D_{0,j}\,\delta_{jk} + d_{0,jk}] & [D_{1,j}\,\delta_{jk} + d_{1,jk}] \\ [D_{1,j}\,\delta_{jk} + d_{1,kj}] & [D_{2,j}\,\delta_{jk} + d_{11,jk}] \end{matrix} \right\|.$$

c also is $2n \times 2n$, symmetric, and Gramian. Again let

$$(7.25) \qquad \lambda = \tfrac{1}{2}E^2.$$

Equations (7.19) and (7.20) can now be stated as a single matric equation:

$$(7.26) \qquad zh = \lambda zc.$$

In general, c will be nonsingular, so that it will have an inverse by which both members of (7.26) can be postmultiplied to yield

$$(7.27) \qquad zhc^{-1} = \lambda z.$$

Therefore z is a latent vector of hc^{-1}, and λ is a latent root. Since we want the largest correlation ratio, we seek the largest of the non-trivial latent roots The largest root in practice will ordinarily be unique. There is then a unique latent vector corresponding to this root, and the elements of this vector provide the most consistent ξ_j and η_j for the population in the sense of least squares.

That c is Gramian and in general nonsingular, that the latent roots of hc^{-1} are all nonnegative, and that the latent vectors of hc^{-1} are all real, requires only proofs analogous to those for the corresponding properties of $A + C$ and $h(A + C)^{-1}$ in the previous section, which need not be repeated here.

As in the previous section, the final numerical steps can be carried out by iterations according to (7.27). Again, the trial vectors should be adjusted to conform to (5.23) to prevent convergence to the trivial solution.

A marginal indication of the consistency of the quantification is the agreement in sign of

$$(p - q)\eta_j + (r - s)\eta_k$$

with

$$\sum_i e_{ijk/pr,qs} - \sum_i e_{ijk/qs,pr}\,,$$

for all comparisons.

Appendix: A distinction between the conventional principle and the present principle. The relationship between the conventional principle of estimating means of hypothetical distributions and the present principle of reproducing the comparisons of each individual will be analyzed here for the case of ordinary comparisons. Only the *principles* will be contrasted here.

In the conventional approach, it is assumed that each of the N individuals has a numerical value for each of the O_j. Let s_{ij} be such a value of O_j for the ith individual. The hypothesis is that person i makes the judgment $O_j > O_k$ if $s_{ij} > s_{ik}$; and the conventional problem is to estimate from the judgments what the relative distances are between the means μ_j, where

$$\mu_j \equiv \frac{1}{N} \sum_i s_{ij} . \tag{A.1}$$

The ranges of the subscripts are: $i = 1, 2, \cdots, N$; $j, k, l = 1, 2, \cdots, n$; and will not be explicitly indicated.

According to the approach of this paper, *if we are to consider hypothetical variables*, the problem would be to determine for each O_j a numerical value x_j such that the differences $(x_j - x_k)$ will best approximate the $(s_{ij} - s_{ik})$ for each individual in the sense of least squares. This will separate "higher than" x-values from "lower than" x-values. If we let

$$Z = \sum_i \sum_j \sum_k [(s_{ij} - s_{ik}) - w_i (x_j - x_k)]^2 , \tag{A.2}$$

where w_i is a constant of proportionality to be determined for each individual separately, then the problem is to determine the x_j and the w_i which will minimize Z.

Differentiating Z with respect to the w_i and x_j respectively, and setting the derivatives equal to zero, yields the stationary equations

$$\sum_i w_i[(s_{ij} - \bar{s}_i) - w_i (x_j - \bar{x})] = 0 \tag{A.3}$$

$$\sum_k (x_k - \bar{x})(s_{ik} - w_i x_k) = 0 , \tag{A.4}$$

where

$$\bar{s}_i \equiv \frac{1}{n} \sum_k s_{ik} , \qquad \bar{x} = \frac{1}{n} \sum_k x_k . \tag{A.5}$$

Since Z is invariant with respect to translations of the x_j (also to translations of the s_{ij}), the origin of the x_j is arbitrary, and there is no loss in generality in setting

$$\bar{x} = 0. \tag{A.6}$$

Then if we let

$$\alpha = \sum_i w_i^2 , \qquad \beta = \sum_k x_k^2 , \tag{A.7}$$

equations (A.3) and (A.4) can be re-written respectively as

$$\sum_i w_i (s_{ij} - \bar{s}_i) = \alpha x_j, \tag{A.8}$$

$$\sum_k x_k s_{ik} = \beta w_i. \tag{A.9}$$

By summing both members of (A.8) over j, we see that

$$\alpha \sum_j x_j = 0. \tag{A.10}$$

Therefore, since in general $\alpha > 0$, we must have $\bar{x} = 0$; and a solution of (A.8) will necessarily be consistent with (A.6).

Using (A.9) in (A.8) yields the stationary equations for the x_j alone:

$$\sum_k x_k \sum_i s_{ik}(s_{ij} - \bar{s}_i) = \alpha\beta x_j. \tag{A.11}$$

This shows that the x_j are elements of a latent vector corresponding to a latent root $\alpha\beta$ of the $n \times n$ matrix defined by the elements S_{jk}, where

$$S_{jk} \equiv \sum_i s_{ik}(s_{ij} - \bar{s}_i) \equiv \sum_i s_{ij} s_{ik} - \frac{1}{n} \sum_l \sum_i s_{ik} s_{il}. \tag{A.12}$$

To determine which one of the latent roots provides the minimum Z, we first notice—by multiplying both members of (A.9) by w_i, summing over i, and using (A.7)—that

$$\sum_i \sum_k x_k s_{ik} w_i = \alpha\beta. \tag{A.13}$$

Then expanding the right member of (A.2) with the aid of (A.9) and (A.13), we obtain

$$Z/2n = \sum_i \sum_j (s_{ij} - \bar{s}_i)^2 - \alpha\beta. \tag{A.14}$$

Clearly, Z will be minimized if we use the largest $\alpha\beta$. Therefore, we seek the latent vector associated with the largest latent root of $\| S_{jk} \|$.

To examine the relation of the elements of this minimizing latent vector to the means μ_j of the hypothetical variables, denote the variances and correlations of the hypothetical variables by:

$$\sigma_j^2 \equiv \frac{1}{N} \sum_i (s_{ij} - \mu_j)^2 \equiv \frac{1}{N} \sum_i s_{ij}^2 - \mu_j^2 \tag{A.15}$$

$$\rho_{jk} \equiv \frac{\sum_i (s_{ij} - \mu_j)(s_{ik} - \mu_k)}{N\sigma_j \sigma_k} \equiv \frac{\frac{1}{N} \sum_i s_{ij} s_{ik} - \mu_j \mu_k}{\sigma_j \sigma_k}. \tag{A.16}$$

Then

$$\sum_i s_{ij} s_{ik} = N(\sigma_j \sigma_k \rho_{jk} + \mu_j \mu_k). \tag{A.17}$$

From (A.17) and the last member of (A.12), we can write

$$\frac{1}{N} S_{jk} \equiv \sigma_j \sigma_k \rho_{jk} + \mu_j \mu_k - \frac{1}{n} \sum_l (\sigma_k \sigma_l \rho_{kl} + \mu_k \mu_l) . \tag{A.18}$$

The elements of the matrix of which the x_j are a latent vector are now expressed in terms of the means, variances, and correlations of the hypothetical variables, according to the right member of (A.18). It is clear that in general, the μ_j are not elements of a latent vector of $\| S_{jk} \|$, so that our approach is in general not equivalent to the conventional approach.

In the special case of equal variances and correlations, such as is often assumed in the conventional approach,[4] we can now see that the μ_j do define a latent vector. For this case, let the common variance be σ^2, and let the common correlation coefficient be ρ. Then

$$\rho_{jk} \equiv \rho + \delta_{jk}(1 - \rho), \tag{A.19}$$

where δ_{jk} is Kronecker's delta, and (A.18) becomes

$$\frac{1}{N} S_{jk} \equiv \sigma^2(1 - \rho)\left(\delta_{jk} - \frac{1}{n}\right) + (\mu_j - \bar{\mu})\mu_k , \tag{A.20}$$

where

$$\bar{\mu} = \frac{1}{n} \sum_j \mu_j . \tag{A.21}$$

From (A.20) and (A.12), (A.11) becomes converted to

$$[\gamma - \sigma^2(1 - \rho)] x_j = (\mu_j - \bar{\mu}) \sum_k \mu_k x_k , \tag{A.22}$$

where

$$\gamma = \alpha\beta/N . \tag{A.23}$$

Multiplying both members of (A.22) by x_j and summing over j shows that

$$(\sum_j \mu_j x_j)^2 = \beta[\gamma - \sigma^2(1 - \rho)] . \tag{A.24}$$

From (A.22) and (A.24) we obtain the elements of the minimizing latent vector for Z to be, in normalized form,

$$\frac{x_j}{\sqrt{\beta}} = \frac{\mu_j - \bar{\mu}}{\sqrt{\gamma - \sigma^2(1 - \rho)}} . \tag{A.25}$$

That this is the minimizing vector follows from the fact that the remaining latent roots must all have $\gamma = \sigma^2(1 - \rho)$ in order to have vectors distinct from (A.25); (A.25) does correspond to the largest nontrivial root, since for it the

[4] More specifically, zero correlations are assumed, but this is not necessary for our purpose.

root satisfies the inequality $\gamma > \sigma^2(1 - \rho)$. (The remaining latent vectors are not uniquely defined, for they all correspond to equal roots.) Therefore, the means of the hypothetical variables are a linear function of the elements of the minimizing latent vector for the case of equal variances and correlations

As a final comment, it should be pointed out that paired comparisons are insufficient to estimate the hypothetical values. Two persons with widely different hypothetical values will make the same judgments provided only that their values have the same rank order Therefore, hypotheses about variables presumed to underlie the comparisons cannot be completely tested only on the basis of the comparisons

Psychologically, it may or may not be proper to assume that judgments of the type $O_j > O_k$ can be expressed as a function of differences $s_{ij} - s_{ik}$. Perhaps, psychologically, comparisons may operate on some more complicated principle. The approach presented in the body of this paper does not assume anything about underlying variables, but simply seeks a set of numerical values that will best help reproduce the observed data for each individual.

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RELATIVE ACCURACY OF SYSTEMATIC AND STRATIFIED RANDOM SAMPLES FOR A CERTAIN CLASS OF POPULATIONS[1]

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1. Summary. A type of population frequently encountered in extensive samplings is one in which the variance within a group of elements increases steadily as the size of the group increases. This class of populations may be represented by a model in which the elements are serially correlated, the correlation between two elements being a positive and monotone decreasing function of the distance apart of the elements. For populations of this type, the relative efficiencies are compared for a systematic sample of every kth element, a stratified random sample with one element per stratum and a random sample.

The stratified random sample is always at least as accurate on the average as the random sample and its relative efficiency is a monotone increasing function of the size of the sample. No general result is valid for the relative efficiency of the systematic sample. In fact, there are populations in the class in which the systematic sample is more accurate than the stratified sample for one sampling rate, but is less accurate than the random sample for another sampling rate. If, however, the correlogram is in addition concave upwards, the systematic sample is on the average more accurate than the stratified sample for any size of sample.

Some numerical results are given for the cases in which the correlogram is (i) linear (ii) exponential.

2. Introduction. We consider a finite population consisting of the elements $x_1, x_2, \cdots, x_{nk}$, where n and k are integers. A systematic sample is drawn by choosing an element at random from the elements $x_1, \cdots, x_k$, and then selecting every kth consecutive element. That is, if x_i is the element first chosen, the systematic sample comprises the elements $x_i, x_{i+k}, \cdots, x_{i+(n-1)k}$. This type of sample has found considerable use in practice, because it is often easier to select and to administer than a random or stratified random sample and because it has an intuitive appeal through spreading the sample evenly over the population. Much remains to be learned, however, about the accuracy of this systematic sample relative to that of comparable random or restricted random samples. Probably the most relevant comparison is that between the systematic sample and the stratified random sample having one element per stratum. In the latter case, the population is divided into the n strata $\{x_1, \cdots, x_k\}$, $\{x_{k+1}, \cdots, x_{2k}\}, \cdots$, and one element is chosen independently at random from each of the strata. This type of sample is similar in many respects to the systematic

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sample. Both divide the population into the same n strata of k elements each, with one element chosen from each stratum. Moreover, neither sample provides the data for an unbiased estimate of the sampling variance of the sample mean, at least in the sense that the estimate is unbiased whatever the form of the population of elements x_i.

The first thorough investigation of the properties of systematic samples was made by W. G. and L. H Madow [1]. In particular, these authors compared the accuracies of a systematic sample and a stratified random sample of the types described above for several types of finite population. Where the elements in the population lie on the line $x_i = i$, they showed that the stratified random sample, with one element per stratum, is more accurate than the systematic sample. If the population has a periodic distribution, the stratified random sample is superior when k is an integral multiple of the period, but the systematic sample is superior when k is an odd multiple of the half-period. The authors also considered the more complex case where the population contains both a trend function and a periodic function.

The object of this paper is to make similar comparisons for another type of population which appears to be fairly frequently encountered in extensive samplings The population is one in which the variance among the elements in any group of contiguous elements increases steadily as the size of the group increases. This type of population has long been regarded as applicable in field experimental work, where the variance among plots within a block is found usually to increase with the size of block. Summarizing data from 40 uniformity trials, Fairfield Smith [2] verified this notion and derived an empirical relationship from which the rate of increase may be estimated. The same type of population is also considered in several recent papers on extensive sample surveys. Thus, in a discussion of methods for sampling farm populations, Jessen [3] postulated a law in which the variance among farms within a grid is a monotone increasing function of the size of the grid and used the law for estimating the optimum number of farms which should be included in a sampling-unit. Mahalanobis [4] independently developed the same law as Fairfield Smith in a comprehensive investigation of large-scale sample surveys. Hansen and Hurwitz [5] referred to the increase in variance within a cluster with growing size of cluster as typical of many actual populations Numerous other references could be given.

3. Specification of the population. Various mathematical models may be constructed to represent the situation in which the variance within any group increases with increasing size of group. For instance, we might consider that the elements x_i are drawn from different populations, the population changing in some regular manner with i. Alternatively, the x_i may be assumed to belong to the same population, but to be serially correlated. For simplicity, we assume further that the serial correlation between x_i and x_{i+u} is some quantity ρ_u which depends only on u. Then if ρ_u is positive and is a monotone decreasing function

of u, it may be expected from intuition (and will be proved later) that the variance within the group of elements $x_i, x_{i+1}, \cdots, x_{i+k}$ is a monotone increasing function of k This model seems appropriate for our purpose, since many writers refer explicitly to positive correlations between the x's as the basis for the phenomenon of increasing variance.

The specification above will be qualified in one respect. To assume that the ρ's are *strictly* monotone for an actual finite population of only moderate size does not seem realistic While the correlogram may exhibit a definite downward trend, yet individual fluctuations about the trend prevent the correlogram from being strictly monotone. It is more reasonable to regard the finite population as being itself a sample from an infinite population in which the ρ's are monotone. This attitude is, I believe, in accord with that of the authors referred to above, who, as I interpret their writings, regard the variance law as holding in an idealized population Thus, comparisons between the systematic and stratified random samples will be made not for a single finite population, but for the average of finite populations drawn from an infinite population with monotone decreasing ρ. Results for an individual finite population will differ from the average results because the r's which appear in the population fluctuate about their expectations ρ. As the finite population becomes larger, its results will tend to coincide with the average results.

Accordingly, the elements x_i, $i = 1, 2, \cdots, nk$, are assumed to be drawn from a population in which

$$E(x_i) = \mu, \; E(x_i - \mu)^2 = \sigma^2, \; E(x_i - \mu)(x_{i+u} - \mu) = \rho_u\sigma^2$$

where $\rho_u \geq \rho_v \geq 0$, whenever $u < v$.

4. Some useful preliminary formulas. If $\bar{x}$ is the mean of a specified finite population, the following algebraic identity, frequently useful in the analysis of variance, is easily established.

$$(1) \qquad (kn) \sum_{i=1}^{kn} (x_i - \bar{x})^2 = \sum_{i=1}^{kn} \sum_{j>i} (x_i - x_j)^2.$$

Since there are $(kn)(kn - 1)/2$ possible pairs of values (x_i, x_j), this gives

$$(2) \quad \sum_{i=1}^{kn} (x_i - \bar{x})^2 = \frac{(kn - 1)}{2} E(x_i - x_j)^2 = \frac{(kn - 1)}{2} E\{(x_i - \mu) - (x_j - \mu)\}^2$$

where E is taken over the finite population. Now expand the quadratic and average over all finite populations. In the $(kn)(kn - 1)/2$ combinations, there are $(kn - 1)$ in which j exceeds i by 1, $(kn - 2)$ in which j exceeds i by 2, and so on. Hence

$$(3) \quad E \sum_{i=1}^{kn} (x_i - \bar{x})^2 = (kn - 1)\sigma^2 \left\{1 - \frac{2}{(kn)(kn - 1)} \sum_{u=1}^{kn-1} (kn - u)\rho_u\right\}.$$

To obtain the corresponding expectation for the sum of squares within a single stratum of k consecutive elements, we need only replace (kn) by k in (3). Since

the result is the same for all n strata, we obtain

$$(4)\qquad E\,(\text{S. S. within strata}) = n(k-1)\,\sigma^2 \left\{1 - \frac{2}{k(k-1)} \sum_{u=1}^{k-1} (k-u)\,\rho_u\right\}.$$

Formula (3) also gives the expected sum of squares within a specified systematic sample if we replace (kn) by n and u by (ku), since there are n elements in the sample and since the correlations between successive elements are ρ_k, ρ_{2k}, $\cdots$ instead of ρ_1, ρ_2, $\cdots$. The result is the same for each of the k systematic samples. Hence

$$(5)\qquad E\,(\text{S. S. within systematic samples}) = k(n-1)\,\sigma^2 \left\{1 - \frac{2}{n(n-1)} \cdot \sum_{u=1}^{n-1} (n-u)\,\rho_{ku}\right\}.$$

5. Average variance for a random sample. The symbols σ_r^2, σ_{st}^2, σ_{sy}^2 will be used to denote the average variances of the means of the random, stratified random and systematic samples, respectively, about the mean of the finite population, this average being taken over all finite populations drawn from the infinite population specified in the previous section. Comparisons with the random sample, though not our main purpose, will be included where they are of interest.

For a single finite population, it has been shown by several writers that the variance of the mean of a random sample is

$$(6)\qquad \frac{1}{n} \cdot \frac{(kn-n)}{(kn-1)} \cdot \frac{1}{kn} \sum_{i=1}^{kn} (x_i - \bar{x})^2$$

where $\bar{x}$ is the mean of the finite population.

From (3), we obtain

$$(7)\qquad \sigma_r^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\left\{1 - \frac{2}{(kn)(kn-1)} \sum_{u=1}^{kn-1} (kn-u)\,\rho_u\right\}.$$

6. Average variance for a stratified random sample. If $\bar{x}_{st}$ is the mean of a typical stratified random sample, the sampling variance of $\bar{x}_{st}$ is by definition

$$(8)\qquad E(\bar{x}_{st} - \bar{x})^2.$$

Consider first the average over a single finite population. Let $\bar{x}_1$, $\bar{x}_2$, $\cdots$, $\bar{x}_n$ be the means of the n strata, respectively, and let x_{1j}, x_{2j}, $\cdots$, x_{nj} be the elements selected from the respective strata. Then (8) may be written

$$(9)\qquad \frac{1}{n^2} E\,\{(x_{1j} - \bar{x}_1) + (x_{2j} - \bar{x}_2) + \cdots + (x_{nj} - \bar{x}_n)\}^2$$

since
$$\sum_{i=1}^{n} x_{ij} = n\bar{x}_{st} \text{ and } \sum_{i=1}^{n} \bar{x}_i = n\bar{x}.$$

Take the average over all k^n samples from the finite population. All cross-product terms vanish, since, for example, x_{1j} appears equally often with $x_{21}, x_{22}, \cdots, x_{2k}$. This gives

$$\frac{1}{kn^2}\sum_{i=1}^{n}\sum_{j=1}^{k}(x_{ij}-\bar{x}_i)^2 \tag{10}$$

for the variance for a single finite population. The sum of squares involved is, of course, simply the sum of squares within strata. Hence, by (4)

$$\sigma_{st}^2 = \frac{\sigma^2}{n}\left(1-\frac{1}{k}\right)\left\{1-\frac{2}{k(k-1)}\sum_{u=1}^{k-1}(k-u)\rho_u\right\}. \tag{11}$$

7. Average variance for the systematic sample. If $\bar{x}_{sy}$ is the mean of a typical sample, the variance for a single finite population is

$$E(\bar{x}_{sy}-\bar{x})^2 = \frac{1}{kn}\{n\Sigma(\bar{x}_{sy}-\bar{x})^2\} \tag{12}$$

where the sum is taken over the k systematic samples. Since the sum of squares among samples is equal to the total sum of squares in the population *minus* the sum of squares within samples, (12) equals

$$\frac{1}{kn}\sum_{i=1}^{kn}(x_i-\bar{x})^2 - \frac{1}{kn}\text{(S. S. within systematic samples)}. \tag{13}$$

To obtain the average over all finite populations we substitute from (3) and (5) for the first and second terms respectively. The result is

$$\begin{aligned}\sigma_{sy}^2 = \frac{(kn-1)}{kn}\sigma^2&\left\{1-\frac{2}{(kn)(kn-1)}\sum_{u=1}^{kn-1}(kn-u)\rho_u\right\}\\ &-\frac{(n-1)}{n}\sigma^2\left\{1-\frac{2}{n(n-1)}\sum_{u=1}^{n-1}(n-u)\rho_{ku}\right\}.\end{aligned} \tag{14}$$

This reduces to

$$\begin{aligned}\sigma_{sy}^2 = \frac{\sigma^2}{n}\left(1-\frac{1}{k}\right)&\left\{1-\frac{2}{kn(k-1)}\sum_{u=1}^{kn-1}(kn-u)\rho_u\right.\\ &\left.+\frac{2k}{n(k-1)}\sum_{u=1}^{n-1}(n-u)\rho_{ku}\right\}.\end{aligned} \tag{15}$$

It should be noted that the formulas and notations above are different from those used by the Madows, who define ρ and σ^2 with reference to a single finite population and discuss the sample variances for a single finite population.

8. Relative accuracies of random and stratified random samples. First, some general comments From (7), (11) and (15) the relative efficiencies of the three types of sample are seen to depend only on the linear functions of the ρ's which appear in σ_r^2, σ_{st}^2, and σ_{sy}^2. It is easy to verify that in each case the sum of the coefficients of the ρ's is unity. For the random sample, the linear function in-

volves every serial correlation up to lag $(kn - 1)$ with coefficients which decrease linearly as the lag increases and are independent of the size of sample, depending only on $N = (kn)$, the number of elements in the finite population. For the stratified random sample, only serial correlations with lags up to $(k - 1)$ appear, k being the number of elements in the stratum. As presented in (15), the formula for the systematic sample is separated into two linear functions. The first is the same function as appears in the formula for the random sample except that all coefficients are $(kn - 1)/(k - 1)$ times as large The second, which carries a positive sign, involves correlations where the lag is a multiple of k.

Thus far the formulae require no restrictions on the ρ's In considering the case where the ρ's are positive and monotone decreasing, the following lemma is helpful.

LEMMA. *If* ρ_i, $(i = 1, \cdots, m)$, *are positive and monotone decreasing, that is,* $\rho_i \geq \rho_{i+1} > 0$ *and if* $(\alpha_1 + \alpha_2 + \cdots + \alpha_m)$ *is zero, the necessary and sufficient conditions that*

$$L = \alpha_1\rho_1 + \alpha_2\rho_2 + \cdots + \alpha_m\rho_m \geq 0, \qquad \textit{for all admissible sets of } \rho\textit{'s}, \tag{16}$$

$$\textit{are } \alpha_1 + \alpha_2 + \cdots + \alpha_i \geq 0, i = 1, 2, \cdots, (m - 1). \tag{17}$$

For let $\rho_i = \rho_{i+1} + \delta_i$, where by hypothesis $\delta_i \geq 0$. Then if we substitute successively for $\rho_1, \rho_2, \cdots, \rho_{m-1}$ in terms of $\delta_1, \delta_2, \cdots, \delta_{m-1}$, we find

$$\begin{aligned} L = \alpha_1\delta_1 + (\alpha_1 + \alpha_2)\delta_2 + (\alpha_1 + \alpha_2 + \alpha_3)\delta_3 + \cdots \\ + (\alpha_1 + \alpha_2 + \cdots + \alpha_{m-1})\delta_{m-1}, \end{aligned} \tag{18}$$

the final term in ρ_m vanishing because $(\alpha_1 + \cdots + \alpha_m)$ is zero. Since all $\delta_i \geq 0$, the sufficiency of (17) is obvious. Also, if for any i the coefficient of δ_i is negative, we can make L negative by choosing that δ_i as positive and all other δ's as zero. This establishes necessity.

COROLLARY. *If* ρ_i *are strongly monotone, i.e.,* $\rho_i > \rho_{i+1}$, *and if at least one of the* α_i *is different from zero, conditions* (17) *are sufficient to establish that* L *exceeds zero.* For in (18) all the δ's are greater than zero and by (17) none of the δ's has a negative coefficient. Further, the coefficient of at least one of the δ's must exceed zero, otherwise all the α's would be zero. Hence $L > 0$.

We now show that if the ρ_u are monotone decreasing,

$$L(k) = \frac{2}{k(k-1)} \sum_{u=1}^{k-1} (k - u)\rho_u \tag{19}$$

is a monotone decreasing function of k. This is the linear function which appears in the variance of the stratified sample.

$$L(k) - L(k+1) = \frac{2}{k(k-1)} \sum_{u=1}^{k-1} (k - u)\rho_u - \frac{2}{(k+1)k} \sum_{u=1}^{k} (k + 1 - u)\rho_u \tag{20}$$

$$= \frac{2}{k(k^2 - 1)} \sum_{u=1}^{k} (k + 1 - 2u)\rho_u. \tag{21}$$

Since the sums of the coefficients of the ρ_u are unity in $L(k)$ and $L(k+1)$, the sum is zero in (21). Hence the lemma may be applied. But it is obvious that the sum of the first i coefficients in (21) exceeds zero, since the coefficients are all positive for $u \leq (k+1)/2$ and all negative for $u > (k+1)/2$. Hence

$$L(k) - L(k+1) \geq 0. \tag{22}$$

Further, by the corollary, if the ρ_u are strongly monotone, $L(k)$ is strongly monotone Since all ρ_u are positive, this result is sufficient to prove that

$$1 - \frac{2}{k(k-1)} \sum_{u=1}^{k-1} (k-u)\rho_u \leq 1 - \frac{2}{(nk)(nk-1)} \sum_{u=1}^{nk-1} (nk-u)\rho_u . \tag{23}$$

Consequently, for any size of sample the average variance of the stratified sample cannot exceed that of the random sample. Further, the relative efficiency of the stratified sample to the random sample is monotone increasing with decreasing size of stratum, i e. with increasing size of sample. There is, of course, nothing unexpected in these results. Equation (22) also establishes the result mentioned in the third section, that with monotone decreasing ρ, the average variance within strata increases steadily as the size of stratum increases. For if $n(k-1)$ degrees of freedom are assigned to the sum of squares within strata, formula (4) above shows that the average variance within strata is

$$\sigma^2 \left\{1 - \frac{2}{k(k-1)} \sum_{u=1}^{k-1} (k-u)\rho_u\right\} = \sigma^2\{1 - L(k)\}. \tag{24}$$

9. Comparison of the systematic and random samples. Upon investigation, it is soon evident that no general results can be established about the efficiency of the systematic sample relative to the random samples, unless further restrictions are made on the form of the population. In order to apply the lemma, we find the sums of the first i coefficients of the linear functions of ρ which appear in the variance formulae (7), (11) and (15) By elementary methods these sums are found to be

$$\begin{aligned}
\textstyle\sum_r &= \frac{i(2nk - i - 1)}{nk(nk-1)} \\
\textstyle\sum_{st} &= \frac{i(2k - i - 1)}{k(k-1)}, \qquad 1 \leq i \leq (k-1) \\
&\quad\; 1 \qquad\qquad\quad , \qquad i \geq k. \\
\textstyle\sum_{sy} &= \frac{i(2nk - i - 1)}{nk(k-1)} - \frac{rk(2n - r - 1)}{n(k-1)},
\end{aligned} \tag{25}$$

where r is the integer such that $(r+1)k > i \geq rk$.

From the lemma, in order to establish $\sigma_{sy}^2 \leq \sigma_{st}^2$, it would be necessary to show that $\Sigma_{sy} \geq \Sigma_{st}$ for any i. Now if i is less than k, so that r is zero, clearly

(26) $$\sum_{sy} > \sum_{st} > \sum_{r}, \qquad i = 1, 2, \cdots, (k-1).$$

except when n is 1, in which case all three are equal.

But if i is an integral multiple of k, say rk, we find

(27) $$\sum_{r} = \frac{r}{n}\left[1 + \frac{(n-r)k}{(nk-1)}\right], \qquad \sum_{st} = 1, \qquad \sum_{sy} = \frac{r}{n},$$

so that

(28) $$\sum_{st} > \sum_{r} > \sum_{sy}.$$

Consequently the conditions of the lemma are not satisfied with regard to the systematic sample and no general theorem exists for all populations with monotone decreasing ρ. The result (26) and the corollary show that for any population in this class which has $\rho_u = 0$, $u > (k-1)$, the systematic sample is more efficient than the stratified random sample. On the other hand, (28) shows that in a population with the first k of the ρ's equal and the rest zero, the systematic sample has a higher variance than a random sample. If these two results are collated for a population with the first j of the ρ's equal and the rest zero, we see that the systematic sample with stratum size j is less accurate than the comparable random sample, while the systematic sample with stratum size $(j+1)$ is more accurate than the comparable stratified random sample. Although such a population may not occur in practice, the result suggests that the graph of the variance of the mean against the size of sample is unlikely to exhibit the same regularity for the systematic as for the random samples.

10. Populations in which the correlogram is concave upwards. Further investigation shows that the deciding factors in determining the relative accuracies of the systematic and random samples are the second differences of the ρ_u rather than the first differences. The following result will be proved.

THEOREM: *For all infinite populations in which*

$$\rho_i \geq \rho_{i+1} \geq 0, \; i = 1, 2, \cdots, (kn-1),$$

and

$$\delta_i^2 = \rho_{i-1} + \rho_{i+1} - 2\rho_i \geq 0, \; i = 2, 3, \cdots, (kn-2),$$

then

$$\sigma_{sy}^2 \leq \sigma_{st}^2 \leq \sigma_r^2$$

for any size of sample Further, $\sigma_{sy}^2 < \sigma_{st}^2$, *unless* $\delta_i^2 = 0$, $i = 2, 3, \cdots, (kn-2)$.

This result can be proved by expressing the linear functions of the ρ_u in terms of second differences and establishing a new lemma applicable to second differences. An alternative approach is simpler and perhaps more instructive.

Since the ρ_u are monotone decreasing, $\sigma_{st}^2 \leq \sigma_r^2$ by the results in section 8. In (13) above, the variance of the mean of a systematic sample for a specified finite population was expressed as

$$\begin{aligned}(29)\qquad &\frac{1}{kn}\sum_{i=1}^{kn}(x_i - \bar{x})^2 - \frac{1}{kn}\text{ (Total S.S. within systematic samples)}\\ &= \frac{1}{kn}\sum_{i=1}^{kn}(x_i - \bar{x})^2 - \frac{1}{n}\text{ (Average S.S. within a systematic sample).}\end{aligned}$$

A corresponding equation holds for stratified random samples. For if x_{1j}, x_{2j}, $\cdots$, x_{nj} are the elements of any stratified random sample with mean $\bar{x}_{st}$

$$\sum_{i=1}^{n}(x_{ij} - \bar{x})^2 = \sum_{i=1}^{n}(x_{ij} - \bar{x}_{st})^2 + n(\bar{x}_{st} - \bar{x})^2. \tag{30}$$

Now take the average over all k^n samples. This gives

$$\frac{1}{k}\sum_{i=1}^{kn}(x_i - \bar{x})^2 = \text{(Average S.S. within samples)} + nE(\bar{x}_{st} - \bar{x})^2. \tag{31}$$

Since the term on the extreme right is n times the variance of the stratified random sample, a result analogous to (29) follows at once.

Consequently, $\sigma^2_{sy} \leq \sigma^2_{st}$ if the average sum of squares *within* a systematic sample is greater than or equal to that *within* a stratified random sample. Now by (2), with n in place of (kn), each of these averages is equal to

$$\frac{(n-1)}{2}E(x_{ij} - x_{lj})^2 \tag{32}$$

where x_{ij}, x_{lj} are the elements in the sample from the ith and the lth strata respectively, the average being taken over all possible pairs of strata.

We consider a fixed pair of strata and let $l - i = u$. For the systematic sample, corresponding elements in the ith and lth strata are always (ku) elements apart. Hence,

$$E_{sy}(x_{ij} - x_{lj})^2 = 2\sigma^2(1 - \rho_{ku}). \tag{33}$$

For the stratified random sample, there are k^2 possible pairs of elements from the two strata. One pair is $(ku - k + 1)$ elements apart, two pairs are $(ku - k + 2)$ elements apart, and so on, the numbers of pairs rising linearly to k and then decreasing linearly to one for the final pair which are $(ku + k - 1)$ elements apart. This gives

$$E_{st}(x_{ij} - x_{lj})^2 = 2\sigma^2\left\{1 - \frac{1}{k^2}\sum_{i=-(k-1)}^{(k-1)}(k - |i|)\rho_{ku+i}\right\}. \tag{34}$$

Hence, to complete the proof that $\sigma^2_{sy} \leq \sigma^2_{st}$, it is sufficient to show that

$$\sum_{i=-(k-1)}^{(k-1)}(k - |i|)\rho_{ku+i} - k^2\rho_{ku} \geq 0 \tag{35}$$

for $u = 1, 2, \cdots, (n-1)$, that is, for any pair of strata. This may be written

$$\sum_{i=1}^{(k-1)}(k - i)(\rho_{ku+i} + \rho_{ku-i} - 2\rho_{ku}) \geq 0. \tag{36}$$

But if $\delta^2_{ku} = \rho_{ku-1} + \rho_{ku+1} - 2\rho_{ku}$ is the second central difference it is easy to show that

$$\rho_{ku+i} + \rho_{ku-i} - 2\rho_{ku} = \sum_{j=-(i-1)}^{(i-1)} (i - |j|)\delta^2_{ku+j} \geq 0, \tag{37}$$

since by hypothesis $\delta^2_j \geq 0$, $j = 2, 3, \cdots, (kn - 2)$. This proves that the variance between the elements of the systematic sample is greater than or equal to that between the elements of the stratified random sample for any fixed pair of strata. The result for the overall average follows. Hence $\sigma^2_{sy} \leq \sigma^2_{st}$. Further, unless $\sigma^2_j = 0$, for all j, clearly $\sigma^2_{sy} < \sigma^2_{st}$, except for samples of one.

The essential point in the proof may be put as follows. The elements in the ith and lth strata are on the average (ku) elements apart for both the systematic and the stratified random sample. When two elements in the latter sample are $(ku + i)$ elements apart, they are less correlated than on the average, since $\rho_{ku+i} \leq \rho_{ku}$, and thus provide more independent information. The variance between the elements exceeds the systematic sample variance by $2\sigma^2(\rho_{ku} - \rho_{ku+i})$. However, such cases are counterbalanced by an equal number of cases in which the elements differ by $(ku - i)$ and the variance is below the systematic sample variance by $2\sigma^2(\rho_{ku-i} - \rho_{ku})$. Because of the concavity of ρ_u, the losses on the average balance or outweigh the gains

For the population discussed in section 9, in which $\rho_u = \rho$, $u = 1, 2, \cdots, j$, $\rho_u = 0$, $u > j$, we have $\delta^2_j < 0$, $\delta^2_{j+1} > 0$, and $\delta^2_u = 0$ otherwise. This reversal of the sign of the second difference is the explanation for the anomalous behavior of the systematic samples with stratum sizes j and $(j + 1)$.

The theorem above does *not* prove that the relative accuracy of the systematic to the stratified random sample is a monotone function of n, nor even that σ^2_{sy} decreases steadily as n increases. Actually, there are populations in the class for which neither result holds, as will be illustrated in the next section.

So far as practical applications are concerned, the restriction that the ρ_u should be concave upwards may not be severe. For instance, this condition is satisfied when the correlogram is linear, i.e. $\rho_u = (l - u)/l$, this being one type of correlogram which Wold [6] has considered applicable to economic data. Concavity also holds for the function $\rho_u = e^{-\lambda u}$ which Osborne [7] has suggested for forestry and land-use surveys and for the relation $\rho_u = \tanh (u^{-3/5})$ which Fisher and Mackenzie [8] used for expressing the correlation between the weekly rain at two weather stations as a function of their distance apart. In fact, if ρ_u is conceived of as positive and continuous for all u, a concave upwards function suggests itself naturally.

11. Linear correlograms. It may be of interest to present some results obtained when the correlogram is (i) linear, (ii) exponential, since both types have been suggested as possible models for populations occurring in practice.

In the linear case,

(38) $$\rho_u = (L - u)/L, u \leq L; \quad \rho_u = 0, u > L.$$

If $L \geq (nk - 1)$, the correlogram is a *straight* line throughout the whole range of the finite population. Since all second differences are zero in this case, we may expect $\sigma_{sy}^2 = \sigma_{st}^2 < \sigma_r^2$. If $L < (nk - 1)$, all second differences vanish except δ_L^2, which is positive. Hence we may expect $\sigma_{sy}^2 < \sigma_{st}^2 < \sigma_r^2$.

The results for these cases are found by elementary summations from the basic formulae (7), (11) and (15). Details of the summations will not be presented. For $L \geq (nk - 1)$, we find

(39) $$\sigma_{sy}^2 = \sigma_{st}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\frac{(k + 1)}{3L}; \qquad \sigma_r^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\frac{(nk + 1)}{3L}.$$

The ratio σ_r^2/σ_{sy}^2 is $(nk + 1)/(k + 1)$, which is approximately equal to n, the size of sample, unless the percentage sampled is large. Thus very large gains in efficiency over random sampling are obtained.

If $L < (nk - 1)$, the formulae are less simple. Consider first $k \geq L$; that is, cases where the percentage sampled is less than $100/L$. If $N = nk$,

(40) $$\sigma_r^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\left\{\frac{3N(N - L) + (L^2 - 1)}{3N(N - 1)}\right\}$$

(41) $$\sigma_{st}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\left\{\frac{3k(k - L) + (L^2 - 1)}{3k(k - 1)}\right\}, \qquad k \geq L$$

(42) $$\sigma_{sy}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\left\{\frac{3N(k - L) + (L^2 - 1)}{3N(k - 1)}\right\}, \qquad k \geq L.$$

It is clear on inspection that $\sigma_{sy}^2 < \sigma_{st}^2$; moreover, it is easy to show that the efficiency of systematic relative to stratified random sampling increases steadily as the size of sample increases.

When the size of sample is increased further so that $k \leq L$, formula (40) remains unchanged, while σ_{st}^2 is now given by the same formula as in (39). The formula for σ_{sy}^2 is more complex. If q is the integral part of the quotient when L is divided by k and r is the remainder, so that $L = (qk + r)$, the formula may be written

(42′) $$\sigma_{sy}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right) \cdot \left\{\frac{qk(k^2 - 1) + 3rk(n - q)(k - r) + r(r^2 - 1)}{3NL(k - 1)}\right\}, \quad k \leq L.$$

It is noteworthy that the last two terms in the numerator inside the curly bracket vanish whenever L is exactly divisible by k. Further, the second term is of order $nk = N$ and, when present, exerts a much greater weight than the first term. Thus σ_{sy}^2 takes a sudden dip whenever L is a multiple of k. In fact, for $L = qk$, (42′) reduces to

(43) $$\sigma_{sy}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\frac{(k + 1)}{3N}, \qquad L = qk,$$

so that the variance goes to zero if N is sufficiently large. By comparison with formula (39) for σ_{st}^2 we see that when $L = qk$ the relative efficiency of systematic to stratified random sampling is N/L, which increases beyond bound if N is sufficiently large. In intermediate cases, when the remainder r does not vanish, the leading term in the relative efficiency for N large is $(k^2 - 1)/3r(k - r)$. This varies somewhat irregularly, depending on the relation between L and k.

To illustrate, numerical values are given below when $L = 10$ and the finite population is large enough so that terms in $1/n$ are negligible.

The quantities v_{st}, v_{sy} are the corresponding variances apart from a factor σ^2/N. The stratified sample variance decreases steadily with increasing percentage sampled. On the other hand the systematic sample variance goes to zero and the relative efficiency to infinity when k is 2, 5 or 10. Moreover, in the intermediate cases $k = 3, 4, 6, 7, 8, 9$, the variance and the relative efficiency show no consistent relation to the percentage sampled. For samples of less than 10 per cent, including the cases outside the limits of the table, the relative efficiency decreases steadily from 4 at $k = 11$ to 1 when k is large.

TABLE 1

Variances except for a factor σ^2/N and relative efficiency for systematic and stratified random samples for a linear correlogram

k	2	3	4	5	6	7	8	9	10	11	20
% Sampled	50	33	25	20	17	14	12	11	10	9	5
v_{st}	.10	.27	.50	.80	1.17	1.60	2.10	2.67	3.30	4.00	11.65
v_{sy}	0	.20	.40	0	.80	1.20	1.20	.80	0	1.00	10.00
v_{st}/v_{sy}	∞	1.33	1.25	∞	1.46	1.33	1.75	3.33	∞	4.00	1.16

12. Exponential correlograms. For the exponential $\rho_u = e^{-\lambda u}$ the results are much more regular Each of the linear functions of the ρ's consists of a finite number of terms of an expansion of the form $(1 - x)^{-2}$. If

$$(44)\qquad f(N, \lambda) = \frac{2}{N(N-1)}\left\{\frac{(N-1)e^{\lambda} - N + e^{-(N-1)\lambda}}{(e^{\lambda} - 1)^2}\right\}$$

which is the sum for σ_r^2, we find

$$(45)\qquad \sigma_r^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\{1 - f(N, \lambda)\}$$

$$(46)\qquad \sigma_{st}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\{1 - f(k, \lambda)\}$$

$$(47)\qquad \sigma_{sy}^2 = \frac{\sigma^2}{n}\left(1 - \frac{1}{k}\right)\left\{1 - \frac{(N-1)}{(k-1)} f(N, \lambda) + \frac{k(n-1)}{(k-1)} f(n, k\lambda)\right\}.$$

It may be shown that the variance of the systematic sample decreases steadily and its efficiency relative to stratified sampling increases steadily as the sample becomes larger.

In order to obtain some idea of the magnitude of the gain in efficiency, consider the case where k and n are large. For this case the relative efficiency, which actually is a function of k, n and λ, turns out to depend almost entirely on the single quantity $(k\lambda)$; or, equally, on the correlation $e^{-k\lambda}$ between the items in successive strata in the systematic sample. If $t = (k\lambda)$, we obtain $\sigma_r^2 = \sigma^2/n$,

$$\sigma_{st}^2 = \frac{\sigma^2}{n}\left\{1 - \frac{2}{t} + \frac{2}{t^2} - \frac{2e^{-t}}{t^2}\right\}, \tag{48}$$

$$\sigma_{sy}^2 = \frac{\sigma^2}{n}\left\{1 - \frac{2}{t} + \frac{2}{(e^t - 1)}\right\}. \tag{49}$$

The relative efficiency is given in Table 2 for a selection of values of e^{-t}, the correlation between the items in successive strata.

The relative efficiency has a limiting value 2 when ρ tends to 1 and decreases slowly towards 1 as ρ falls to zero. The gains in efficiency are quite substantial if ρ exceeds 0.1.

TABLE 2

Relative efficiency of systematic and stratified random samples for an exponential correlogram

ρ	.9	.8	.7	.6	.5	.4	.3	.2	.1
$\sigma_{st}^2/\sigma_{sy}^2$	1.96	1.90	1.84	1.78	1.71	1.64	1.55	1.46	1.33

It was pointed out in section 1 that no unbiased estimate of error is available from a single sample for either the systematic or the stratified random sample. This does not mean that no estimate of error can be attempted. However, any estimate must depend on certain assumptions about the form of the population which is being sampled and is likely to be vitiated insofar as these assumptions are false. If, for instance, the correlogram were assumed to be exponential, formula (47), or (49) in the particular case with n, k large, would appear to be the appropriate basis for the estimation of error from a single systematic sample. Consider the simpler case in which (49) is valid. The correlation between successive items in the systematic sample provides an estimate of e^{-t} and hence of t Also, if terms in $1/n$ are negligible, the mean square within the systematic sample is found to be an unbiased estimate of σ^2. By substitution in (49) a consistent estimate of the variance of a single systematic sample would be secured, provided that the exponential assumption were correct. The gains in efficiency over stratified and random sampling could also be estimated.

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OPERATING CHARACTERISTICS FOR THE COMMON STATISTICAL TESTS OF SIGNIFICANCE

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1. Summary. Methods making possible quick calculation of operating characteristics or power curves of common tests of significance involving the χ^2, F, t, and normal distributions are presented. In addition, a comprehensive set of curves illustrating graphically the power of each test for the 5% significance level are included We are interested in the power of: (1) the χ^2-test to determine whether an unknown population standard deviation is greater or less than a standard value, (2) the F test to determine whether one unknown population standard deviation is greater than another (one-sided alternative), and (3) the t-test and normal test to determine whether an unknown population mean differs from a standard or two unknown population means differ from each other. Such operating characteristics have application for the quality control engineer and statistician in the design of sampling inspection plans using variables where they may be used to determine the sample size that will guarantee a specified consumer's and producer's risk. On the other hand they are of use in displaying the power of a test if the sample size has already been set. Finally, they are a necessary adjunct to the proper interpretation of the common tests of significance.

2. Introduction. In the application of the common statistical tests of significance there has been a great need for readily accessible information on the power of the test employed to distinguish between the null hypothesis and pertinent alternative hypotheses for given sample size. In this connection, two important applications arise. On one hand it becomes important for the sampler to know, for a given sample size and critical region, something about the power of the test in rejecting the stated hypothesis when some alternative hypothesis is true. On the other hand, if the sampler wants a given degree of assurance in rejecting the null hypothesis when a particular alternative is true, he would like to know the minimum sample size which would accomplish this when the probability of rejecting the null hypothesis when true is given. In particular, the need for such information arises most frequently in setting sample sizes to distinguish effectively, on the basis of single sample results, between (1) population standard deviations and (2) population means. If the sample size has already been set, as is the case with most specifications, quick information on whether or not it is large enough to keep the risk of accepting poor material down to a reasonable figure is highly desirable. Such probabilities will be recognized, of course, as the Type I and Type II errors of the Neyman-Pearson theory. Such risks must be given proper consideration in the interpretation of a significance test or in designing the provisions of an acceptance test.

Needless to say, the appropriate expressions for the power functions of the χ^2-test, F-test, normal-test, and t-test have been derived at one time or another in the literatuer. However, insofar as the practical statistician or quality control engineer is concerned, such information has not been employed to advantage widely since no informative graphs or extensive tables of power functions for the common statistical tests of significance have been presented. Due to the practical importance of questions of this type, the authors believe there is need for operating characteristics or graphical power functions of the common statistical tests of significance. This paper supplies such a need over a useful range of sample sizes and alternative hypotheses for the 5% significance level.

3. Definitions. In the following account, we will refer to one or both of the normal populations, π_1 and π_2. We will let x_1 be a variate from π_1 whose expected value or mean is μ_1 and standard deviation σ_1. By n_1 we will mean the number of observations drawn at random from π_1 and our sample statistics will be defined in the usual fashion:

$$\bar{x}_1 = \sum_1^{n_1} x_1/n_1, \qquad s_1^2 = \sum_1^{n_1} (x_1 - \bar{x}_1)^2/(n_1 - 1).$$

Similar definitions apply to the normal population π_2 with the appropriate subscript for sample statistics and population values. In dealing with a single population we will drop the subscripts from the sample statistics.

We also define

σ = a standard or arbitrary value of the standard deviation,

a = a standard or given level,

$$s_{12}^2 = \frac{\sum_1^{n_1} (x_1 - \bar{x}_1)^2 + \sum_1^{n_2} (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$ when two normal populations are encountered.

H_0 will be used to denote the null hypothesis and H_1 any one of a set of alternative hypotheses. The probability of rejecting the null hypothesis H_0 when it is true (Type I error) will be denoted by α, and the probability of accepting the null hypothesis when some alternative hypothesis H_1 is true (Type II error) will be denoted by β.

4. Power function of the χ^2-test. The statistic $\chi^2 = \dfrac{(n-1)s^2}{\sigma^2}$ (dropping subscripts of sample statistics) is used to accept or reject the hypothesis that the standard deviation, σ_1, of the normal population sampled is some specified or given value, σ.

Our hypotheses are

$$H_0: \sigma_1 = \sigma$$

$$H_1: \sigma_1 = \lambda\sigma, \ (\lambda > 0).$$

A. To determine whether or not $\sigma_1 > \sigma$. We choose a significance level, α, and compute $\chi^2 = \frac{(n-1)s^2}{\sigma^2}$. If $\chi^2 > \chi^2_\alpha$, where the percentage point α is determined by

$$\frac{(\frac{1}{2})^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} \int_{\chi^2_\alpha}^{\infty} u^{(n-3)/2} e^{-u/2}\, du = \alpha \tag{1}$$

we reject H_0 and conclude that $\sigma_1 > \sigma$.

To set up the power function we note that:

If H_0 is true

$$Pr\left\{\frac{(n-1)s^2}{\sigma^2}\right\} > \chi^2_\alpha = \alpha$$

If H_1 is true

$$Pr\left\{\frac{(n-1)s^2}{\sigma^2}\right\} > \chi^2_\alpha = 1 - \beta, \qquad (1 - \beta = \alpha, \text{ if } \lambda = 1).$$

However, since

$$Pr\left\{\frac{(n-1)s^2}{\sigma_1^2} > \chi^2_{1-\beta}\right\} = 1 - \beta$$

or

$$Pr\left\{\frac{(n-1)s^2}{\sigma^2} > \lambda^2 \chi^2_{1-\beta}\right\} = 1 - \beta$$

we have the relation

$$\lambda^2 \chi^2_{1-\beta} = \chi^2_\alpha \quad \text{or} \quad \lambda = \sqrt{\frac{\chi^2_\alpha}{\chi^2_{1-\beta}}}.$$

Therefore, for a given significance level, α (Type I error), and various Type II errors, β, we can make use of the Tables of Percentage Points of the χ^2-distribution [1] and compute enough of the points (λ, β) to plot the power curves depicted in Fig. 1. The Type I error, α, has been set at the practical level of .05 for Fig. 1.

B. To detect $\sigma_1 < \sigma$. We compute

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

and if $\chi^2 < \chi^2_{1-\alpha}$ we reject H_0, concluding that $\sigma_1 < \sigma$.

By reasoning similar to that in A. we arrive at the relationship

$$\chi^2_{1-\alpha} = \lambda^2 \chi^2_\beta \quad \text{or} \quad \lambda = \sqrt{\frac{\chi^2_{1-\alpha}}{\chi^2_\beta}}.$$

Again, by use of the Table of Percentage Points of the χ^2-Distribution the operating characteristics of Fig. 2 are obtained. We have chosen the practical level of $\alpha = .05$ for Fig. 2.

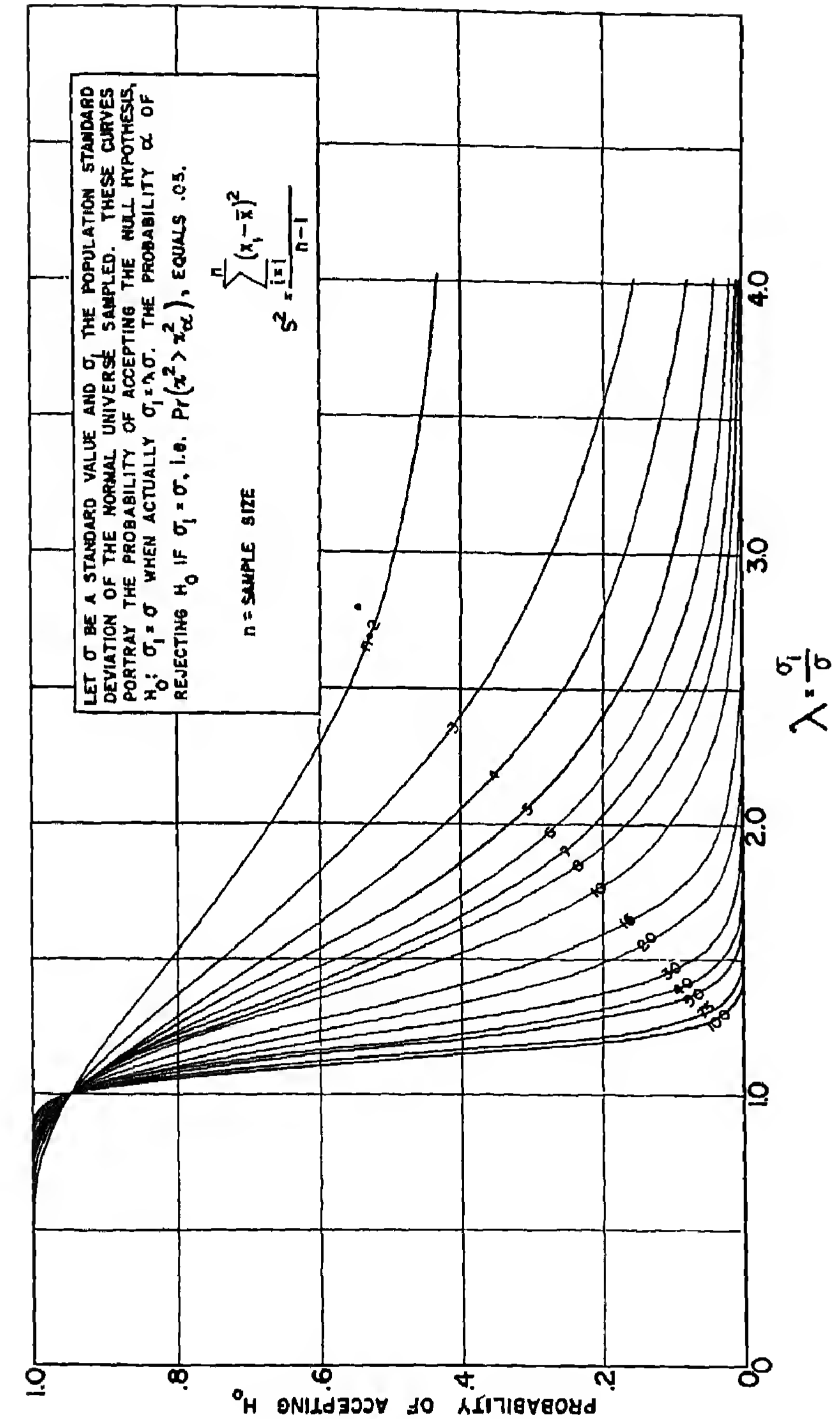

FIG. 1. OPERATING CHARACTERISTICS OF THE χ^2-TEST $\left[\chi^2 = \frac{(n-1)s^2}{\sigma^2}\right]$ FOR TESTING $\sigma_1 = \sigma$ AGAINST $\sigma_1 > \sigma$

Example:[1] A Rifle Association is purchasing small arms ammunition for match purposes It is the desire of the rifle club that the dispersion in muzzle velocity of a lot of ammunition intended for match purposes be kept down to a practical minimum. Acceptance or rejection of an ammunition lot must, of course, be made on a sampling basis since the ballistic acceptance test is destructive in nature Moreover, for practical reasons acceptance of a given lot is to be on the basis of a single sample. The Association specifies that they are not willing to accept more than 5% of the lots whose standard deviation in muzzle velocity is 6 ft./sec. The ammunition manufacturer agrees that he will accept these terms provided not more than 5% of the lots whose standard deviation in muzzle velocity is 4 ft./sec will be rejected. Under these agreements, it is desired to know what sample size is necessary to provide the stated assurances for the Rifle Association and the ammunition manufacturer.

In this problem, $\alpha = .05$, $\beta = .05$, and $\lambda = 1.5$. Referring to Fig. 1, we find the required sample size is approximately 35.

On the other hand, if a sample size had already been set, the appropriate curve in Fig. 1 could be examined to determine whether it provided sufficient protection against the acceptance of inferior ammunition.

5. Power function of the F-test. In discussing the power function of the F-test we will focus our attention on the problem of comparing the standard deviations of two normal populations.

A. To determine whether or not the standard deviation, σ_1, of one normal population is greater than the standard deviation, σ_2, of another normal population. We choose a significance level, α, and compute $F = s_1^2/s_2^2$. If $F > F_\alpha$, where the percentage point F_α is determined by

$$\begin{aligned}(2)\quad &\frac{\Gamma[\frac{1}{2}(n_1 + n_2 - 2)]}{\Gamma[\frac{1}{2}(n_1 - 1)]\Gamma[\frac{1}{2}(n_2 - 1)]} (n_1 - 1)^{\frac{1}{2}(n_1-1)} (n_2 - 1)^{\frac{1}{2}(n_2-1)} \\ &\qquad \cdot \int_{F_\alpha}^{\infty} \frac{u^{\frac{1}{2}(n_1-3)}}{[(n_1 - 1)u + n_2 - 1]^{\frac{1}{2}(n_1+n_2-2)}}\, du = \alpha,\end{aligned}$$

we conclude that $\sigma_1 > \sigma_2$.

Our hypotheses are

$$H_0\colon \sigma_1 = \sigma_2$$

$$H_1\colon \sigma_1 = \lambda\sigma_2,\ (\lambda > 1).$$

To set up the power function of the F-test we note that:
If H_0 is true

$$Pr\{s_1^2/s_2^2 > F_\alpha\} = \alpha.$$

[1] This example is used to illustrate the use of the power of the χ^2-test and is not advocated as a most powerful sampling technique (See ref. [10]).

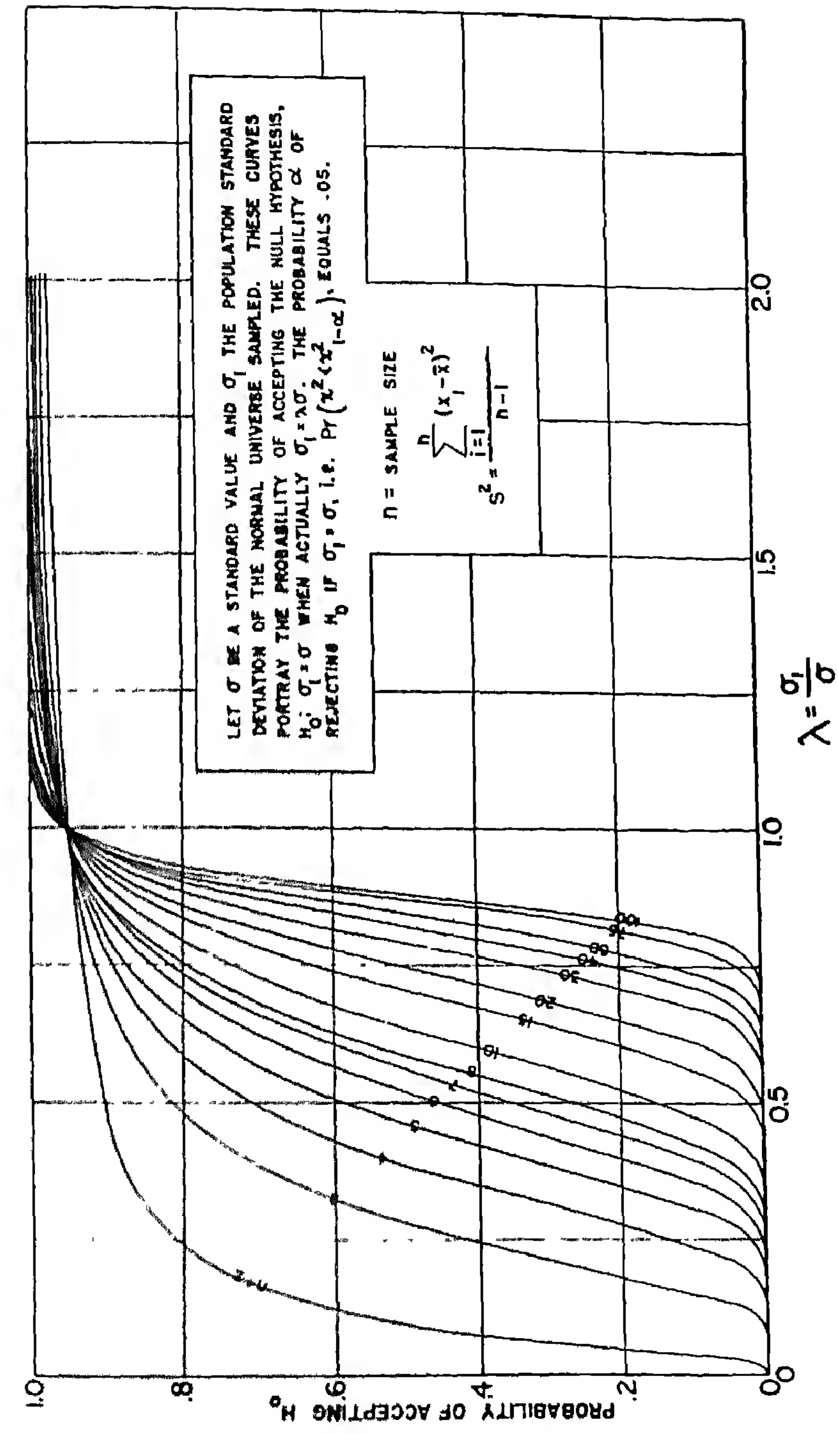

Fig. 2. Operating Characteristics of the χ^2-Test $\left[\chi^2 = \frac{(n-1)s^2}{\sigma^2}\right]$ for Testing $\sigma_1 = \sigma$ against $\sigma_1 < \sigma$

If H_1 is true

$$Pr\{s_1^2/s_2^2 > F_\alpha\} = 1 - \beta, \qquad (1 - \beta = \alpha \text{ if } \lambda = 1).$$

However, since

$$Pr\left\{\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} > F_{1-\beta}\right\} = 1 - \beta$$

or

$$Pr\{s_1^2/s_2^2 > \lambda^2 F_{1-\beta}\} = 1 - \beta,$$

we have the relation $\lambda^2 F_{1-\beta} = F_\alpha$ or $\lambda = \sqrt{\dfrac{F_\alpha}{F_{1-\beta}}}$.

Therefore, for a given Type I error, α, and various Type II errors, β, we can make use of the Table of Percentage Points of the F-Distribution [2] and compute sufficient points (λ, β) to plot the operating characteristics depicted in Figs. 3, 4, and 5 In these figures, α has been set at the practical level of .05.

It should be emphasized that the operating characteristics presented in this paper are applicable only when one is interested in the one-sided alternative that $\sigma_1 > \sigma_2$ and not $\sigma_1 < \sigma_2$. Under these circumstances, the exact formation of the F ratio will be set beforehand and will not depend upon test results (for example, placing the greatest mean square in the numerator). In those cases where one is interested in the two-sided alternative, a two-tail F-test such as described by H. Scheffé [3] should be used. It is hoped that at a later date operating characteristics of such a test calculated in a manner similar to the example in [3] will be presented.

Example: It became necessary for a manufacturer to make a choice between a new type casting and one produced under standard design practices. One of the bases of comparison was dispersion in tensile strength. It was considered that if the standard deviation of the standard casting were larger than the new type, definite preference should be given to the latter. When the question of a practical criterion for rejecting the standard casting was considered, it was decided that if its true standard deviation in tensile strength were actually $1\frac{1}{2}$ times that of the new type there should be a 90% chance of rejection. It would be of little practical importance to detect any ratio less than $1\frac{1}{2}$ in this particular case. It was also decided that the 5% significance level would suffice insofar as rejection of equal quality was concerned. A preliminary sample size of 20 was selected, and the question arose as to how well a sample of this size gave the protection desired.

The question can be answered immediately by reference to Fig. 3 (here s_1^2 is computed from the standard casting data, of course) where it is seen that a sample size of 20 will fail to detect the stated difference 47% of the time. In order to achieve the desired protection, it is seen at once from Fig. 3 that a sample size of over 50 will be necessary. The exact sample size, determined with the aid of the formulas above, is found to be 54.

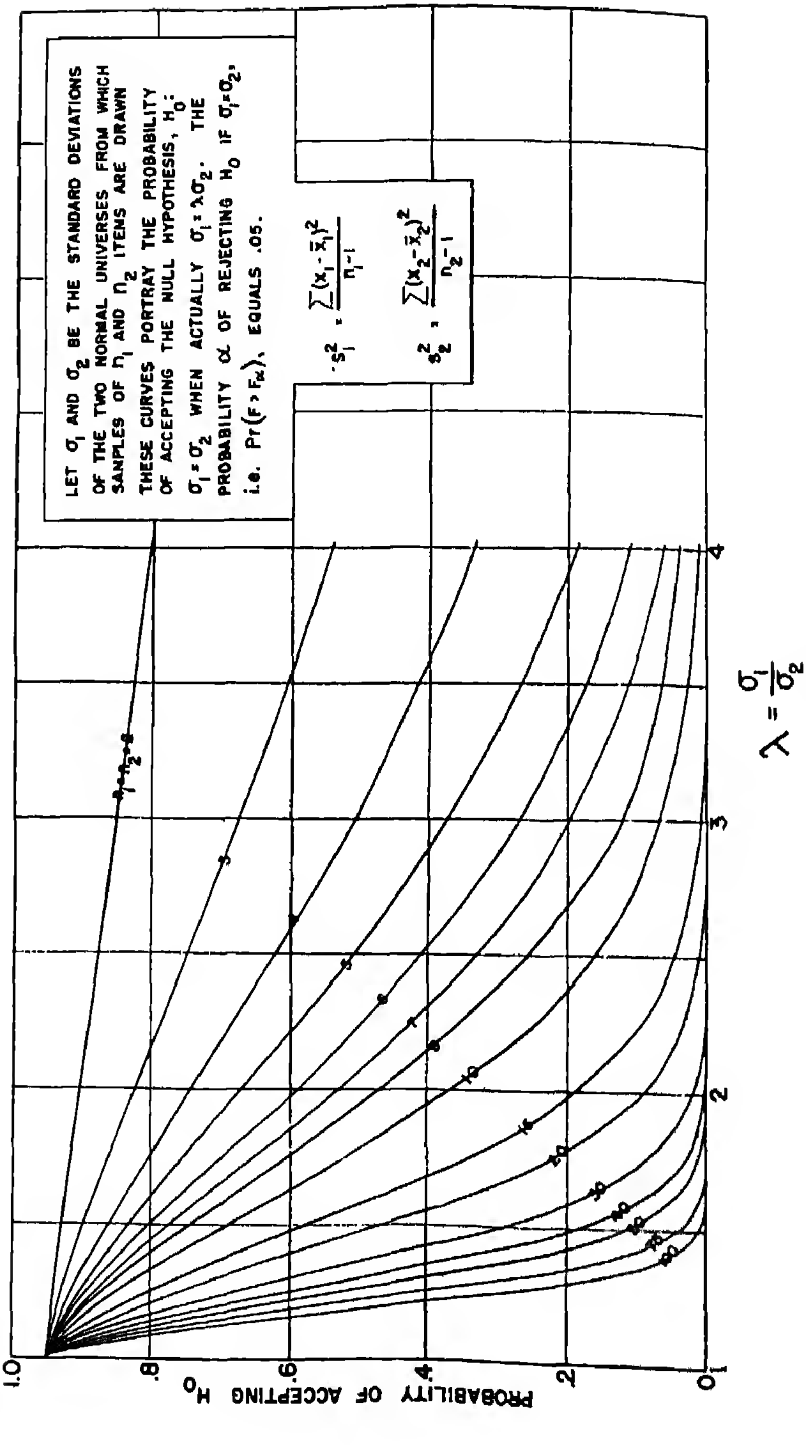

FIG. 3. OPERATING CHARACTERISTICS OF THE F-TEST $\left[F = \frac{s_1^2}{s_2^2}\right]$ FOR TESTING $\sigma_1 = \sigma_2$ AGAINST $\sigma_1 > \sigma_2$
$(n_1 = n_2)$

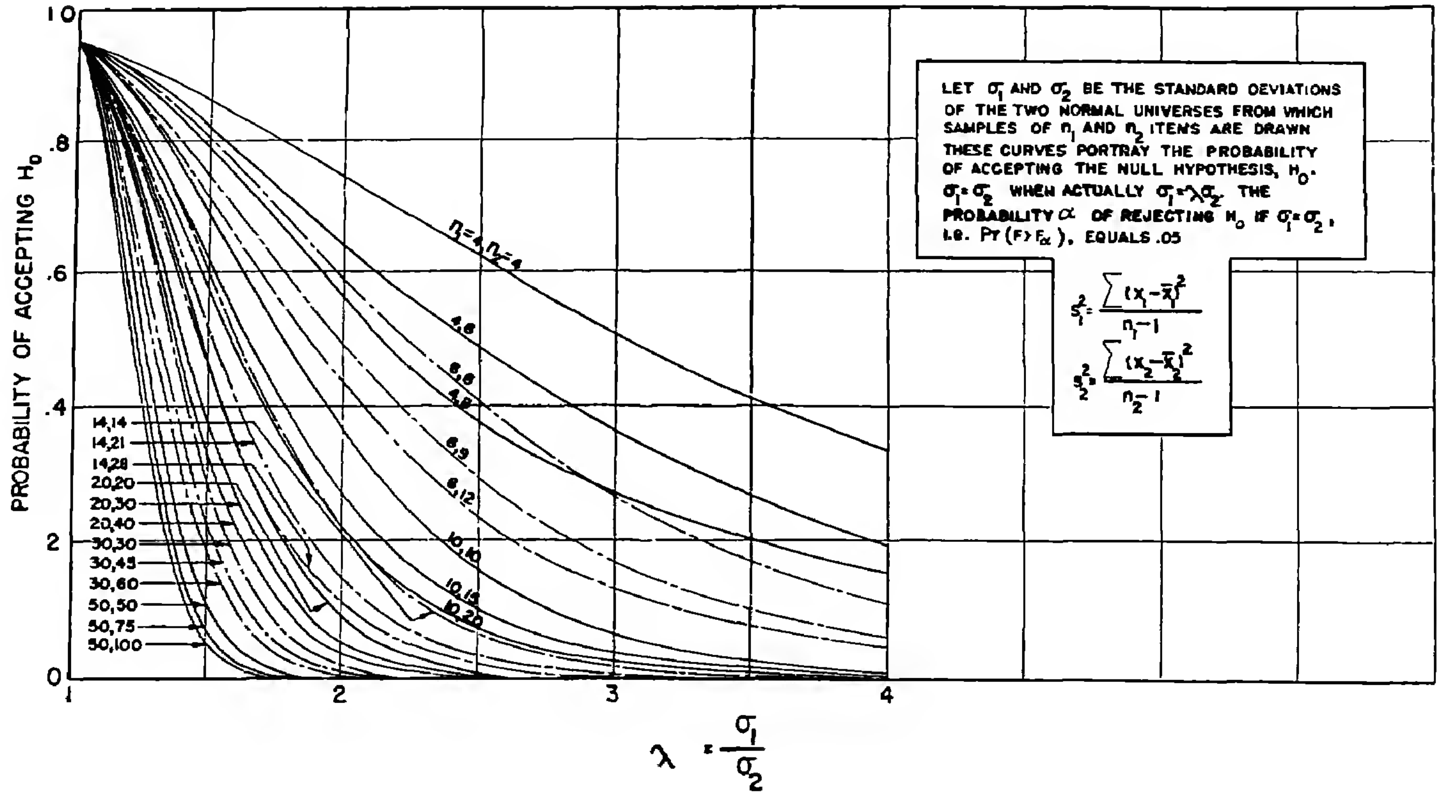

FIG. 4. OPERATING CHARACTERISTICS OF THE F-TEST $\left[F = \frac{s_1^2}{s_2^2}\right]$ FOR TESTING $\sigma_1 = \sigma_2$ AGAINST $\sigma_1 > \sigma_2$

$(n_1 = n_2\,,\ 3n_1 = 2n_2\,,\ 2n_1 = n_2)$

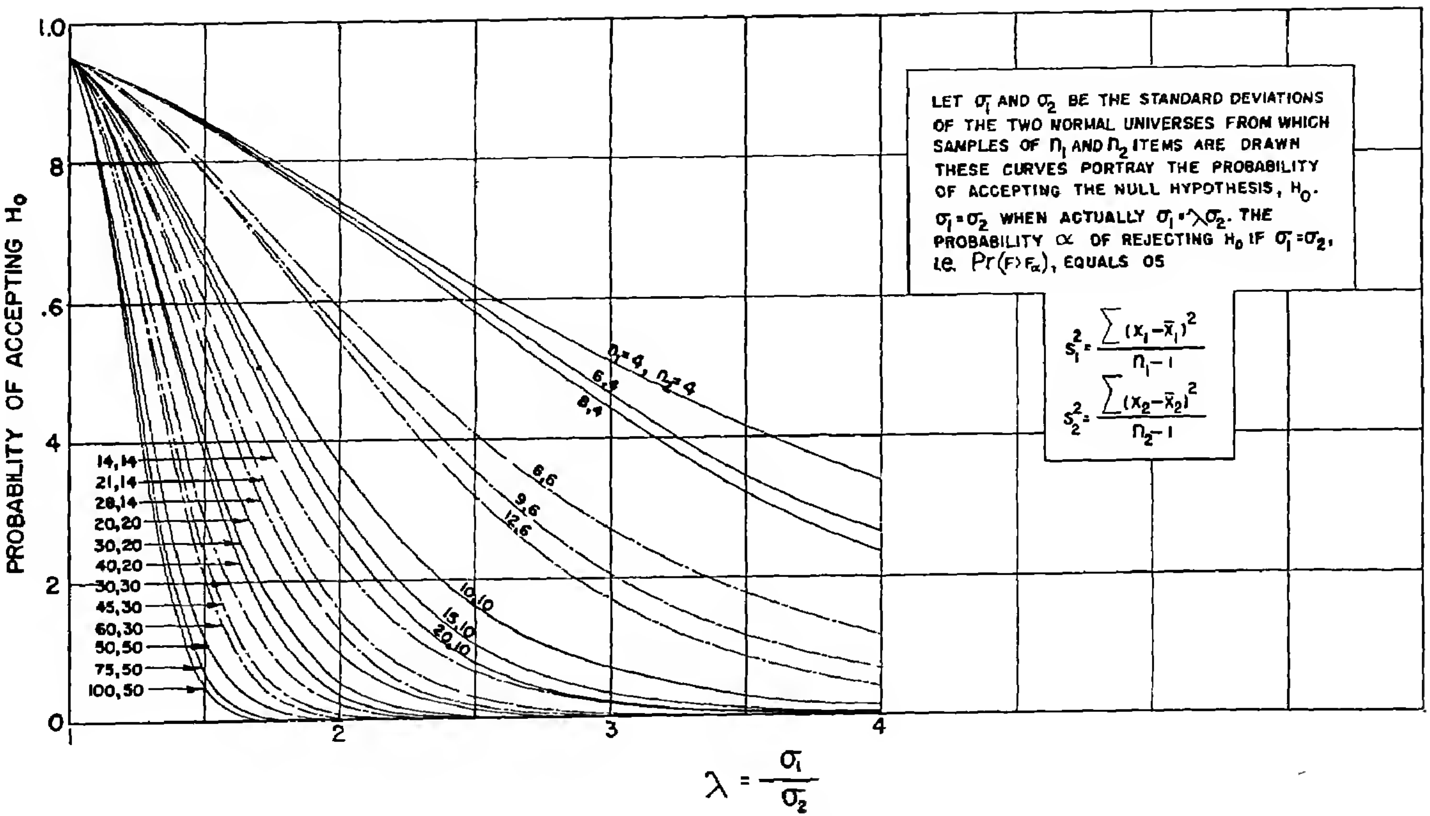

FIG. 5. OPERATING CHARACTERISTICS OF THE F-TEST $\left[F=\frac{s_1^2}{s_2^2}\right]$ FOR TESTING $\sigma_1=\sigma_2$ AGAINST $\sigma_1>\sigma_2$
$(n_1=n_2,\ 2n_1=3n_2,\ n_1=2n_2)$

B. *Analysis of variance.* We shall consider the analysis of variance layout where a sample of n items is drawn from each of m normal populations with common variance σ^2. It is required to decide on the basis of the sample results whether or not there is any variation among the true means of the m normal populations sampled.

Let x_{ij} be the jth item drawn at random from the ith population,

$$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}, \quad \text{and} \quad \bar{x} = \frac{1}{m}\sum_{i=1}^{n} \bar{x}_i .$$

The F-test utilizes the comparison of the variation among the sample means (external variance) with that among the items within the samples (internal variance) in order to test the equality of population means by making use of the ratio

$$F = \frac{n \sum_{i=1}^{m} (\bar{x}_i - \bar{x})^2 m(n-1)}{\sum_{i,j} (x_{ij} - \bar{x}_i)^2 (m-1)} .$$

If $F > F_\alpha$, where F_α is defined as in 5.A, we conclude that the population means are not equal.

In our approach we will assume that the m true lot means represent a sample from a super-population, also normal, with variance equal to $\theta^2\sigma^2$. Since the sampling variance of the means is σ^2/n, the total variance among the sample means equals

$$\sigma^2/n + \theta^2\sigma^2 = \lambda^2\sigma^2/n, \qquad (\lambda^2 = 1 + n\theta^2).$$

Hence, our hypotheses are

$$H_0: \theta = 0$$

$$H_1: \theta > 0.$$

Since F/λ^2 follows the F-distribution with $m - 1$ and $m(n - 1)$ degrees of freedom the operating characteristic, i.e. the probability for various θ of accepting H_0, may be obtained from the curves already graphed by setting $n_1 = m$, $n_2 = nm - m + 1$, and $\lambda^2 = 1 + n\theta^2$.

In the design of experiments when the number of populations is indefinite (for example, daily tests) and the total sample size mn is limited, the above procedure will enable one to determine what values of m and n give the most powerful operating characteristic for the given amount of sampling. For example, for $mn = 24$ operating characteristics for all possible pairings were computed and charted. They were observed to cross one another, each combination in turn becoming most powerful for a given interval of θ. The following table gives the best pairings for various intervals of θ:

m	n	θ
2	12	00– 32
3	8	.32– 60
4	6	.60– .91
6	4	91–1.37
8	3	1.37–2.50
12	2	2.50–

In contrast to the above discussion, mention should be made of P. C. Tang's approach [4] to the power function of the analysis of variance. The basic difference lies in the method of expressing the alternative hypothesis. Tang expresses it in terms of the variance of a finite number of population means We express it in terms of normally distributed population means. We believe our approach has considerable practical value in control chart analyses where we are interested in the quality of the flow of production of a large number of lots. In addition, our approach obviates the difficulties imposed by the non-central χ^2-distribution.

6. Power function of the normal test.

A. The statistic $u = \dfrac{\sqrt{n}(\bar{x} - a)}{\sigma_1}$ is used to accept or reject the hypothesis that the mean, μ, of the normal population sampled, is some specified standard level, a, when the population standard deviation is known (for example, from past data).

Our hypotheses are

$$H_0: \mu = a$$

$$H_1: | \mu - a | = \lambda\sigma_1, \ (\lambda > 0).$$

To test the hypothesis $\mu = a$, we choose a significance level, α, and compute u. If $| u | > u_\alpha$, where the percentage point, u_α, is determined by

$$\frac{1}{\sqrt{2\pi}} \int_{-u_\alpha}^{+u_\alpha} e^{-\frac{1}{2}\chi^2} d\chi = 1 - \alpha, \tag{3}$$

we reject H_0 and conclude that $\mu \neq a$.

To set up the power function we note that:

If H_0 is true

$$Pr\{-u_\alpha < u < +u_\alpha\} = 1 - \alpha$$

If H_1 is true

$$Pr\left\{-u_\alpha < \frac{\sqrt{n}(\bar{x} - a)}{\sigma_1} < u_\alpha\right\} = \beta, \qquad (1 - \beta = \alpha \text{ if } \lambda = 0),$$

$$= Pr\left\{-u_\alpha + \lambda\sqrt{n} < \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma_1} < u_\alpha + \lambda\sqrt{n}\right\}$$

where $\lambda = \dfrac{| \mu - a |}{\sigma_1}$

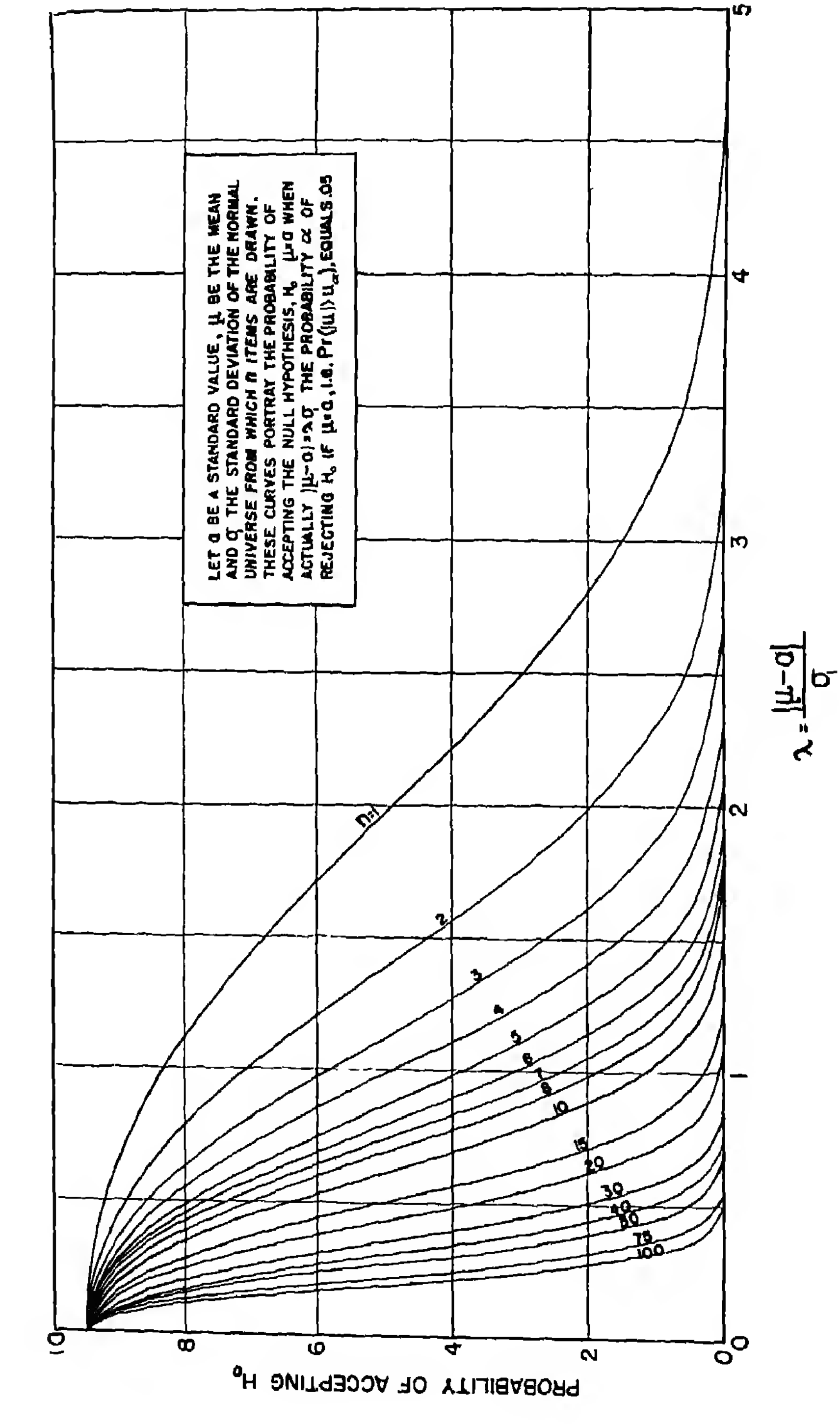

FIG. 6. OPERATING CHARACTERISTICS OF THE NORMAL TEST $\left[u = \frac{\sqrt{n}(\bar{x} - a)}{\sigma_1}\right]$ FOR TESTING $\mu = a$ AGAINST $\mu \neq a$

In the latter expression the statistic $\frac{\sqrt{n}(\bar{x} - \mu)}{\sigma_1}$ is normally distributed with zero mean and unit variance. The required probabilities are found easily from tables of areas under the normal frequency curve. By computing enough points (λ, β) the operating characteristics depicted in Fig. 6 were constructed.

It should be noted that the β corresponding to a pair of values n' and λ' may be obtained from any other operating characteristic by use of the relation $\lambda = \lambda'\sqrt{n'/n}$ For example, if it is desired to find the Type II error for a sample size of $n' = 12$ and $\lambda' = 1$, select any operating characteristic, say for $n = 3$, as the reference curve, compute $\lambda = 1\sqrt{12/3} = 2$, and find from the curve for $n = 3$ that $\beta = .07$ In Fig. 6, however, individual operating characteristics are plotted for convenience and to provide a picture of the comparative efficiency of various sample sizes.

Example. Pressure-measuring instruments are being tested against a standard level It has been decided that instruments whose true mean reading is as much as 10 pounds per square inch away from the standard level should be rejected 95% of the time. On the other hand only 5% of instruments whose true mean reading equals that of the standard should be rejected From past data, it is known that all test instruments of the type being considered have a stable standard deviation of 5 psi. If rejection or acceptance is to occur on the basis of a single sample and the normal criterion of significance, what sample size should be chosen to accomplish this purpose? Referring to Fig. 6 with $\lambda = 10/5 = 2$ it is seen that a sample size of 4 provides the required assurance.

B. In sampling two normal populations π_1 and π_2, the statistic

$$u = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$$

is used to accept or reject the hypothesis that $\mu_1 = \mu_2$. For generality it will be assumed that the population standard deviations σ_1 and σ_2 may not be equal, although they are known accurately.

Our hypotheses are

$$H_0: \mu_1 = \mu_2$$

$$H_1: |\mu_1 - \mu_2| = \lambda\sigma_1 .$$

Significance is determined in the same manner as in 5.A., and the power function is set up in identical fashion. The value β is found to be the area under the standardized normal curve between the abscissas.

$$\pm u_\alpha + \lambda \sqrt{\frac{n_1 n_2}{k^2 n_1 + n_2}}$$

where $\sigma_2 = k\sigma_1$. The value of β may easily be read from Fig. 6 for any λ', n_1, n_2, and k by selecting the curve for a convenient sample size, n, on Fig. 6 and taking

$$\lambda = \frac{\lambda'}{\sqrt{n}} \sqrt{\frac{n_1 n_2}{k^2 n_1 + n_2}}.$$

7. Power function of the t-test.

A. The statistic $t = \dfrac{\sqrt{n}(\bar{x} - a)}{s}$ is used to accept or reject the hypothesis that the mean, μ, of the normal population sampled, is equal to some specified level, a, when the population standard deviation, σ_1, is unknown.

Our hypotheses are

$$H_0 : \mu = a$$

$$H_1 : |\mu - a| = \lambda\sigma_1, \ (\lambda > 0).$$

In order to test the hypothesis $\mu = a$ we choose a significance level, α, and compute the statistic $t = \dfrac{\sqrt{n}(\bar{x} - a)}{s}$. If $|t| > t_\alpha$, where the percentage point, t_α, is determined by

$$\text{(4)} \qquad \frac{\Gamma(n/2)}{\Gamma[\frac{1}{2}(n-1)]\sqrt{n-1}\sqrt{\pi}} \int_{-t_\alpha}^{+t_\alpha} \left(1 + \frac{x^2}{n-1}\right)^{-n/2} dx = 1 - \alpha,$$

we reject H_0 and conclude that $\mu \neq a$.

To set up the power function we note that:

If H_0 is true

$$\Pr\{-t_\alpha < t < +t_\alpha\} = 1 - \alpha.$$

If H_1 is true,

$$\Pr\{-t_\alpha < t < t_\alpha\} = \beta, \qquad (1 - \beta = \alpha \text{ if } \lambda = 0).$$

However, we have the identity

$$Pr\left\{-t_\alpha \frac{s}{\sigma_1} + \lambda\sqrt{n} < \frac{\sqrt{n}(\bar{x} - \mu)}{\sigma_1} < +t_\alpha \frac{s}{\sigma_1} + \lambda\sqrt{n}\right\} = Pr\{-t_\alpha < t < +t_\alpha\}$$

where $\lambda = \dfrac{|\mu - a|}{\sigma_1}$. Hence, for any fixed $\dfrac{s}{\sigma_1}$, the above probability may be denoted by say $h(s/\sigma_1)$ or, using the notation of section 4, $h\left(\sqrt{\dfrac{\chi^2}{n-1}}\right)$, and evaluated as the area under the standardized normal curve between the abscissas indicated. Then

$$\beta = \int_0^\infty h\left(\sqrt{\frac{\chi^2}{n-1}}\right) f(\chi^2)\, d(\chi^2)$$

where $f(\chi^2)$ is the probability density function of χ^2 for $n - 1$ degrees of freedom. This is one method of evaluating β and it was used for calculating the operating characteristics for $n < 5$.

It has been noted that such a formula had been employed by Neyman and Tokarska [6] in calculating Type II errors where only one tail of the t-curve is used as the region of rejection. Probabilities calculated in this manner are

provided by Neyman and Tokarska for degrees of freedom $n = 1$ to 30 and Type I errors of 01 and .05. As soon as the area in one tail of the non-central t-distribution becomes negligible these curves are equivalent to the test treated herein with an α of .02 and 10 respectively. An idea of the critical values of λ at which this occurs may be obtained from a table in a succeeding footnote in which they are quoted for $\alpha = .05$. The values are surprisingly small, such that almost all of Neyman's figures can be interpreted for a two-tail region of rejection.

Using C. C. Craig's development of the non-central t [7] we obtain[2]

$$\beta = Pr\left\{-t_\alpha < \frac{\sqrt{n}(\bar{x} - \mu)/\sigma_1 + \sqrt{n}\lambda}{s/\sigma_1} < +t_\alpha\right\}$$

$$= e^{-\frac{1}{2}n\lambda^2} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}n\lambda^2)^r}{r!} I\left[(r + 1/2), \tfrac{1}{2}(n - 1);\ \frac{t_\alpha^2}{n - 1 + t_\alpha^2}\right]$$

where $I(p, q;x)$ represents the Incomplete-Beta Function Ratio [7]. This may be conveniently used for those values of n where the necessary values are obtainable from Tables of the Incomplete-Beta Function ratio [8] and for small values of λ where the above series converges rapidly.

The method actually used for $n > 4$, however, made use of the tables prepared by Johnson and Welch [9]. Replacing their λ by π to avoid confusion with our notation, these tables give values of π tabulated against f, t, and ϵ such that

$$Pr\left\{t = \frac{z + \delta}{\sqrt{w}} > t_0\right\} = \epsilon$$

where z is a normally distributed variate with zero mean and unit variance, fw is distributed according to the χ^2-distribution with f degrees of freedom, and $\delta = t_0 - \pi\sqrt{1 + t_0^2/2f}$. We want

$$\beta = 1 - \Pr\{t < -t_\alpha\} - \Pr\{t > t_\alpha\}.$$

For those values of λ and n for which $\Pr\{t < -t_\alpha\}$ is negligible[3] we can, for any given ϵ, take $t_0 = t_\alpha$ and $f = n - 1$ and read π from the tables, then deter-

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[2] It should be noted that Craig's formula as published is in error in having $\frac{1}{2}(r + 1)$ as the parameter in the incomplete beta function instead of $r + \frac{1}{2}$.

[3] Values of λ for which $Pr\{t < -t_{.05}\} = 005$ are listed below.

$f = n - 1$	λ
4	.34
5	.30
6	.27
7	.25
8	.23
9	216
16	.159
36	103
144	.051
∞	.000

mine δ and finally λ from the relation $\lambda = \delta/\sqrt{n}$ After computing $\beta = 1 - \epsilon$, the point (λ, β) on the operating characteristic may be graphed. At the few places where $\Pr\{t < -t_\alpha\}$ is not negligible and β is needed we can for a given λ take

$$\pi = \frac{t_0 - \delta}{\sqrt{1 + t_0^2/2f}}$$

and then by reading π for various values of ϵ, f, t_0 make an inverse interpolation for ϵ thus setting values for $\Pr\{t > -t_\alpha\}$ and $\Pr\{t > t_\alpha\}$. Finally

$$\beta = \Pr\{t > -t_\alpha\} - \Pr\{t > +t_\alpha\}.$$

It was found that for $n > 10$ a good approximation for computing operating characteristics is given by

$$\beta = \Pr\{-t_\alpha + \lambda\sqrt{n} < t < +t_\alpha + \lambda\sqrt{n}\}$$

in which the variable t is distributed as central t with $n - 1$ degrees of freedom. This formula proved to be quite useful in preparation of the operating characteristics for the t-test.

Fig 7 presents operating characteristics of the t-test calculated by these methods. It should be noted that in using the t-test, alternative hypotheses are expressed as so many multiples of the unknown population standard deviation away from the level stated in the null hypothesis In some applications the alternatives may be naturally so expressed. In many applications, however, it may be desired to control the distance $\mu - a$ regardless of the standard deviation of the lot sampled. In this case, one could place confidence limits on the estimate of σ, determine the λ value corresponding to each estimate, and finally obtain limits on the sample sizes or risks involved [4]

B. For the case of two normal populations, the statistic

$$t = \frac{\bar{x}_1 - \bar{x}_2}{s_{12}\sqrt{1/n_1 + 1/n_2}}$$

is used to accept or reject the hypothesis that $\mu_1 = \mu_2$ when the two normal population standard deviations are unknown but equal to say, σ_1.

Our hypotheses are

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : |\mu_1 - \mu_2| = \lambda\sigma_1 .$$

Significance is determined in the same manner as in par 6.A., and, by reasoning similar to that in the preceding section, it is found that β for a given λ' can be read from Fig. 7 by taking

$$\lambda = \frac{\lambda'}{\sqrt{n}} \sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$

[4] For a test of this nature in which the power of the test depends only on the absolute value of the distance $\mu - a$ see [10]

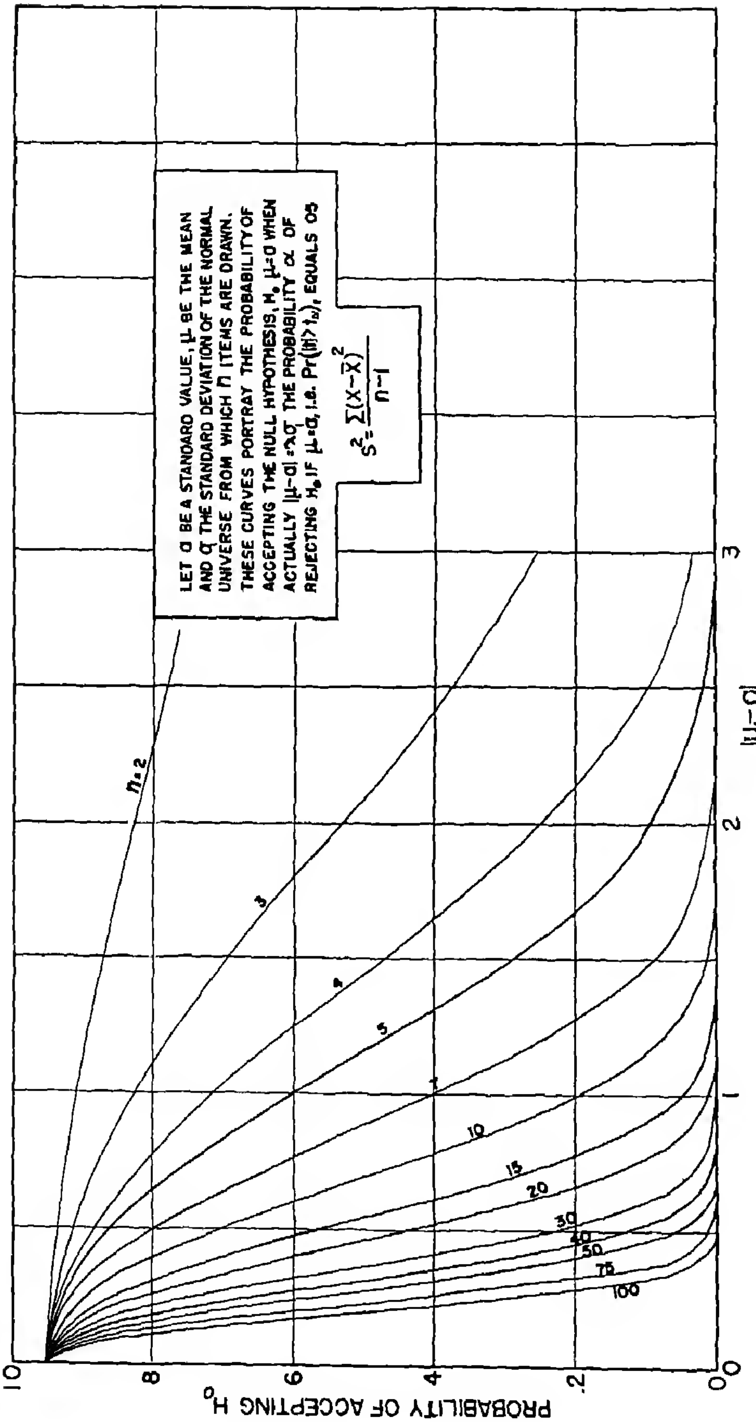

FIG. 7. OPERATING CHARACTERISTICS OF THE t-TEST $\left[t = \frac{\sqrt{n}(\bar{x} - a)}{s}\right]$ FOR TESTING $\mu = a$ AGAINST $\mu \neq a$

and $n = n_1 + n_2 - 1$. Before a statistical test of this nature is applied the data should be examined to verify consistency with the assumption that $\sigma_1 = \sigma_2$

Example: An analysis of the difference in tensile strength between two types of castings is being conducted. A sample of 10 items is selected from each type of casting and the t-test employed to establish superiority of one over the other. Experience has shown that the variability in tensile strength for one type of casting is comparable to that of the other type. If α is set equal to .05, what percentage of the time would our significance test fail to detect a superiority of one standard deviation in tensile strength? $n = 10 + 10 - 1 = 19$ and $\lambda = .513$. Referring to Fig. 7 for this λ and n, it is seen that the percentage β is approximately 45.

In this paper we have presented power curves or operating characteristics of the common significance tests employed but a single sample of items. The power of the tests obtained here does not represent the limit that can be obtained for the average amount of inspection performed, say, over many consecutive lots. Tests, sequential in character [11], have been shown to be much more efficient. Nevertheless, single sampling is often the only practical procedure available. Again, the data may be brought to the analyst as single sample results collected supplementary to other purposes or prescribed by a standard procedure. Finally, in performing a significance test, it is quite important to be able to give constructive advice when the data indicate practical differences although no statistical significance is found [6]

Although sequential tests using variables have been devised, no investigation of double sampling schemes for variables similar to the Dodge-Romig [12] plans for attributes has, as yet, been designed with the exception of [9]. It is believed, however, that such plans would have considerable application for industry in combining efficiency with practicability.

The graphs of the operating characteristics in this report have been made by calculating a sufficient number of points to draw them in by use of French curves. Considering this method of plotting slight error should be allowed for in reading probabilities of acceptance from the graphs, especially where the curves are steep.

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MINIMAL VARIANCE AND ITS RELATION TO EFFICIENT MOMENT TESTS

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1. Summary. When a curve is fitted to a set of data by moments, the usual procedure used in testing the hypothesis that the population is of the given form with the parameters as computed from the moments is to compare the higher moments with their expected values as determined by the hypothesis Generally speaking, moments about the mean are computed although the reason for this is not clear. To shed some light on this question, the sample given in the introduction is fitted to two curves. Moments about various points are compared with their expected values and the discrepancy in standard units examined. This discrepancy is found to vary widely and to have a maximum. The notion of equivalent moment tests is introduced, and on this basis the most efficient moment test is defined in such a way that of all equivalent moment tests, this one is most likely to reject a false hypothesis.

For any moment it is shown that there is a point about which its variance is a minimum. The conditions are found which determine the position of this point for second and third moments. It is proved that for symmetrical populations the variance is minimal when the moments are computed about the mean of the population. If the population is an asymmetrical Pearson frequency function, it is proved that the point about which the third moment variance is minimal differs more from the mean than does the corresponding point for second moments. The condition is pointed out for which this is true in the general case.

The third and fourth standard semi-invariants of second moments of minimal variance are computed and compared to those of the second moment about the mean. The ratios of these are displayed for some populations to illustrate how this may be used to investigate when the approach to normality is more rapid in one case than in the other. Some examples are presented to contrast these and other tests.

2. Introduction. In testing the hypothesis that a given set of observations is a random sample from a completely specified population (either a priori or specified by a consideration of the sample), generally the Chi-square test is applied or certain functions of the moments are compared with their expected values and the significance of their departure as determined by the hypothesis is examined.

In the Neyman-Pearson theory it is required that the functional form be known. The hypothesis then is some statement concerning the parameters. The main principle there used is that the test used should be such that, while keeping the probability of rejecting the hypothesis when true at a certain sig-

nificance level, it will minimize the chance of accepting the hypothesis when some alternative is true.

However, if the functional form is regarded as unknown, the alternative hypotheses are then usually unknown. The test then must be one that does not depend on alternatives In the light of incomplete knowledge of the distribution of sample statistics, and since moments of moments are practically the only ones known, we shall here use the principle of comparing observed moments with their expected values. It is known that the distribution of moments in large samples is asymptotic to the normal distribution if the appropriate moments of the population exist [1]. Here we shall confine ourselves to such populations and large samples.

To introduce the idea which underlies the theory here presented, consider a simple example. Suppose a sample is given and the hypothesis is of the form $f(x, \theta)$ with $\theta = \theta_0$. Furthermore, suppose the first moment of the sample is equal to its expected value. If a second-moment test is used, this means that one computes the arithmetic mean of the squares of the deviations of the elements of the sample about some point, and compares this with the theoretical moment about the same point. Generally speaking, the point used is the mean of the population or the mean of the sample. However, the point may be chosen in any manner. For each such choice a test can be devised such that the probability of rejecting the hypothesis when true is ϵ. All such tests are called equivalent moment tests. Among these equivalent moment tests, one particular second-moment will have the minimal variance. This one is here called the most efficient moment test.

This test has the property that the range of values of the second moment for which the hypothesis is accepted is as small as possible. Thus of all equivalent second-moment tests, this one is most likely to reject a false hypothesis.

This idea may be easily extended to moments of higher order, in all of which the concept of minimal variance is fundamental. The point of view may be taken that the point about which the moments are computed should be such that the variance is a minimum, or what is equivalent, the variance of moments about the origin is minimized by choosing the origin properly

An example is here presented to bring this out more clearly. A sample of 1,000 items is given and fitted by the first two moments to two different frequency functions (The sample items are not given here; they are to be found in *Tables for Statisticians* [2]). The third and fourth moments have been computed and the discrepancies in standard units as determined by the hypotheses are exhibited in a table.

This sample of 1,000 items considered as a sample from an infinite population has these moments:

$$m_1' = 139.288$$
$$m_2' = 19692.452$$
$$m_3' = 2827467\ 388$$
$$m_4' = 412561061.04$$

By fitting the first two moments of the sample to curve A,

$$y = \frac{a^{n+1}}{\Gamma(n+1)} x^n e^{-ax}$$

we get $a = 0.4781516735$ and $n = 65.60079029$; to curve B,

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

we get $\mu = 139.288$ and $\sigma^2 = 291\ 305056$.

The discrepancy between the observed and theoretical rth moment about any point is measured by

$$t = \frac{m_r'' - \mu_r''}{\sqrt{\dfrac{\mu_{2r}'' - \mu_r''^2}{n}}}$$

in which m_r'' is the rth moment of the sample of n about this point, and μ_r'' is the rth moment of the population about the same point.

The values of $|t|$ have been computed corresponding to various points for the third and fourth moments. These are exhibited in four tables, given below.

Examination of the table for the discrepancy between the observed and theoretical third moments for curve B, shows that when this moment is computed about $x = 0$, the hypothesis is accepted at the 1% level, this is also true for $x = 39.3$, but for $x = 139.3$ the hypothesis would be rejected at that level. It is evident that some rule must be established to decide what point is to be used to make the test.

If the curve is fitted by the first two moments the value $m_3'' - \mu_3''$ is the same for every point. This is easily demonstrated, for if m_3'' and μ_3'' are measured about a point h units to the right of the origin, $m_3'' = m_3' - 3hm_2' + 3h^2m_1' - h^3$ and $\mu_3'' = \mu_3' - 3h\mu_2' + 3h^2\mu_1' - h^3$. Now, $m_2' = \mu_2'$ and $m_1' = \mu_1'$. It follows that $m_3'' - \mu_3'' = m_3' - \mu_3'$

The maximum value of $|t|$ is attained when the variance of third moments is a minimum. In this manner it is assured that the range of values for which the third moment is accepted shall be a minimum.

If the third moments agree, or the agreement is sufficiently close such that the hypothesis cannot be rejected, $m_4'' - \mu_4''$ is constant or varies only slightly from point to point, so that minimizing the variance yields the maximum value of t.

As is seen from the tables above, when the moments are compared at the different points, the hypothesis may be accepted for one point and rejected for another. By the principle of using the point which yields the minimal variance, the hypothesis will be rejected more often than for other points. Thus, of all equivalent moment tests, this one is most likely to reject a false hypothesis

The problem of determining for various moments how the origin may be chosen such that the variance of the distribution of these moments shall be a minimum is now considered.

3. First moments. In the case of the first moment, whose expected value is the mean of the population, the variance is given by $\frac{1}{n}(\mu_2' - \mu_1'^2)$. It is obvious that the choice of origin does not affect the variance of the first moment, since it is well known that $\mu_2' - \mu_1'^2$ is invariant with respect to choice of origin.

4. Minimal variance of second moments. The variance of second moments about an arbitrary origin is $\frac{1}{n}(\mu_4' - \mu_2'^2)$. Expressed in terms of μ_1' and central

TABLES

Curve A.

Third moments		Fourth moments	
Point	t	Point	t
0	.0365	0	.197
50	.084	50	.697
100	.33	100	4.74
120	.77	120	14.17
130	1.28	130	26.76
140	1.91	140	49.03
142	1.95	145	45.26
145	1.90	150	42.89
150	1.60	160	21.31
160	.95	180	6.25
170	.57	200	2.51
180	.37	300	.183
200	.18		

Curve B.

Third moments		Fourth moments	
Point	t	Point	t
0	085	0	.02
39.3	.19	39.3	.13
89.3	.69	99 3	.88
109.3	1 16	109.3	1.09
119.3	2.39	119.3	2.00
129.3	4.05	129.3	3.18
139.3	5.57	133.3	3.83
149.3	4.05	135.3	3.96
159.3	2.39	137.3	3.93
169.3	1.16	139.3	3.67
179.3	.98	140.3	3.46
189.3	.69	143 3	2.72
199.3	.50	148.3	1.59
209.3	.38	159.3	39
239.3	.19	179.3	.13
		239.3	.07

moments, this may be written

(1) $$\mu_2(m_2') = \frac{1}{n}(\mu_4 - \mu_2^2 + 4\mu_3\mu_1' + 4\mu_2\mu_1'^2).$$

Here it is evident that the variance of second moments does depend on the choice of origin, and is not invariant under translation.

The minimum value of $\mu_2(m_2')$ is given by $\mu_1' = -\frac{\mu_3}{2\mu_2}$ and is $\frac{1}{n}\left(\mu_4 - \mu_2^2 - \frac{\mu_3^2}{\mu_2}\right)$. Then we may write

(2) $$\mu_2(m_2^*) = \frac{1}{n}\left(\mu_4 - \mu_2^2 - \frac{\mu_3^2}{\mu_2}\right).$$

Throughout this paper m_2^* denotes the second moment of the sample about an origin chosen such that $\mu_1' = -\frac{\mu_3}{2\mu_2}$, which is the value of μ_1' which minimizes (1); m_2^0 denotes the second moment about an origin chosen such that $\mu_1' = 0$; m_2 denotes the second moment about the mean of the sample. It may be noted that in large samples the distributions of m_2^0 and m_2 are approximately the same.

It is clear from (2) that if $\mu_3 = 0$, or, if the population is symmetric, i.e. $f(-x) = f(x)$, then $\mu_2(m_2^*) = \mu_2(m_2^0)$. However, if $\mu_3 \neq 0$ then $\mu_2(m_2^*) < \mu_2(m_2^0)$.

5. A moment inequality. Since the quantity given by (2) is essentially non-negative, an inequality is obtained valid for any distribution in which the first four moments exist, viz.

$$\mu_4 - \mu_2^2 - \frac{\mu_3^2}{\mu_2} \geq 0, \qquad \mu_2 \neq 0 \tag{3}$$

or in standard moments

$$\alpha_4 - \alpha_3^2 - 1 \geq 0 \tag{4}$$

This is a stronger inequality than the one given by Bertelsen [3], i.e. $\alpha_3^2 - \alpha_4 - 2 < 0$ or the one generally known, $\alpha_4 \geq \alpha_3^2$, [4]. This inequality, however, was known to K. Pearson [5, p 432], although he derived it from a different point of view.

6. Minimal variance of higher moments. The variance of the distribution of rth moments of random samples about an arbitrary origin always has a minimum. The variance of m_r' is given by

$$\mu_2(m_r') = \frac{1}{n}(\mu_{2r}' - \mu_r'^2). \tag{5}$$

This expression when expanded in powers of μ_1' is always a polynomial of even degree with the coefficient of the highest power a positive number. Furthermore, by differentiating $\mu_2(m_r')$ with respect to μ_1' and equating the derivative to zero, the value of μ_1' which minimizes $\mu_2(m_r')$ will be found among the solutions of that equation

For third moments of samples the variance is given by

$$\mu_2(m_3') = \frac{1}{n}[\mu_6' - \mu_3'^2]$$

which, when expressed in terms of moments about the mean and powers of the mean, becomes

$$\mu_2(m_3') = \frac{1}{n}[\mu_6 - \mu_3^2 + 6(\mu_5 - \mu_3\mu_2)\mu_1' + (15\mu_4 - 9\mu_2^2)\mu_1'^2 + 18\mu_3\mu_1'^3 + 9\mu_2\mu_1'^4]. \tag{6}$$

Differentiating with respect to μ_1' and equating to zero, we have

$$(7) \qquad 6\mu_2\mu_1'^3 + 9\mu_3\mu_1'^2 + (5\mu_4 - 3\mu_2^2)\mu_1' + (\mu_5 - \mu_3\mu_2) = 0.$$

By straightforward application of the methods of solving cubics, it is easy to show by means of (3) that (7) has one real root only, which moreover is $\lesseqgtr -\frac{\mu_3}{2\mu_2}$ according as

$$\alpha_5 - \alpha_3(\tfrac{5}{2}\alpha_4 - \tfrac{3}{2}\alpha_3^2 - 1) \gtreqless 0.$$

Since it can also be shown by means of (3) that the second derivative of (6) is positive, this root of (7) will minimize $\mu_2(m_3')$.

These facts demonstrate:

THEOREM I. *The point about which the arithmetic mean of the cubes of the variates has minimal variance is to the right, at, or to the left of the corresponding point for the squares according as*

$$(8) \qquad \alpha_5 - \alpha_3(\tfrac{5}{2}\alpha_4 - \tfrac{3}{2}\alpha_3^2 - 1) \lesseqgtr 0.$$

By examination of (7) it is readily seen that if $\alpha_5 = \alpha_3$ or if the population is symmetric, the real root will be zero; so that for such a population the variance of third moments is a minimum when moments are taken about the mean of the population. If $\alpha_5 \neq \alpha_3$ the variance of third moments will be a minimum when taken about some other point

For fourth moments of samples the variance is of the sixth degree in μ_1' and its derivative therefore of the fifth degree. There is not much to be said in a general way except that if $\alpha_7 = \alpha_4\alpha_3$ or if the population is symmetric, $\mu_1' = 0$ will cause this derivative to vanish.

If the distribution is a Pearson frequency function, from the recursion formula for the moments [6, p 24],

$$\alpha_5 = \alpha_3\left(\frac{2\alpha_4 + 4 + 2\delta}{1 - \delta}\right)$$

where

$$\delta = \frac{2\alpha_4 - 3\alpha_3^2 - 6}{\alpha_4 + 3}.$$

The criterion (8) can be written

$$(9) \qquad \alpha_3\left(\frac{2\alpha_4 + 4 + 2\delta}{1 - \delta}\right) + \alpha_3 + \tfrac{3}{2}\alpha_3^3 - \tfrac{5}{2}\alpha_4\alpha_3.$$

It will now be shown that (9) $\gtreqless 0$ according as $\alpha_3 \gtreqless 0$, since (9) is $\alpha_3 D$ where

$$(10) \qquad D = \frac{2\alpha_4 + 4 + 2\delta}{1 - \delta} + 1 + \tfrac{3}{2}\alpha_3^2 - \tfrac{5}{2}\alpha_4.$$

It suffices to show that $D > 0$ for all Pearson curves. Using the method of Lagrange multipliers, it is possible to show that within the permissible range of values of the variables involved, the g.l.b. of D is $\frac{1}{2}$, and so $D > 0$. It has been proved that the variance of the squares is a minimum when $\mu_1' = \frac{-\alpha_3}{2}\sigma$. It has just been shown that the sign of (9) agrees with that of α_3. These, together with Theorem I, demonstrate

THEOREM II. *For Pearson frequency functions, $\alpha_3 \neq 0$, the point about which the variance of cubes is a minimum deviates more from the mean than does the corresponding point for the squares.*

7. Symmetric populations. For the distribution of rth moments of samples

$$\mu_2(m_r') = \frac{1}{n}(\mu_{2r}' - \mu_r'^2). \tag{11}$$

To find the minimum of (11) expand in terms of central moments and powers of μ_1', differentiate with respect to μ_1', and equate to zero. This yields:

$$\begin{aligned}(2r-2)r^2\mu_2\mu_1'^{2r-3} + \cdots + K\mu_1'^{K-1}\Bigg[\binom{2r}{K}\mu_{2r-K} \\ - \sum_{i=0}^{K}\binom{r}{i}\binom{r}{K-i}\mu_{r-i}\mu_{r-K+i}\Bigg] + \cdots + 2r(\mu_{2r-1} - \mu_r\mu_{r-1}) = 0.\end{aligned} \tag{12}$$

For each power of μ_1', the coefficient is an isobaric moment function and is of even weight when the power of μ_1' is odd, and of odd weight when the power of μ_1' is even. If the population is symmetric the coefficients of even powers will vanish as will the constant term. Then μ_1' will be a factor, the other factor being a polynomial with only even powers of μ_1'. In this latter factor, where K is even, the coefficient of $K\mu_1'^{K-2}$ is

$$\binom{2r}{K}\mu_{2r-K} - \sum_{i=0}^{K}\binom{r}{i}\binom{r}{K-i}\mu_{r-i}\mu_{r-K+i}\,. \tag{13}$$

Since

$$\binom{x+y}{n} = \sum_{m=0}^{n}\binom{x}{n-m}\binom{y}{m},$$

(13) may be written

$$a\mu_{2r-K} + \sum_{i=0}^{K} b_i(\mu_{2r-K} - \mu_{r-i}\mu_{r-K+i}), \qquad r-i,\ K \text{ even},$$

where a, b_i are non-negative integers.

It can be immediately established by use of an inequality due to Tchebycheff [7, pp. 43, 168] that $\mu_{2K+2l}' \geq \mu_{2K}'\cdot\mu_{2l}'$ and therefore (13) is positive or zero.

To sum up, if the odd moments vanish (12) will have a factor μ_1' and a factor

which is a polynomial with even powers only of μ_1' with positive coefficients; therefore there is one and only one solution, $\mu_1' = 0$. This establishes

THEOREM III. *For a symmetrical population, the distribution of* rth *moments of samples has minimal variance when the origin is the population mean.*

8. Distribution of second moments. To study in more detail the distributions of m_2^* and m_2^0 the higher moments are computed and compared. Applying the formula for the distribution of rth moments we obtain, for m_2^0

$$\begin{aligned}
\mu_1'(m_2^0) &= \mu_2 \\
\mu_2(m_2^0) &= \frac{1}{n}(\mu_4 - \mu_2^2) \\
\alpha_3(m_2^0) &= \frac{\alpha_6 - 3\alpha_4 + 2}{\sqrt{n}\,(\alpha_4 - 1)^{3/2}} \qquad (14) \\
\alpha_4(m_2^0) - 3 &= \frac{1}{n}\left[\frac{\alpha_8 - 4\alpha_6 + 6\alpha_4 - 3}{(\alpha_4 - 1)^2} - 3\right]
\end{aligned}$$

etc.

For the distribution of m_2^*, we get

$$\begin{aligned}
\mu_1'(m_2^*) &= \mu_2 + \frac{\mu_3^2}{4\mu_2^2} \\
\mu_2(m_2^*) &= \frac{1}{n}\left(\mu_4 - \mu_2^2 - \frac{\mu_3^2}{\mu_2}\right) \\
(15) \quad \alpha_3(m_2^*) &= \frac{\alpha_6 - 3\alpha_4 + 2 + 3\alpha_3^3 - 3\alpha_5\alpha_3 + 3\alpha_4\alpha_3^2 - \alpha_3^4}{\sqrt{n}\,(\alpha_4 - \alpha_3^2 - 1)^{3/2}} \\
\alpha_4(m_2^*) - 3 &= \frac{1}{n}[(\alpha_8 - 4\alpha_6 + 6\alpha_4 - 3 + 12\alpha_5\alpha_3 \\
&\qquad - 6\alpha_3^2 - 4\alpha_7\alpha_3 + 6\alpha_6\alpha_3^2 - 12\alpha_4\alpha_3^2 + 4\alpha_3^4 - 4\alpha_3^3\alpha_5 \\
&\qquad + \alpha_4\alpha_3^4)(\alpha_4 - \alpha_3^2 - 1)^{-2} - 3]
\end{aligned}$$

etc.

Computing the ratios of α_3's, we have

$$(16) \quad \frac{\alpha_3(m_2^*)}{\alpha_3(m_2^0)} = \left[1 - \frac{\alpha_3\{3(\alpha_5 - \alpha_3) - \alpha_3(3\alpha_4 - \alpha_3^2)\}}{\alpha_6 - 3\alpha_4 + 2}\right]\left(1 - \frac{\alpha_3^2}{\alpha_4 - 1}\right)^{-3/2}.$$

Similarly

$$\begin{aligned}
&\frac{\alpha_4(m_2^*) - 3}{\alpha_4(m_2^0) - 3} \\
(17) \quad &= \left[1 - \frac{\alpha_3(4\alpha_7 + 6\alpha_4\alpha_3 + 4\alpha_3^2\alpha_5 + 12\alpha_3 - 12\alpha_5 - 6\alpha_6\alpha_3 - \alpha_3^3 - \alpha_3^3\alpha_4)}{\alpha_8 - 4\alpha_6 - 3\alpha_4^2 + 12\alpha_4 - 6}\right] \\
&\qquad \cdot \left(1 - \frac{\alpha_3^2}{\alpha_4 - 1}\right)^{-2}.
\end{aligned}$$

It is evident that when $\alpha_3 = 0$, the ratio in each case is unity. These ratios seem too involved to make any other general statements, but for particular types of populations these ratios in terms of the parameters are considerably simplified.

To illustrate this statement, consider

$$f_x = \frac{e^{-M}M^x}{x!}.$$

From the foregoing formulas we compute

$$\mu_1'(m_2^*) = M + \tfrac{1}{4}, \qquad \mu_1'(m_2^0) = M$$

$$\mu_2(m_2^*) = \frac{2M^2}{n}, \qquad \mu_2(m_2^0) = \frac{2M^2 + M}{n}$$

$$\frac{\alpha_3(m_2^*)}{\alpha_3(m_2^0)} = \sqrt{\frac{2}{M}}\,\frac{(2M+1)^{5/2}}{8M^2 + 22M + 1} \tag{18}$$

$$\frac{\alpha_4(m_2^*) - 3}{\alpha_4(m_2^0) - 3} = \frac{(12M^2 + 36M + 2)(2M+1)^2}{M(48M^3 + 384M^2 + 112M + 1)}. \tag{19}$$

The minimum value of (18) is 0.71 for $M = 1.22$ and (18) is < 1 for $M > 0.31$. The minimum of (19) is 0.70 and is < 1 for $M > 0.62$. For the Poisson distribution, then, not only is the variance of m_2^* less than that of m_2^0, but at least as far as the first four moments are concerned, the distribution of m_2^* approaches normality more rapidly than does m_2^0 for all values of $M > 0.62$.

When one follows the same procedure for $\frac{1}{\Gamma(p)}\, x^{p-1} e^{-x}$ it is found that not only is the variance of m_2^* less than that of m_2^0, but as far as the first four moments are concerned, the distribution of m_2^* approaches normality more rapidly than does m_2^0, for values of $p > 0.7$.

In the case of higher moments, it seems desirable to solve the necessary equations in each particular case, since the equations are somewhat involved.

9. Examples. A few examples are exhibited to illustrate the foregoing ideas and to contrast with some of the other methods.

1. A sample of 1,000 is obtained with the following distribution

x:	0	1	2	3	4
f:	625	269	91	11	4

The hypothesis being tested is that the population is $f_x = \frac{e^{-M}M^x}{x!}$, with $M = 0.5$.

$\bar{x} = 0.5$ and therefore the mean does not differ from its expected value.

By using the m_2^* test, we compute $t = 2.06$. If m_2^* is distributed normally, the hypothesis is rejected at the 5% level. By using the m_2^0 test, we find $t = 1.45$, and therefore by this test the hypothesis is not rejected at the 5% level.

Applying the χ^2 test, we find that the hypothesis is not rejected at the 5% level.

2. We return now to the sample mentioned in the introduction.

Since the parameters in population A were found by fitting the first two moments, the tests will be made on the higher moments. From the definition of m_2^0 and m_2^* it is clear what is meant by m_3^0, m_3^*, m_4^0 and m_4^*.

Consider the discrepancy of third moments in standard units t as a function of h, the distance from the origin. It is easy to see that

$$t = (m_3' - \mu_3')/\sqrt{G},$$

where

$$G = \frac{1}{n}[\mu_6' - \mu_3'^2 - 6h(\mu_5' - \mu_3'\mu_2') + 3h^2(5\mu_4' - 3\mu_2'^2 - 2\mu_3'\mu_1') - 18h^3(\mu_3' - \mu_2'\mu_1') + 9h^4(\mu_2' - \mu_1'^2)].$$

For the m_3^0 test, $h = 139.288$. The value of h which minimizes the variance is a solution of $6(\mu_2' - \mu_1'^2)h^3 - 9(\mu_3' - \mu_2'\mu_1')h^2 + (5\mu_4' - 3\mu_2'^2 - 2\mu_3'\mu_1')h - (\mu_5' - \mu_3'\mu_2') = 0$, which, for this population is $h = 142.66$. Using these values and computing, we find, for the m_3^0 test, $t = 1.90$ and for the m_3^* test, $t = 1.95$.

Using the same methods applied to fourth moment tests, we obtain for the m_4^0 test, $h = 139.288$ and $t = 48.7$, and for the m_4^* test, $h = 143.73$ and $t = 52.4$.

The χ^2 test cannot be used here since the moments alone are given; furthermore there is some difficulty in interpreting it under these conditions.

In this particular example, the third moment test would not reject the hypothesis at the 1% level, while the fourth moment test would reject at that level.

3. Since population B is symmetric, it is known that the m_3^0 and m_3^* tests are identical; similarly for m_4^0 and m_4^*. For the m_3^* test, $t = 5.57$, which would reject the hypothesis at the 1% level. The fourth moment test would not be applied in practice.

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TOLERANCE LIMITS FOR A NORMAL DISTRIBUTION[1]

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Summary. The problem of constructing tolerance limits for a normal universe is considered. The tolerance limits are required to be such that the probability is equal to a preassigned value β that the tolerance limits include at least a given proportion γ of the population. A good approximation to such tolerance limits can be obtained as follows: Let $\bar{x}$ denote the sample mean and s^2 the sample estimate of the variance. Then the approximate tolerance limits are given by

$$\bar{x} - \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, rs \quad \textit{and} \quad \bar{x} + \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, rs$$

where n is one less than the number N of observations, $\chi^2_{n,\beta}$ denotes the number for which the probability that χ^2 with n degrees of freedom will exceed this number is β, and r is the root of the equation

$$\frac{1}{\sqrt{2\pi}} \int_{1/\sqrt{N}-r}^{1/\sqrt{N}+r} e^{-t^2/2}\, dt = \gamma.$$

The number $\chi^2_{n,\beta}$ can be obtained from a table of the χ^2 distribution and r can be determined with the help of a table of the normal distribution.

1. Introduction. The problem of setting tolerance limits for a distribution on the basis of an observed sample was discussed by S. S. Wilks [1], [2] and by one of the present authors [3], [4]. For a univariate distribution the problem may be formulated briefly as follows: Let x be the chance variable under consideration and let $x_1, \cdots, x_N$ be a sample of N independent observations on x. Two functions, L_1 and L_2, of the sample are to be constructed such that the probability that the limits L_1 and L_2 will include at least a given proportion γ of the population is equal to a preassigned value β. The limits L_1 and L_2 are called tolerance limits.

The following two cases have been treated in the literature: (1) Nothing is known about the distribution of x, except perhaps that it is continuous, or that it admits a continuous probability density function. (2) The functional form of the distribution of x is known and only the values of a finite number of parameters involved in the distribution of x are unknown. We shall refer to (1) as the non-

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parametric case and to (2) as the parametric case. An exact solution of the problem for univariate distributions in the non-parametric case has been given by S. S. Wilks [1] His results have been extended to multivariate distributions by one of the present authors [3]. An asymptotic solution of the problem in the parametric case, which may be used for large samples, was given in [4].[2]

In the present paper we shall deal with the problem of setting tolerance limits for a normal distribution with unknown mean and variance. Approximation formulas are obtained which differ from the exact values by a magnitude of the order $1/N^2$. They give much closer approximations to the exact values than those which can be obtained by applying the general asymptotic results in [4] to the normal distribution In addition, the approximation formulas in the present paper have the advantage of considerable simplicity and can easily be computed with the help of tables of the normal and χ^2 distributions. To estimate the closeness of the approximation of the formulas given in this paper, a method of computing upper and lower limits for the exact values has been derived. Computations show that the approximation is good even for small values of N. A few numerical examples are given in section 7.

2. Precise formulation of the problem and notation. Let $x_1, \cdots, x_N$ be N independent observations from a normal population with mean μ and variance σ^2, both unknown. We shall denote by $\bar{x}$ the arithmetic mean of the observations and by s^2 the sample estimate of the population variance σ^2, i.e.,

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \tag{2.1}$$

and

$$s^2 = \frac{\sum (x_i - \bar{x})^2}{n}, \quad \text{where } n = N - 1. \tag{2.2}$$

For any positive λ we shall denote by $A(\bar{x}, s, \lambda)$, or more briefly by A, the proportion of the normal universe included between the limits $\bar{x} - \lambda s$ and $\bar{x} + \lambda s$, i.e.,

$$A = A(\bar{x}, s, \lambda) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{\bar{x}-\lambda s}^{\bar{x}+\lambda s} e^{-(1/2\sigma^2)(t-\mu)^2}\, dt. \tag{2.3}$$

A is a chance variable, since the limits of integration are chance variables. In this paper we shall deal with the problem of determining the value of λ so that the probability that A exceeds a preassigned value γ is equal to a preassigned value β. The desired tolerance limits will then be given by $\bar{x} - \lambda s$ and $\bar{x} + \lambda s$, respectively. In practice, the values β and γ will usually be chosen near unity, frequently $\geq .95$.

[2] Although the results obtained in the non-parametric case could be applied to the parametric case as well, it would not be satisfactory to do so, since for the parametric case methods having greater efficiency can be devised by taking into account the available information regarding the functional form of the distribution.

It can be verified that the distribution of A does not depend on the unknown parameters μ and σ. Thus we can assume without loss of generality that $\mu = 0$ and $\sigma = 1$.

For any given positive value λ we shall denote by $P(\gamma,\lambda)$ the probability that $A > \gamma$. For a given value $\bar{x}$ we shall denote by $P(\gamma,\lambda \mid \bar{x})$ the conditional probability that $A > \gamma$ under the condition that the sample mean has a given value $\bar{x}$ It is clear that $P(\gamma,\lambda)$ is equal to the expected value of $P(\gamma,\lambda \mid \bar{x})$, i.e.,

$$P(\gamma,\lambda) = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(\gamma, \lambda \mid \bar{x})\, e^{-\frac{1}{2}N\bar{x}^2}\, d\bar{x}\,. \tag{2.4}$$

3. Method of computing $P(\gamma,\lambda \mid \bar{x})$ for any given values γ,λ and $\bar{x}$. Since $A = A(\bar{x},s,\lambda)$ is a strictly increasing function of s, the equation in s

$$A(\bar{x},s,\lambda) = \gamma \tag{3.1}$$

has exactly one root in s. Denote this root by

$$s = r(\bar{x},\gamma,\lambda). \tag{3.2}$$

Thus, $r(\bar{x},\gamma,\lambda)$ is that value for which

$$\frac{1}{\sqrt{2\pi}} \int_{\bar{x}-\lambda r(\bar{x},\gamma,\lambda)}^{\bar{x}+\lambda r(\bar{x},\gamma,\lambda)} e^{-t^2/2}\, dt = \gamma. \tag{3.3}$$

It is clear that $\lambda r(\bar{x},\gamma,\lambda)$ does not depend on λ We shall write

$$\lambda r(\bar{x},\gamma,\lambda) = r(\bar{x},\gamma). \tag{3.4}$$

Obviously $r(\bar{x},\gamma)$ is that value for which

$$\frac{1}{\sqrt{2\pi}} \int_{\bar{x}-r(\bar{x},\gamma)}^{\bar{x}+r(\bar{x},\gamma)} e^{-t^2/2}\, dt = \gamma. \tag{3.5}$$

For given values of $\bar{x}$ and γ the value $r(\bar{x},\gamma)$ can be obtained from a table of the normal distribution.

Since $A(\bar{x},s,\lambda)$ is a strictly increasing function of s, the inequality $A(\bar{x},s,\lambda) > \gamma$ is equivalent to the inequality $s > r(\bar{x},\gamma,\lambda) = r(\bar{x},\gamma)/\lambda$. Hence, since $\bar{x}$ and s are independently distributed, we have

$$P(\gamma,\lambda \mid \bar{x}) = P(s > r(\bar{x},\gamma)/\lambda) \tag{3.6}$$

where $P(s > c)$ denotes the probability that $s > c$ for any constant c. In genral, for any relation R we shall denote by $P(R)$ the probability that R holds.

Since ns^2 has the χ^2 distribution with $n = N - 1$ degrees of freedom, we have

$$P\left(s > \frac{r(\bar{x},\gamma)}{\lambda}\right) = P\left(\chi_n^2 > \frac{nr^2(\bar{x},\gamma)}{\lambda^2}\right) \tag{3.7}$$

where χ_n^2 stands for a random variable which has the χ^2 distribution with n degrees of freedom. The probability on the right-hand side of (3.7) can be obtained from a table of the χ^2 distribution

Hence, we see that the computation of $P(\gamma,\lambda \mid \bar{x})$ for given values γ,λ and $\bar{x}$ can be carried out in two simple steps. First we determine the value of $r(\bar{x},\gamma)$ from a table of the normal distribution and then read the value of

$$P\left(\chi^2_n > \frac{nr^2(\bar{x},\gamma)}{\lambda^2}\right)$$

from a table of the χ^2 distribution.

4. Proof that the difference $P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) - P(\gamma,\lambda)$ is of the order $1/N^2$. It is clear that $P(\gamma,\lambda \mid \bar{x})$ is an even function of $\bar{x}$. Hence, in the expansion of $P(\gamma,\lambda \mid \bar{x})$ in a power series in $\bar{x}$, only even powers will occur. Terminating the Taylor expansion (in section 8 we prove its validity) at the fourth term, we have

$$(4.1)\quad P(\gamma,\lambda \mid \bar{x}) = P(\gamma,\lambda \mid 0) + \frac{\bar{x}^2}{2}\frac{\partial^2 P(\gamma,\lambda \mid \bar{x})}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + \frac{\bar{x}^4}{4!}\frac{\partial^4 P(\gamma,\lambda \mid \bar{x})}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}$$

where $0 \leq \xi \leq \bar{x}$.

The expected value of $P(\gamma,\lambda \mid \bar{x})$ (considering $\bar{x}$ as a random variable) is equal to $P(\gamma,\lambda)$. Since the expected value of $\bar{x}^2$ is $1/N$ and the expected value of

$$\frac{\bar{x}^4}{4!}\frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}$$

is of the order $1/N^2$ (this is proved in section 9), we obtain from (4.1)

$$(4.2)\qquad P(\gamma,\lambda) = P(\gamma,\lambda \mid 0) + \frac{1}{2N}\frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + 0\left(\frac{1}{N^2}\right).$$

On the other hand, substituting $1/\sqrt{N}$ for $\bar{x}$ in (4.1) we obtain

$$(4.3)\qquad P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) = P(\gamma,\lambda \mid 0) + \frac{1}{2N}\frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + \frac{1}{4!N^2}\frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi'}$$

where $0 \leq \xi' \leq 1/\sqrt{N}$. Hence, since the second term of the right member of (4.3) is of the order $1/N^2$,

$$(4.4)\qquad P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) = P(\gamma,\lambda \mid 0) + \frac{1}{2N}\frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + 0\left(\frac{1}{N^2}\right).$$

From (4.2) and (4.4) it follows that

$$(4.5)\qquad P(\gamma,\lambda) - P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right) = 0\left(\frac{1}{N^2}\right).$$

Thus, this difference approaches zero rapidly as $N \to \infty$.

5. Computation of the value λ for which $P\left(\gamma,\lambda \,\middle|\, \frac{1}{\sqrt{N}}\right)$ takes a preassigned value β Denote by $\chi^2_{n,\beta}$ that value for which $P(\chi^2_n > \chi^2_{n,\beta}) = \beta$. This value can

be obtained from a table of the χ^2 distribution. From (3.6) and (3.7) it follows that the required value λ^* of λ is given by the root of the equation

$$\frac{n}{\lambda^2} r^2\left(\frac{1}{\sqrt{N}}, \gamma\right) = \chi^2_{n,\beta}. \tag{5.1}$$

Thus, the desired value of λ^* is given by

$$\lambda^* = \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, r\left(\frac{1}{\sqrt{N}}, \gamma\right). \tag{5.2}$$

The value $r\left(\frac{1}{\sqrt{N}}, \gamma\right)$ is defined by (3.5) and can be obtained from a table of the normal distribution.[3]

6. Lower and upper limits for $P(\gamma,\lambda)$ As mentioned in section 2, $P(\gamma,\lambda)$ is equal to the expected value of $P(\gamma,\lambda \mid \bar{x})$. Thus,

$$P(\gamma,\lambda) = \frac{\sqrt{N}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} P(\gamma,\lambda \mid \bar{x}) e^{-\frac{1}{2}N\bar{x}^2}\, d\bar{x}. \tag{6.1}$$

To obtain upper and lower limits for $P(\gamma,\lambda)$, we shall construct upper and lower limits for the integral on the right-hand side of (6.1). It can easily be seen that $P(\gamma,\lambda \mid \bar{x})$ is a strictly decreasing function of $\bar{x}^2$. Hence, to obtain lower and upper limits for the integral in the right member of (6.1) we can proceed as follows: Choose a positive constant d and a positive integer k. Denote by a_i the probability that $id \leq \bar{x} \leq (i+1)d$, $(i = 0, 1, \cdots, k-1)$, and let a_k be the probability that $\bar{x} > kd$. Then $2\sum_{i=0}^{k} a_i P(\gamma,\lambda \mid id)$ is an upper bound, and $2\sum_{i=1}^{k} a_{i-1} P(\gamma,\lambda \mid id)$ is a lower bound of the integral in question. Thus

$$P(\gamma,\lambda) \geq 2\sum_{i=1}^{k} a_{i-1} P(\gamma,\lambda \mid id) \tag{6.2}$$

and

$$P(\gamma,\lambda) \leq 2\sum_{i=0}^{k} a_i P(\gamma,\lambda \mid id). \tag{6.3}$$

The two limits can be brought arbitrarily close to each other by choosing d sufficiently small and k sufficiently large. A method of computing $P(\gamma,\lambda \mid \bar{x})$ for any given value $\bar{x}$ has been described in section 3 and the quantities a_i can be obtained from a table of the normal distribution. The amount of computational work, however, increases rapidly with increasing k.

[3] The Statistical Research Group computed, under the supervision of Albert H. Bowker, a table of tolerance limit factors λ (see formula 5.2) for $\beta = .75, .90, .95, .99$; $\gamma = .75, .90, .95, .99, .999$, $N = 2$ (1) 102 (2) 180 (5) 300 (10) 400 (25) 750 (50) 1000. Mr. Bowker also developed an asymptotic formula for λ (published elsewhere in this issue of the *Annals*) which, when $\beta \leq .99$, $\gamma \leq .999$, and $N \geq 160$, agrees with (5.2) to within 1 unit in the third significant figure. The Applied Mathematics Panel plans to publish the table and a brief explanation of tolerance limits in the volume entitled *Techniques of Statistical Analysis* described in the footnote on page 217.

7. Approximate determination of the tolerance limits. The exact tolerance limits are given by $\bar{x} - \lambda s$ and $\bar{x} + \lambda s$ where λ is the root of the equation in λ

$$P(\gamma,\lambda) = \beta. \tag{7.1}$$

This equation has exactly one root in λ, since $P(\gamma,\lambda)$ is a strictly increasing function of λ. Denote this root by $\lambda = \lambda(\beta,\gamma)$. Thus, the exact tolerance limits are given by $\bar{x} - \lambda(\beta,\gamma)s$ and $\bar{x} + \lambda(\beta,\gamma)s$.

We have seen in section 4 that $P\left(\gamma,\lambda \mid \frac{1}{\sqrt{N}}\right)$ closely approximates $P(\gamma,\lambda)$, the difference being of the order $1/N^2$. Thus, a close approximation to $\lambda(\beta,\gamma)$ can be obtained by solving the equation in λ,

$$P\left(\gamma, \lambda \mid \frac{1}{\sqrt{N}}\right) = \beta. \tag{7.2}$$

This equation has again exactly one root in λ, since $P\left(\gamma,\lambda \mid \frac{1}{\sqrt{N}}\right)$ is a strictly increasing function of λ. Denote the root of equation (7.2) by $\lambda = \lambda^*(\beta,\gamma)$. Thus approximate tolerance limits are given by $\bar{x}-\lambda^*(\beta,\gamma)s$ and $\bar{x}+\lambda^*(\beta,\gamma)s$. In section 5 it has been shown that

$$\lambda^*(\beta,\gamma) = \sqrt{\frac{n}{\chi^2_{n,\beta}}}\, r \tag{7.3}$$

where $n = N-1$, $\chi^2_{n,\beta}$ is that number for which the probability that χ^2 with n degrees of freedom exceeds this number is β, and r is the root of the equation

$$\frac{1}{\sqrt{2\pi}} \int_{1/\sqrt{N}-r}^{1/\sqrt{N}+r} e^{-t^2/2}\, dt = \gamma\,. \tag{7.4}$$

The number $\chi^2_{n,\beta}$ can be obtained from a table of the χ^2 distribution and r can be determined from a table of the normal distribution.

Since $\lambda^*(\beta,\gamma)$ is only an approximation to $\lambda(\beta,\gamma)$, $P[\gamma,\lambda^*(\beta,\gamma)]$ will differ slightly from β. To judge the goodness of the approximation of $\lambda^*(\beta,\gamma)$ to the exact value $\lambda(\beta,\gamma)$, it is desirable to derive upper and lower limits for the difference $P[\gamma,\lambda^*(\beta,\gamma)] - \beta$. Such limits can be obtained by computing upper and lower limits for $P[\gamma,\lambda^*(\beta,\gamma)]$ using the method described in section 6.

We cite here a few numerical examples to show the goodness of the approximation.

N	γ	β	$\lambda^*(\beta,\gamma)$	Upper limit of $P[\gamma,\lambda^*(\beta,\gamma)]$	Lower limit of $P[\gamma,\lambda^*(\beta,\gamma)]$
2	.95	95	37.674	.95202	.95077
9	.95	.99	4.550	.98989	.98908
25	.95	.95	2.631	.95161	.94393
25	.95	.99	2.972	.99024	.98813

8. Validity of the Taylor expansion of $P(\gamma,\lambda \mid \bar{x})$. We shall show that $P(\gamma,\lambda \mid \bar{x})$ has derivatives of all orders at every point $\bar{x}$, γ and λ being fixed. This is sufficient to validate the Taylor expansion used in section 4.

For typographical convenience write

$$r(\bar{x},\gamma) = R.$$

We have

$$\frac{1}{\sqrt{2\pi}} \int_{\bar{x}-R}^{\bar{x}+R} e^{-\frac{1}{2}t^2}\, dt = \gamma. \tag{8.1}$$

Differentiating (8.1) with respect to $\bar{x}$ we obtain

$$\left(1 + \frac{dR}{d\bar{x}}\right) e^{-\frac{1}{2}(\bar{x}+R)^2} = \left(1 - \frac{dR}{d\bar{x}}\right) e^{-\frac{1}{2}(\bar{x}-R)^2} \tag{8.2}$$

whence

$$\frac{dR}{d\bar{x}} = \tanh \bar{x}R\,. \tag{8.3}$$

Now the analytic function $\tanh z$ of the complex variable z has only purely imaginary singularities. Hence R possesses derivatives of all orders for all real values of $\bar{x}$.

Now

$$P(\gamma,\lambda \mid \bar{x}) = P\left(s > \frac{R}{\lambda}\right) = 1 - k \int_0^R t^{n-1}\, e^{-nt^2/(2\lambda^2)}\, dt$$

where k is a constant. Hence from (8.3)

$$\frac{\partial P}{\partial \bar{x}} = -kR^{n-1}\, e^{-nR^2/(2\lambda^2)} \tanh \bar{x}R\,. \tag{8.4}$$

The right member of (8.4) is a product of functions which are analytic in the entire (complex) R plane by a function which possesses derivatives of all orders for every real $\bar{x}$. Since R possesses a derivative (with respect to $\bar{x}$) for all real $\bar{x}$, it follows that P possesses derivatives of all orders for every real $\bar{x}$.

9. Proof that

$$E\left[\frac{\bar{x}^4}{4!} \frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}\right] = 0\left(\frac{1}{N^2}\right).$$

Since R is a minimum at $\bar{x} = 0$ it follows that $P(\gamma,\lambda \mid \bar{x})$ has a maximum there. Hence, from (4.1), the quantity

$$\bar{x}^2 \left(\frac{1}{2}\right) \frac{\partial^2 P}{\partial \bar{x}^2}\bigg|_{\bar{x}=0} + \frac{\bar{x}^4}{4!} \frac{\partial^4 P}{\partial \bar{x}^4}\bigg|_{\bar{x}=\xi}$$

is never positive. Therefore

$$\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi} \leq -\frac{12}{\bar{x}^2}\left.\frac{\partial^2 P}{\partial \bar{x}^2}\right|_{\bar{x}=0}.$$

Consequently $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded above for $|\bar{x}| \geq \delta$, where $\delta > 0$ is arbitrarily small. Since P possesses everywhere derivatives of all orders, the fourth derivative is continuous and hence bounded above for $|\bar{x}| \leq \delta$. From this we obtain that $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded above for every real $\bar{x}$.

Since $P(\gamma,\lambda \mid \bar{x})$ is always positive we have, from (4.1), that

$$\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi} \geq -\frac{12\left(2P + \bar{x}^2 \left.\frac{\partial^2 P}{\partial \bar{x}^2}\right|_{\bar{x}=0}\right)}{\bar{x}^4}.$$

For $|\bar{x}|$ greater than a sufficiently large number C, the left member of the above inequality is thus bounded below. For $|\bar{x}| \leq C$ we have that $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded below because $\frac{\partial^4 P}{\partial \bar{x}^4}$ is continuous. Hence $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded below for every real $\bar{x}$.

Since $\left.\frac{\partial^4 P}{\partial \bar{x}^4}\right|_{\bar{x}=\xi}$ is bounded above and below for every real $\bar{x}$, the desired result follows.

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APPROXIMATE FORMULAS FOR THE PERCENTAGE POINTS AND NORMALIZATION OF t AND χ^2 [1]

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1. Introduction. The χ^2 Distribution and Student's t-distribution are functions of a parameter n (degrees of freedom) and approach the normal distribution as n approaches infinity. The normal distribution is a good approximation to these distributions for large n. For small or moderate n, a better approximation may be obtained by using a function of t(or χ^2) which approaches the normal distribution more rapidly as n increases. Hotelling and Frankel [7] pointed out that an additional advantage of the normalization of a distribution is that further statistical tests are possible with the normalized variate. Normalizing t(or χ^2) is equivalent to transforming it into a function which is normally distributed to a required degree of approximation; that is, a normally distributed variate of zero mean and unit variance is expressed as a function of t(or χ^2) in powers of $1/n$.

The reverse problem of expressing t(or χ^2) as a function of a normally distributed variate of zero mean and unit variance in powers of $1/n$ is also of practical importance in connection with significance tests for which the significance levels, or percentage points, of the t and χ^2 distributions are required.

Cornish and Fisher [1] (see also [2]) have given a method for the normalization of distributions which approach normality as the number of degrees of freedom, n, increases and whose cumulants are expressed in power series of $1/n$, so that the order of magnitude of the rth cumulant is that of $n^{-(r-1)}$. A method has also been given for expressing a variate with such a distribution as a function of a normally distributed variate of zero mean and unit variance in powers of $1/n$.

It is the purpose of this note to apply the Cornish-Fisher method (1) to the derivation of asymptotic formulas for the percentage points of the t and χ^2 distributions and (2) to the normalization of these distributions. Tables are given which indicate the accuracy of these approximations and compare them with other approximations. Tables are also given to facilitate the calculation of the approximations for the percentage points of t and χ^2.

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[1] This paper reports work done in the Statistical Research Group, Division of War Research, Columbia University under contract OEMsr-618 with the Applied Mathematics Panel, National Defense Research Committee, Office of Scientific Research and Development. The work was first reported in an unpublished memorandum, "Application of the Cornish-Fisher method to an approximation of the significance levels of t and χ^2" (SRG number 507, April 28, 1945)

[2] Henry Goldberg died April 19, 1945.

2. The Cornish-Fisher method.[3] Consider the random variable y with probability distribution function $f(y)$, expected value $E(y)$, and variance $\sigma^2(y)$. Let K_r denote the rth cumulant of y and a_r denote the rth relative cumulant of y; i.e., $a_r = \frac{K_r}{K_2^{r/2}}$. Let x denote a normally distributed variate with zero mean and unit variance.

For every p, $(0 \leq p \leq 1)$, let y_p be defined by

$$\int_{-\infty}^{y_p} f(y)\, dy = p$$

and x_p by

$$\int_{-\infty}^{x_p} \frac{1}{\sqrt{2\pi}} e^{-(r^2/2)}\, dr = p.$$

That is, corresponding to every y_p, there is an x_p having the same probability integral (p). The Cornish-Fisher Method for expressing a normally distributed variate with zero mean and unit variance as a function of a standardized variate with the same probability integral gives

$$x_p \sim b_0 + b_1 z_p + b_2 z_p^2 + b_3 z_p^3 + b_4 z_p^4 + b_5 z_p^5 + \cdots \tag{1}$$

where z_p is the standardized variate corresponding to y_p; i.e.,

$$z_p = \frac{y_p - E(y)}{\sigma(y)}$$

and the b_i are defined in terms of the relative cumulants.

Cornish and Fisher give also the following expansion for a standardized variate as a function of a normally distributed variate:

$$z_p \sim c_0 + c_1 x_p + c_2 x_p^2 + c_3 x_p^3 + c_4 x_p^4 + c_5 x_p^5 + \cdots \tag{2}$$

where the c_i are defined in terms of the relative cumulants.

3. An approximation for the percentage points of Student's t-distribution. The standardized variate $z = t\left(\frac{n-2}{n}\right)^{\frac{1}{2}}$ can be expressed as a function of the normal variate, x, in powers of $1/n$ by using the Cornish-Fisher equation (2). Omitting terms of degree greater than two in $1/n$ gives, after simplification, the following asymptotic expansion for t:

$$t \sim x + \frac{x^3 + x}{4n} + \frac{5x^5 + 16x^3 + 3x}{96n^2} + \cdots. \tag{3}$$

[3] Churchill Eisenhart suggested the use of the Cornish-Fisher Method for obtaining percentage points of the chi-square distribution not given in existing tables, a problem which arose in several connections, including the computation of a table of factors for tolerance limits for normal distributions according to two formulas devised in the Statistical Research Group, one by A. Wald and J. Wolfowitz and the other by Albert H. Bowker, both of which are published elsewhere in this issue of the *Annals of Math. Stat.* The table will be included in a volume by the Statistical Research Group, *Techniques of Statistical Analysis*, to be published by the McGraw-Hill Book Company in 1946; its preparation, including the work reported in the present paper, was directed by Albert H. Bowker; the Statistical Research Group was directed by W. Allen Wallis.

For simplicity, the subscript p which appears in the Cornish-Fisher equation (2) has been dropped. It should be understood, however, that the x and t used in expansion (3) have the same probability integral. It is interesting to note that the first two terms were derived by Peiser [4].

TABLE 1

*Table of Polynomials Required for the Approximation for the Percentage Points of the t-distribution**

Probability Integral (p)	$x_p = x$	$f_1(x)$	$f_2(x)$
.999	3.090232	8.150129	19.692529
.9975	2.807034	6.231221	12.850916
.995	2.575829	4.916548	8.834762
.99	2.326348	3.729074	5.719746
975	1.959964	2.372271	2 822499
.95	1.644854	1.523769	1.420203
.90	1.281552	.846585	.570891
.75	.674490	.245335	.079490

* This table can be used for determining x, $f_1(x)$ and $f_2(x)$ corresponding to the complements of the selected values of p by using the relations

$$\begin{aligned} x_{1-p} &= -x_p \\ f_1(-x) &= -f_1(x) \\ f_2(-x) &= -f_2(x). \end{aligned}$$

To facilitate the use of the approximation, tables of the required polynomials in x have been computed for selected probability integrals. The approximation can be written

$$t \sim x + \frac{f_1(x)}{n} + \frac{f_2(x)}{n^2} + \cdots$$

where

$$f_1(x) = \frac{x^3 + x}{4}$$

and

$$f_2(x) = \frac{5x^5 + 16x^3 + 3x}{96}.$$

Table 1 gives values of x_p (or x), $f_1(x)$ and $f_2(x)$ for selected values of the probability integral p Table 2 gives approximate and exact percentage points of t for selected values of p and degrees of freedom. The exact values were taken from Merrington [5]. Table 2 shows the high degree of accuracy of the three

TABLE 2

Comparative Table of Approximate and Exact Values of the Percentage Points of the t-distribution

Probability Integral (p)	Degrees of Freedom	Approximate Percentage Point			Exact Percentage Point
		Normal	2 Term	3 Term	
.9975	1	2 8070	9.0383	21 8892	127.32
	2		5.9226	9.1354	14.089
	10		3.4302	3.5587	3.5814
	20		3.1186	3.1507	3.1534
	40		2.9628	2.9708	2.9712
	60		2.9109	2.9145	2.9146
	120		2.8590	2.8599	2.8599
.9950	1	2.5758	7 4924	16.3271	63.657
	2		5 0341	7.2428	9.9248
	10		3.0675	3 1558	3.1693
	20		2 8217	2.8437	2.8453
	40		2.6987	2.7043	2.7045
	60		2.6578	2.6602	2.6603
	120		2.6168	2.6174	2.6174
.9750	1	1.9600	4.3322	7.1547	12.706
	2		3.1461	3 8517	4.3027
	10		2.1972	2.2254	2.2281
	20		2.0786	2.0856	2.0860
	40		2.0193	2.0210	2.0211
	60		1.9995	2.0003	2.0003
	120		1.9797	1.9799	1.9799
.9500	1	1.6449	3.1686	4 5888	6.3138
	2		2.4067	2.7618	2.9200
	10		1.7972	1.8114	1.8125
	20		1.7210	1.7246	1.7247
	40		1.6829	1.6838	1.6839
	60		1.6702	1.6706	1.6707
	120		1.6576	1.6577	1.6577
.7500	1	0.6745	.9198	.9993	1.0000
	2		.7972	.8170	.8165
	10		.6990	.6998	.6998
	20		.6868	.6870	.6870
	40		.6806	.6807	.6807
	60		.6786	.6786	.6786
	120		.6765	.6765	.6765

term approximation for $n \geq 10$ and the superiority of this approximation over the two-term approximation derived by Peiser.

4. An approximation for the percentage points of the χ^2 distribution. The standardized variate $z = \dfrac{\chi^2 - n}{\sqrt{2n}}$ can be expressed as a function of the normal variate, x, in powers of $1/n$ by using the Cornish-Fisher equation (2). Retain-

TABLE 3

*Table of Polynomials Required for the Approximation for the Percentage Points of the χ^2 distribution**

Probability Integral (p)	$G_1(x)$	$G_2(x)$	$G_3(x)$	$G_4(x)$	$G_5(x)$
.999	4.370248	5.699690	.619006	−1.602112	1.273498
9975	3.969745	4.586292	.193953	−1.113149	.875184
.995	3.642773	3.756598	−.073888	−.802518	.622768
.99	3.289953	2.941263	−.290266	−.541971	.411597
.975	2.771808	1.894306	−.486382	−.272398	.194832
.95	2.326174	1.137029	−.554981	−.122957	.077898
.90	1.812388	.428250	−.539450	−.017722	.002186
.75	.953873	− 363376	−.346842	.060220	−.030881

* This table can be used for determining the $G_i(x)$ for values of x corresponding to the complements of the selected values of p by using the relations

$$x_{1-p} = -x_p$$
$$G_i(-x) = (-1)^i G_i(x), \text{ for } i = 1, \dots, 5.$$

ing terms in $n^{-3/2}$ gives, after simplification, the following asymptotic expansion for χ^2:

$$\chi^2 \sim n + G_1(x)n^{\frac{1}{2}} + G_2(x) + \frac{G_3(x)}{n^{\frac{1}{2}}} + \frac{G_4(x)}{n} + \frac{G_5(x)}{n^{\frac{3}{2}}} + \cdots \tag{4}$$

where

$$G_1(x) = \sqrt{2}\,x$$

$$G_2(x) = \tfrac{2}{3}(x^2 - 1)$$

$$G_3(x) = \frac{1}{9\sqrt{2}}(x^3 - 7x)$$

$$G_4(x) = -\frac{1}{405}(6x^4 + 14x^2 - 32)$$

$$G_5(x) = \frac{1}{4860\sqrt{2}}(9x^5 + 256x^3 - 433x).$$

TABLE 4

Comparative Table of Various Approximate and Exact Values of the Percentage Points of the χ^2 Distribution

Approximation	Degrees of Freedom	Probability Integral (p)									
		005	.01	.05	10	25	75	90	.95	.99	.995
Exact Value	1	.0000	.0002	.0039	.0158	.1015	1.3233	2.7055	3.8415	6.6349	7.8794
Cornish-Fisher		*	*	.1650	.1354	.1207	1.2730	2.6857	3.8632	6.8106	8.1457
Peiser		1.1877	.9416	.3658	.1553	.0296	1.2437	2.7012	3.9082	6 9409	8 3255
Wilson-Hilferty		*	*	.0000	.0052	.0972	1.3156	2.6390	3.7468	6.5858	7.9048
Fisher		1.2416	.8796	.2079	.0396	.0530	1.4020	2.6027	3.4976	5.5323	6 3933
Exact Value	2	0100	0201	.1026	.2107	5754	2.7726	4.6052	5.9915	9.2103	10.5966
Cornish-Fisher		.0357	.0773	.1507	.2370	.5739	2.7595	4.6018	6.0004	9.2632	10.6749
Peiser		.6572	.4938	.2398	.2466	.5329	2.7403	4.6099	6 0343	9.3887	10.8560
Wilson-Hilferty		.0001	.0029	.0790	.1968	.5857	2.7628	4.5590	5.9369	9.2205	10,6729
Fisher		.3560	.1766	.0038	.1015	.5592	2.8957	4.5409	5.7017	8 2353	9.2789
Exact Value	10	2.1558	2.5582	3.9403	4 8652	6 7372	12.5489	15.9871	18 3070	23 2093	25 1882
Cornish-Fisher		2.1606	2.5621	3.9418	4.8657	6.7369	12.5484	15.9872	18.3077	23 2120	25.1921
Peiser		2 2605	2.6293	3.9565	4 8676	6.7299	12.5434	15.9889	18.3175	23.2532	25 2527
Wilson-Hilferty		2.0937	2.5122	3.9315	4.8695	6 7506	12.5386	15.9677	18.2918	23.2394	25 2523
Fisher		1.5897	2.0656	3.6830	4.7350	6.7874	12.6675	15 9073	18.0225	22.3463	24.0452
Exact Value	20	7.4339	8 2604	10.8508	12.4426	15.4518	23 8277	28.4120	31.4104	37.5662	39.9968
Cornish-Fisher		7.4020	8.2614	10.8511	12.4427	15 4517	23.8276	28.4120	31.4106	37.5670	40 0309
Peiser		7.4491	8.2930	10.8582	12.4436	15.4483	23.8249	28.4129	31.4159	37.5895	40.0641
Wilson-Hilferty		7.3835	8.2257	10.8470	12.4480	15.4619	23.8194	28.3989	31.4017	37 5914	40.0461
Fisher		6.7314	7.6779	10.5807	12.3179	15.5153	23 9397	28.3245	31.1249	36.7340	38.9035

* Computed percentage point is negative.

TABLE 4 (CONT.)

Approximation	Degrees of Freedom	Probability Integral (p)									
		005	.01	05	10	.25	75	.90	.95	.99	995
Exact Value	40	20.7065	22.1643	26.5093	29.0505	33.6603	45.6160	51.8050	55.7585	63.6907	66.7659
Cornish-Fisher		20.6835	22.1645	26.5094	29.0505	33.6603	45.6160	51.8051	55.7585	63.6909	66.7896
Peiser		20.7060	22.1797	26.5128	29.0510	33.6586	45.6146	51.8055	55.7613	63.7029	66.8072
Wilson-Hilferty		20.6690	22.1394	26.5080	29.0555	33.6676	45.6097	51.7963	55.7534	63.7104	66.8024
Fisher		19.9230	21.5289	26.2330	28.9305	33.7325	45.7225	51.7119	55.4726	62.8830	65.7119
Exact Value	60	35.5346	37.4848	43.1879	46.4589	52.2938	66.9814	74.3970	79.0819	88.3794	91.9517
Cornish-Fisher		35.5155	37.4850	43.1880	46.4589	52.2938	66.9814	74.3970	79.0820	88.3795	91.9709
Peiser		35.5303	37.4949	43.1902	46.4592	52.2927	66.9805	74.3973	79.0838	88.3877	91.9829
Wilson-Hilferty		35.5034	37.4647	43.1874	46.4633	52.2998	66.9762	74.3900	79.0782	88.3961	91.9820
Fisher		34.7185	36.8285	42.9095	46.3411	52.3697	67.0853	74.3013	78.7960	87.5834	90.9164
Exact Value	80	51.1720	53.5400	60.3915	64.2778	71.1445	88.1303	96.5782	101.879	112.329	116.321
Cornish-Fisher		51.1555	53.5401	60.3915	64.2778	71.1445	88.1303	96.5782	101.879	112.329	116.338
Peiser		51.1664	53.5475	60.3931	64.2781	71.1437	88.1295	96.5784	101.881	112.335	116.347
Wilson-Hilferty		51.1448	53.5227	60.3912	64.2819	71.1497	88.1256	96.5723	101.876	112.344	116.348
Fisher		50.3375	52.8718	60.1120	64.1614	71.2225	88.2325	96.4809	101.594	111.540	115.297
Exact Value	100	67.3276	70.0648	77.9295	82.3581	90.1332	109.141	118.498	124.342	135.807	140.169
Cornish-Fisher		67.3276	70.0649	77.9295	82.3581	90.1332	109.141	118.498	124.342	135.807	140.184
Peiser		67.3363	70.0708	77.9308	82.3583	90.1326	109.141	118.498	124.343	135.812	140.192
Wilson-Hilferty		67.3032	70.0494	77.9294	82.3618	90.1378	109.137	118.493	124.340	135.820	140.193
Fisher		66.4809	69.3888	77.6493	82.2427	90.2126	109.242	118.400	124.056	135.023	139.154

As before, the subscript p which appears in the Cornish-Fisher equation (2) has been dropped. The x and χ^2 which are used in expansion (4) have the same probability integral. The first four terms were derived by Peiser [4].

Table 3 gives values of the $G_i(x)$ for selected values of the probability integral p. Table 4 compares various approximations with the exact percentage

TABLE 5

Comparative Table of Approximate and Exact Values of the Probability Integral of t

t	Probability Integral of t							
	$n = 1$		$n = 2$		$n = 10$		$n = 20$	
	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact
0.1	.5311	.5317	.5351	.5353	.5388	.5388	.5393	.5393
1	.7734	.7500	.7917	.7887	.8296	.8296	.8354	.8354
3	1.0000	.8976	1.0000	.9523	.9954	.9933	.9967	.9965
5	1.0000	.9372	1.0000	.9811	1.0000	.9997	1.0000	1 0000
6	1.0000	.9474	1.0000	.9867	1.0000	.9999	1.0000	1.0000

TABLE 6

Comparative Table of Approximate and Exact Values of the Probability Integral of χ^2

χ^2	Probability Integral of χ^2							
	$n = 2$		$n = 10$		$n = 20$		$n = 29$	
	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact
1	.3963	.3935	.0010	.0002	.0000	.0000	.0000	.0000
5	.9646	.9179	.1098	.1088	.0004	.0003	.0000	.0000
10	1.0000	.9933	.5594	.5595	.0323	.0318	.0005	.0004
20	1.0000	1.0000	.9768	.9707	.5420	.5421	.1071	.1071
30	1.0000	1.0000	1.0000	.9991	.9305	.9301	5860	.5860
50							.9916	.9910

points of χ^2 for selected values of p and degrees of freedom. The Peiser four-term approximation, the Wilson-Hilferty approximation,

$$\chi_p^2 = n\left(1 - \frac{2}{9n} + x_p\sqrt{\frac{2}{9n}}\right)^3$$

and the Fisher approximation,

$$\chi_p^2 = \tfrac{1}{2}(x_p + \sqrt{2n - 1})^2$$

are given for comparison. The exact values were taken from Thompson [6]. Table 4 shows the high degree of accuracy, and the general superiority of the Cornish-Fisher approximation, for $n \geq 10$. For low probabilities (.005) the Peiser approximation is often better than the full series, for small n, (1, 2), the Wilson-Hilferty approximation is often better.

5. Normalization of t and χ^2. The Cornish-Fisher equation (1) applied to the t-distribution or, alternatively, a formal reversion of the power series (3) gives the asymptotic expansion

$$x \sim t\left[1 - \frac{t^2+1}{4n} + \frac{13t^4+8t^2+3}{96n^2} + \cdots\right]. \tag{5}$$

Expansion (5) agrees with the first three terms of an expansion derived by Hotelling and Frankel [7].

Applying the Cornish-Fisher equation (1) to the χ^2 distribution gives the expansion

$$\begin{aligned} x \sim \frac{1}{38880\sqrt{2}\,n^{\frac{1}{2}}}\Big\{&-68649n + [128469\chi^2 + 29056] \\ &- \frac{2}{n}[53553\chi^4 + 2208\chi^2 - 386] + \frac{2}{n^2}[34257\chi^6 + 792\chi^4 + 238\chi^2] \\ &- \frac{1}{n^3}[25221\chi^8 + 304\chi^6] + \frac{3993}{n^4}\chi^{10} + \cdots\Big\}. \end{aligned} \tag{6}$$

6. Accuracy of the normalizations of t and χ^2. The accuracy of the normalization (5) of t may be judged from Table 5, which compares the approximate value of the probability integral with the exact value. The approximate value is the normal probability integral corresponding to the value of x computed from (5) for the given values of t and n. The exact values were obtained from Student's tables [8]. For fixed n, the approximation improves as t decreases from moderate to small values. The approximation appears to improve as t increases from moderate values (about 3) to large values because of the more rapid approach to unity of the probability integral of a normal variate.

The accuracy of the normalization (6) of χ^2 may be judged from Table 6, which compares the approximate value of the probability integral with the exact value. The approximate value is the normal probability integral corresponding to the value of x computed from (6) for the given values of χ^2 and n. The exact values were obtained from the table of Pearson [9].

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THE EFFECT ON A DISTRIBUTION FUNCTION OF SMALL CHANGES IN THE POPULATION FUNCTION

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1. Summary. It is generally assumed in the application of distribution theory that, if the actual population function is not very different from the one used in the theory, then the true sampling distribution of a statistic will not be very different from the one obtained in the theory. But elsewhere in mathematics we do not assert that a conclusion will be only slightly modified by a small deviation in the hypothesis. This paper presents some theorems which are useful in determining the maximum effect on a sampling distribution of certain kinds of small changes in the population function In particular, if the population is denoted by the function $\phi(t)$, if a sample of n independent measurements $(t_1, \cdots, t_n)$ is taken from this population, if a statistic $x = g(t_1, \cdots, t_n)$ is formed from the sample, and if $D(x)$ denotes the distribution of this statistic; then, when $\phi(t)$ is changed by a small proportionate amount to $\phi_1(t)$, $D(x)$ will be changed to $D_1(x)$, and the relation between D and D_1 will be subject to the inequality:

$$\left| \int_a^b (D - D_1)dx \right| \leqq \epsilon \int_a^b D(x)dx,$$

where

$$\epsilon = (1 + \delta)^n - 1, \qquad \text{and} \qquad |\phi_1/\phi - 1| < \delta.$$

2. It is generally assumed in the application of distribution theory that, if the actual population function is not very different from the one used in the theory, then the true sampling distribution of a statistic will be not very different from the one obtained in the theory. For example, we commonly apply to practical problems the distribution theory that has been obtained on the hypothesis that the population is normally distributed even though we know that our actual populations are only approximately normal in form, and we commonly assume that our results are approximately correct. But elsewhere in mathematics we do not assert that a conclusion will be only slightly modified if we only slightly modify the hypothesis. An example of our unwillingness to do this in other branches of mathematics is illustrated in the following example.

Example 1. Let $y = \phi(t)$ have the derivative $y' = \phi'(t)$. Let $\phi(t)$ be replaced by $\phi_1(t)$, where $\phi_1 - \phi = s(t)\phi(t)$, and $|s(t)| \leq \epsilon$, ϵ being small. We have thus chosen to make $(\phi_1 - \phi)$ small relative to ϕ rather than small absolutely so that this example may be useful in another connection. The derivative of ϕ_1 may of course differ very greatly from $\phi'(t)$, as for example in some of the approximations made by a few terms of a Fourier series; and it would be a major error to assume that the two derivatives are approximately equal. How can we

be sure that, in the process of finding a distribution function, we are not making an error of the same[1] sort?

The following theorems partly answer this question. The theorems will first be stated and proved in great generality. Then we shall return to the functions in Example 1 as a special case. We shall be concerned with a sample consisting of a single observation of n measurements $(t_1, \cdots, t_n)$ drawn from the multivariate universe $\psi(t_1, \cdots, t_n)$, or, more briefly, with the vector T as a sample from the n-way universe $\psi(T)$. Throughout this paper ψ and ψ_1 shall be functions which are non-negative and whose integrals over the entire spaces of their definition are unity. Let the statistics $(x_1, \cdots, x_m)$, or more briefly the vector X, be constructed from T thus:

$$x_1 = g_1(T), \cdots, x_m = g_m(T). \tag{1}$$

If now p represents any measurable point set in X space and if dX is used for $(dx_1 \cdots dx_m)$ and dT for $(dt_1 \cdots dt_n)$, a fundamental theorem [1] of distribution theory asserts that, if q is the point set in T space for which X is in p, then the distribution $D(X)$ is determined by the equation,

$$\int_p D(X)\, dX = \int_q \psi(T)dT, \text{ if these integrals exist.} \tag{2}$$

THEOREM 1. *Using the foregoing notation, let $\psi(T)$ be replaced by $\psi_1(T)$ and let $\psi_1(T) - \psi(T) = \psi(T)S(T)$, where $|S| \leq \epsilon$, and as a consequence let $D(X)$ be replaced by $D_1(X)$; then*

$$\left| \int_p D_1(X)dX - \int_p D(X)dX \right| \leq \epsilon \int_p D(X)dX \leq \epsilon. \tag{3}$$

To prove these inequalities we merely need to notice that the point set q depends on the g's but not on the universe, and that therefore we may use the same p and q as in (2) in the following equation which determines D_1:

$$\int_p D_1(X)dX = \int_q \psi_1(T)dT. \tag{4}$$

Subtracting (2) from (4) we obtain

$$\left| \int_p D_1 dX - \int_p DdX \right| = \left| \int_p (D_1 - D)dX \right| = \left| \int_q (\psi_1 - \psi)dT \right| \tag{5}$$

$$= \left| \int_q \psi S dT \right| \leq \epsilon \left| \int_q \psi dT \right| = \epsilon \left| \int_p DdX \right| \leq \epsilon,$$

[1] The general question being raised here has been approached heretofore from different points of view. In particular, other exact population functions besides the normal have been studied, and in some cases the distribution theory has not been greatly disturbed as a result Also, the effects of slight changes in the parameters of a population function have been studied.

since ψ is never negative, and the integral of D is never greater than unity. It should be noticed that the final inequality of (5) is independent of the g's, although this is not true of the preceding inequalities, which do depend on the g's because they involve p and q.

COROLLARY. *In particular*[2] *let* $\psi = \phi(t_1) \cdots \phi(t_n)$, *where* $\phi(t)$ *defines a one-way universe function, and* $t_1, \cdots, t_n$ *are independent samples from it. Let* $x = g(t_1, \cdots, t_n)$. *Then, if* $\phi(t)$ *is replaced by* $\phi_1(t)$, *and if* $\phi_1 - \phi = s(t)\phi(t)$, *and if* $|s(t)| \leq \delta$, *and if* $D(x)$ *is the distribution of* x *before the replacement, and* $D_1(x)$ *is the corresponding distribution after the replacement,*

$$\left| \int_a^b (D_1 - D)dx \right| \leq \epsilon \left| \int_a^b D dx \right| \leq \epsilon,$$

where

$$\epsilon = (1 + \delta)^n - 1, \text{ and } -\infty \leq a < b \leq \infty.$$

This corollary follows from the theorem because of the universe,

$$\psi(t_1, \cdots, t_n) = \phi(t_1) \cdots \phi(t_n),$$

and

$$\psi_1(t_1, \cdots, t_n) = \phi(t_1) \cdots \phi(t_n)[1 + s(t_1)] \cdots [1 + s(t_n)],$$

so that, in the notation of the theorem,

$$\psi_1(T) = \psi(T) + \psi(T)S(T),$$

where

$$S(T) = [s(t_1) + \cdots + s(t_n)] + [s(t_1)s(t_2) + \cdots + s(t_{n-1})s(t_n)] + \cdots + [s(t_1) \cdots s(t_n)].$$

Hence

$$\left| S \right| \leq \left| n\delta + \frac{n!}{2!(n-2)!}\delta^2 + \cdots \delta^n \right| = (1 + \delta)^n - 1 = \epsilon.$$

The interval (a, b) now replaces the point set p of the theorem.

This theorem and its corollary are powerful in that they may be applied to all statistics, but they are weak because of the restrictions on $S(T)$ and $s(t)$. It is to be noted also that the corollary is ineffective when n is large, a difficulty which seems to the author to be implicit in the sampling process. The restrictions on $s(t)$ make it impracticable to apply the corollary to the following example since, as will be observed, if $|t| > c$, $\phi_1 - \phi = -\phi$, and so then $|s| = 1$; and when $\delta = 1$, $\epsilon = 2^n - 1$.

Example 2. Let $\phi(t) = (2\pi)^{-1/2}e^{-t^2/2}$ in $(-\infty, \infty)$, and let $\phi_1(t) = A(2\pi)^{-1/2} e^{-t^2/2}$ in $(-c, c)$ and let $\phi_1(t) = 0$ if $|t| > c$, where c is not infinite and A is so chosen that the integral of ϕ_1 over $(-\infty, \infty)$ is unity.

This type of example is important because, in the attempt to apply the theory of normal distributions to practical matters, the first discrepancy that appears

[2] One could as well use $\phi^1(t_1) \cdots \phi^{(n)}(t_n)$, but we choose the simpler case on account of its importance.

is that in the theory the given distribution is infinite in extent while in practice it is finite. The following theorem generalizes the preceding one so as to permit it to apply to this example.

THEOREM 2. *Let all of T-space be divisible into two parts, Q_0 and Q_1, satisfying the following conditions. In Q_0 let $\psi_1(T) - \psi(T) = S(T)\psi(T)$, and let $|S(T)| \leq \epsilon$. In Q_1 let $\psi_1(T) = 0$, and let*

$$\int_{Q_1} \psi(T)d(T) \leq \epsilon_1 .$$

Then

$$\left| \int_p D_1 dX - \int_p DdX \right| \leq \epsilon \int_p DdX + \epsilon_1 \leq \epsilon + \epsilon_1 .$$

It is not required that Q_0 or Q_1 be the totality of points for which its attendant conditions are true.

PROOF. As before, if the integrals exist,

$$\int_p D_1 dX = \int_q \psi_1 dT, \quad \text{and} \quad \int_p DdX = \int_q \psi dT.$$

Hence

$$\int_p D_1 dX - \int_p DdX = \int_q (\psi_1 - \psi)dT = \int_{q_0} (\psi_1 - \psi)dT + \int_{q_1} (\psi_1 - \psi)dT,$$

where q_0 is that part of q which is in Q_0, and q_1 is that part of q which is in Q_1.

$$\left| \int_p D_1 dX - \int_p DdX \right| \leq \left| \int_{q_0} (\psi_1 - \psi)dT \right| + \left| \int_{q_1} (\psi_1 - \psi)dT \right| . \tag{6}$$

$$\left| \int_{q_0} (\psi_1 - \psi)dT \right| = \left| \int_{q_0} S\psi dT \right| \leq \epsilon \int_{q_0} \psi dT \tag{7}$$

$$\leq \epsilon \int_q \psi dT = \epsilon \int_p DdX, \text{ because } \psi \geq 0.$$

$$\left| \int_{q_1} (\psi_1 - \psi)dT \right| = \left| \int_{q_1} \psi dT \right| \leq \left| \int_{Q_1} \psi dT \right| \leq \epsilon_1 , \tag{8}$$

because $\psi_1 = 0$ in q_1. The inequalities (7) and (8), when substituted in (6), prove the theorem.

COROLLARY. *In particular, let ψ, and x be defined as in the corollary to Theorem 1, and let $\phi_1(t)$ be so defined that, if $|t| \leq c$, $\phi_1(t) - \phi(t) = s(t)\phi(t)$, where as before $|s(t)| \leq \delta$, and $\epsilon \doteq (1 + \delta)^n - 1$; and, if $|t| > c$, let $\phi_1(t) = 0$. Also let $\int_{Q_1} \phi(t) \cdots \phi(t_n)dT \leq \epsilon_1$ where Q_1 is the set where $|t_i| > c$ for at least one value of i. Then*

$$\left| \int_a^b D_1(x)dx - \int_a^b D(x)dx \right| \leq \epsilon \int_a^b D(x)dx + \epsilon_1 \leq \epsilon + \epsilon_1 ,$$

provided these integrals exist.

PROOF. This corollary is implied in the theorem if we let $\psi(T) = \phi(t_1) \cdots \phi(t_n)$ and $\psi_1(T) = \phi_1(t_1) \cdots \phi_1(t_n)$, and then let Q_0 be the point set in T-space where $|t_i| \leq c$ for all values of i, and Q_1 be the point set where $|t_i| > c$ for at least one value of i. As in the corollary to Theorem 1, p becomes the interval (a, b).

Example 3. Let ϕ and ϕ_1 be as in Example 2, and choose $c = 3$. Then $A = 1/0.9973 = 1.0027$, and

$$\int_{Q_1} \phi(t_1)\cdots\phi(t_n)dT = 1 - (.9973)^n.$$

This quantity may be taken as ϵ_1. Also

$$|(\phi_1 - \phi)/\phi| = |A - 1| = 0.0027.$$

This quantity may be taken as δ. Then $\epsilon = (1.0027)^n - 1$. Hence

$$\left| \int_a^b D_1(x)dx - \int_a^b D(x) \right| dx \leq \epsilon \int_a^b D(x)dx + \epsilon_1 .$$

If n is not large, an approximate value for both ϵ and ϵ_1 is $0.003n$. This quantity is not particularly small unless n is small, but it could not be expected to be very small since the corollary pertains to all statistics of the form $x = g(t_1, \cdots, t_n)$.

Example 4. In one of the author's earlier papers [2] he found the distribution of the geometric mean, $x = (t_1 \cdots t_n)^{1/n}$, of n observations chosen from the universe described by the so-called curve of equal facility, whose equation is

$$y = \frac{1}{tc\sqrt{2\pi}} e^{-(1/2c^2)(\log t/G)^2}$$

The author stated that there was about as good justification for assuming that the distribution of statures was given by that universe as for assuming that it was normal. After one more theorem we shall now be able to state that, if one wishes to cling to the assumption that the distribution of statures is normal, then the distribution of the geometric mean is close to the distribution found in that earlier paper. We do need another theorem for this because we should be dealing with two distributions, $\phi_1(t)$ and $\phi(t)$, which do not obey the requirements of the corollary of Theorem 1, because they approach zero at different rates as t becomes infinite, and do not obey the requirements of the corollary of Theorem 2 because neither vanishes throughout the infinite intervals for which $|t| > c$. But the following theorem and corollary will take care of this and of similar cases. It will be observed that Theorem 3 includes Theorem 2 as a special case.

THEOREM 3. *Using the foregoing notation, let all of T-space be divisible into two parts Q_0 and Q_1 satisfying the following conditions. In Q_0 let $\psi_1(T) - \psi(T) = S(T)\psi(T)$, and let $|S(T)| \leq \epsilon$. Let $T = Q_0 + Q_1$ and*

$$\int_{Q_1} \psi_1(T)dT + \int_{Q_1} \psi(T)dT \leq \epsilon_1 .$$

Then

$$\left| \int_P D_1(X)dX - \int_P D(X)dX \right| \le \epsilon \int_P D(X)dX + \epsilon_1 \le \epsilon + \epsilon_1 .$$

PROOF. As before,

$$\int_P D_1 dX - \int_P D dX = \int_Q (\psi_1 - \psi) dT = \int_{q_0} (\psi_1 - \psi) dT + \int_{q_1} (\psi_1 - \psi) dT$$

$$\left| \int_P D_1 \, dX - \int_P D \, dX \right| \le \left| \int_{q_0} (\psi_1 - \psi) \, dT \right| + \left| \int_{q_1} (\psi_1 - \psi) \, dT \right| = \mathrm{I} + \mathrm{II}.$$

$$\mathrm{I} \le \epsilon \int_P D \, dX \le \epsilon.$$

$$\mathrm{II} \le \int_{q_1} \psi_1 \, dT + \int_{q_1} \psi \, dT \le \int_{Q_1} \psi_1 \, dT + \int_{Q_1} \psi \, dT \le \epsilon_1 .$$

These inequalities together prove the theorem.

COROLLARY. *In particular, let ψ, ϕ_1, and x be as in the corollary of Theorem 2, except that now, instead of requiring $\phi_1(t)$ to vanish when $|t| > c$ we shall let Q_1 and ϵ_1 be so chosen that*

$$\int_{Q_1} \phi_1(t_1) \cdots \phi_1(t_n) dT + \int_{Q_1} \phi(t_1) \cdots \phi(t_n) dT \le \epsilon_1 .$$

Then

$$\left| \int_a^b D_1(x)dx - \int_a^b D(x)dx \right| \le \epsilon \int_a^b D(x)dx + \epsilon_1 \le \epsilon + \epsilon_1 .$$

As before stated, the inequalities of this paper apply to all statistics for which the integrals involved exist. It seems probable that closer inequalities could be devised by placing appropriate restrictions on the g functions which define these statistics.

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AN EXPERIMENTAL DESIGN FOR SLOPE-RATIO ASSAYS

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1. Summary. When the response to a drug is a linear function of arithmetic dosage units, the relative potency of two preparations can be computed as a slope-ratio assay. Their dosage-response curves are computed by solving three simultaneous equations to obtain the common intercept a', the slope of the standard, b_1, and the slope of the unknown, b_2. The method is applicable to certain microbiological assays for the vitamins. Usually several unknowns are assayed at one time with a single standard. Their calculation is simplified when such assays meet the following requirements: (1) restriction of treatments to the zone within which the response is related linearly to the dose, (2) equal spacing of doses on an arithmetic scale beginning with the negative control, (3) an equal number (k) of doses of standard and of each unknown and (4) r replicates for each dose of unknown, h' replicates for the negative control and h replicates for each dose of the standard.

2. Method of Analysis. The design and analysis of assays for measuring drug potency has been developed largely about the linear relation between response and the logarithm of the dose of many drugs. An alternative procedure is available when some measure of the response is related linearly to arithmetic dosage units. Recently Finney [5] has applied the technique to microbiological assays of the vitamins. The relationship is also suitable for experiments with toxic agents on micro-organisms, where the length of exposure to treatment is the dose. Since potency is measured from the ratio of the slope of the dosage-response curve for an unknown to that for the standard preparation, Wood [6] has termed the method a "slope-ratio assay."

The validity of quantitative biological assays depends upon a qualitative similarity between the standard and the active agent of the unknown. When the response is related linearly to the log-dose, this is determined by testing the parallelism of the lines fitted separately to the results for the standard and to those for the unknown preparation. If the departure from parallelism is within the sampling error, the combined slope is determined from the data on both preparations and used in computing potency and its error. The analogous test in slope-ratio assays is the convergence of the lines relating response to arithmetic dose at zero content of drug, using drug as a generic term which includes vitamins, poisons and physical agents. When the curves for the standard and the unknown are computed separately, their zero intercept should agree within the experimental error. In assays meeting this requirement, the curves are computed so that they are forced to intersect at zero dose. The curves

$$y_1 = a' + b_1x_1$$

and

$$y_2 = a' + b_2 x_2$$

are fitted by solving three simultaneous equations to obtain the three statistics, a', b_1 and b_2 which are the best estimates of their respective parameters. Finney [5] has illustrated the technique with data from the microbiological assay of nicotinic acid and given a suitable test for convergence as well as the error of the estimated potency.

The calculation described by Finney is flexible but not adapted for routine use. With certain restrictions in design, the calculation can be reduced to a practicable form for the assay of $(m - 1)$ unknowns against a standard preparation. These restrictions are as follows:

1. Doses both of standard and of unknowns must fall within the range for which some function of the response is related linearly to an arithmetic scale of dosage units with convergence at zero dose.
2. Within this range the doses (x) of standard and of all the unknowns must be spaced similarly and preferably equally on an arithmetic scale, beginning with the negative control $(x = 0)$.
3. The doses of each unknown must match those of the standard in respect to both number (k) and their expected potencies, so far as the latter can be judged in advance. Within an assay group there may be h' replicates of the negative control, h replicates of each dose of the standard and r replicates of each dose of each unknown.
4. Some element of randomization must be introduced within an assay group in respect to the preparation of the tubes, their handling and the reading of the results. Replicates of any given dose or of the negative control must not be prepared together.

3. Computational Procedure. The simplified calculation of potency and its error depends upon substituting the assumed for the actual doses. When spaced equally on an arithmetic scale, they may be coded by using the numbers 1, 2, 3, $\cdots k$, k being equal throughout the assay. The sums of the coded doses, S_1, and of their squares, S_2, are then the same for each preparation and may be entered in the equations for computing the inverse matrix, of which the first three are

$$\begin{array}{llcccc}
 & & & i = 0 & i = 1 & i = 2 \\
 & Nc_{0i} + hS_1c_{1i} + rS_1c_{2i} + \cdots & = & 1, & 0, & 0, \cdots \\
(1) & hS_1c_{0i} + hS_2c_{1i} & = & 0, & 1, & 0, \cdots \\
 & rS_1c_{0i} \qquad\qquad + rS_2c_{2i} & = & 0, & 0, & 1, \cdots
\end{array}$$

where the total number of observations is $N = h' + kh + kr(m - 1)$. Multiplying the last two rows by $-S_1/S_2$ and adding the products, we have

$$\left\{N - \frac{hS_1^2}{S_2} - (m - 1)\frac{rS_1^2}{S_2}\right\} c_{0i} = 1, \qquad -\frac{S_1}{S_2}, \qquad -\frac{S_1}{S_2}, \cdots$$

where the subscript $_1$ refers to the standard and the assay includes $_2$ to $_m$ unknown preparations. Substituting

$$D = NS_2 - hS_1^2 - r(m-1)S_1^2,$$

this leads to the following reciprocal coefficients:

$$\begin{aligned}
c_{00} &= S_2/D & & \\
c_{0i} &= c_{i0} = -\ S_1/D, & & i = 1, 2, \cdots m, \\
c_{11} &= 1/hS_2 + S_1^2/DS_2 & & \\
c_{ii} &= 1/rS_2 + S_1^2/DS_2, & & i = 2, 3, \cdots m, \text{ and} \\
c_{ij} &= S_1^2/DS_2 & & \text{for } i, j = 1, 2, \cdots m, \text{ where } i \neq j.
\end{aligned}$$

The reciprocal coefficients are computed from the sums of the doses and their squares, which are the same for all preparations. The doses are multiplied by the responses observed at each dosage level to obtain $T_i = S(xy_i)$ for any given preparation. For the standard there will be h responses at each dose and for each unknown r responses. Let $T = S(T_i)$ be the sum of these products over all m preparations. The total response for all N observations $S(y)$, including the negative control, the standard, and all the unknowns, is designated as T_y.

Using normal regression theory, the common intercept is computed as

$$a' = c_{00}T_y + c_{0i}T.$$

Substituting the above reciprocal coeficients,

$$a' = (S_2T_y - S_1T)/D. \tag{2}$$

The slope of the standard is computed with the reciprocal coefficients as

$$b_1 = c_{01}T_y + c_{11}T_1 + c_{1i}T - c_{1i}T_1.$$

We may take advantage of the identities

$$c_{01} = -\frac{S_1}{S_2}c_{00} \quad \text{and} \quad c_{1i} = -\frac{S_1}{S_2}c_{0i}$$

to obtain

$$b_1 = (c_{11} - c_{1i})T_1 - \frac{S_1}{S_2}a'$$

reducing to

$$b_1 = \frac{T_1}{hS_2} - \frac{a'S_1}{S_2}. \tag{3}$$

Similarly the slope of each unknown is equal to

$$b_i = c_{0i}T_y + c_{1i}T_1 + c_{ii}T_i + c_{ij}T - c_{ij}\{T_1 + T_i\}$$

where $i, j = 2, 3, \cdots m$ and $j \neq i$. Since $c_{1i} - c_{ij} = 0$, this may be reduced to

$$(4) \qquad b_i = \frac{T_i}{rS_2} - \frac{a'S_1}{S_2}, \qquad i = 2, 3, \cdots m.$$

The computation is further simplified if the k doses of all preparations are spaced not only similarly on an arithmetic scale but also at equal intervals. In this case

$$S_1 = k(k+1)/2 \qquad \text{and} \qquad S_2 = k(k+1)(2k+1)/6.$$

Substituting in equations (2), (3) and (4), the common intercept, the slope of the standard and that of each unknown may be computed as

$$(5) \qquad a' = \frac{2(2k+1)T_y - 6T}{N(k-1) + 3h'(k+1)}$$

$$(6) \qquad b_1 = \frac{3}{2k+1}\left\{\frac{2T_1}{hk(k+1)} - a'\right\}$$

$$(7) \qquad b_i = \frac{3}{2k+1}\left\{\frac{2T_i}{rk(k+1)} - a'\right\}.$$

In computing the slope for each unknown in an assay the only variable is T_i. The intercepts and the slope can be checked by substitution in the equation

$$(8) \qquad 2Na' + hk(k+1)b_1 + rk(k+1)(b_2 + \cdots + b_m) = 2T_y.$$

In terms of coded doses, the potency of an unknown (i) relative to that of the standard ($_1$) is computed as

$$(9) \qquad J'_i = \frac{b_i}{b_1}.$$

Each J' is converted to original units by multiplying it by the ratio of the dosage intervals, I_s/I_u, the potency being

$$(10) \qquad J = \frac{b_u I_s}{b_s I_u}.$$

The variance measuring the distribution of the observations about the m lines may be determined as

$$(11) \qquad s^2 = \frac{S(y^2) - a'T_y - b_1T_1 - \cdots - b_mT_m}{N - m - 1}.$$

The variation about the individual lines is assumed not to vary from one preparation to another. This is more likely to be true when the assumed potencies differ but little from those computed from the assay, so that J' differs relatively little from unity.

The confidence limits for potency as estimated from the ratio of the slopes may be computed from Fieller's basic formula [4]. For confidence limits, X_L,

at an appropriate level of significance, such as $P = 0.05$, t is read from the Student-distribution for $N - m - 1$ degrees of freedom and entered with s^2 from equation (11) in the equation

$$X_L^2(b_1^2 - c_{11}s^2t^2) - 2X_L(b_1b_i - c_{1i}s^2t^2) + (b_i^2 - c_{ii}s^2t^2) \leq 0, \tag{12}$$

where i indicates one of the 2 to m unknown preparations. When solved for 0, the limits may be written

$$X_L = \frac{b_1b_i - c_{1i}s^2t^2}{b_1^2 - c_{11}s^2t^2} \tag{13}$$
$$\frac{\pm\, st\sqrt{(c_{11} - c_{1i})b_i^2 + (c_{ii} - c_{1i})b_1^2 + c_{1i}(b_1 - b_i)^2 - (c_{11}c_{ii} - c_{1i}^2)s^2t^2}}{b_1^2 - c_{11}s^2t^2}$$

where $c_{11} - c_{1i} = 1/hS_2$, $c_{ii} - c_{1i} = 1/rS_2$ and $c_{11}c_{ii} - c_{1i}^2 = \dfrac{(r + h)S_1^2 + D}{rhDS_2}$.
In all critical cases, the exact limits should be computed.

In most slope-ratio assays the individual slopes differ very significantly from zero. Under these circumstances the approximate limits may be computed with reasonable accuracy from the variance of the estimated potency by the familiar formula for the variance of a ratio [1].

$$\begin{aligned} V(J') &= \frac{b_i^2 s^2}{b_1^2}\left\{\frac{c_{11}}{b_1^2} + \frac{c_{ii}}{b_i^2} - \frac{2c_{1i}}{b_1 b_i}\right\} \\ &= \frac{s^2}{b_1^4}\{(c_{11} - c_{1i})b_i^2 + (c_{ii} - c_{1i})b_1^2 + c_{1i}(b_1 - b_i)^2\}. \end{aligned} \tag{14}$$

The discrepancies between the approximate and the exact limits are evident from a comparison of equations (13) and (14). When the doses are spaced at equal arithmetic intervals, equation (14) can be reduced to the more convenient form

$$s_{J'}^2 = \frac{6s^2}{b_1^2(2k + 1)}\left\{\frac{h + rJ'^2}{rhk(k + 1)} + \frac{3(1 - J')^2}{N(k - 1) + 3h'(k + 1)}\right\}. \tag{15}$$

A major limitation to slope-ratio assays is the frequent curvature in the relation between response and arithmetic dosage units. For this reason it is advisable to use routinely four or more doses of each preparation. Occasionally an assay in which there is curvature at the highest dosage level may be salvaged by computing the potencies from the data of the smaller doses. The agreement of a given assay with the postulate upon which it is based may be tested objectively by an analysis of variance, segregating the sums of squares (a) for the agreement of the negative control with the intercept, (b) for the agreement of the individual curves at the intercept, (c) for agreement of the observations with straight lines fitted individually and (d) for the variation among the h replicates of the standard, the h' replicates of the negative control and the r replicates of the unknowns. The calculation of such an analysis is greatly facilitated by the recommended

design. Since it follows the usual pattern, it will not be described here. The procedure has been tested with the data from an experiment on the depth dose of x-rays [2] and has been applied to microbiological assays [3] in papers where the reader will find the technique exemplified.

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NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

COMPUTATION OF FACTORS FOR TOLERANCE LIMITS ON A NORMAL DISTRIBUTION WHEN THE SAMPLE IS LARGE[1]

By Albert H. Bowker

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In their paper [1], Wald and Wolfowitz discuss the problem of finding tolerance limits of the form $\bar{x} \pm \lambda s$ for a normal distribution. They propose the following large sample formula for λ which appears to be satisfactory for all practical purposes for $N \geqq 2!$

$$\lambda = \sqrt{\frac{n}{\chi_\beta^2}}\, r\left(\frac{1}{\sqrt{N}}, \gamma\right) \tag{1}$$

where N is the number of observations $(n = N - 1)$, γ is the tolerance coefficient, β is the confidence coefficient, r is defined by

$$\frac{1}{\sqrt{2\pi}} \int_{(1/\sqrt{N})-r}^{(1/\sqrt{N})+r} e^{-t^2/2}\, dt = \gamma$$

and χ_β^2 has the property that $P(\chi^2 > \chi_\beta^2) = \beta$ for n degrees of freedom. To compute λ, tables [2] or known approximations [3] for χ_β^2 are customarily used, but the computation of r, even for large N, is tedious, involving an iterative procedure. The purpose of this note is to obtain an expansion of r in terms of $1/\sqrt{N}$ and to combine this expansion with a known one for χ_β^2 to obtain an asymptotic formula for λ.

To derive a large sample formula for r, consider the function

$$f(x, y) = \frac{1}{\sqrt{2\pi}} \int_{x-y}^{x+y} e^{-t^2/2}\, dt - \gamma = 0 \tag{2}$$

where for convenience $\frac{1}{\sqrt{N}}$ and r are replaced by x and y. It is desired to express y as a power series in x. Let y_0 be defined by $f(0,y_0) = 0$. Since $f(x,y)$ is a con-

[1] This paper reports work done in the Statistical Research Group, Division of War Research, Columbia University, under Contract OEMsr-618 with the Applied Mathematics Panel, National Defense Research Committee, Office of Scientific Research and Development. The work was first reported in an unpublished memorandum, "Computation of Factors for Tolerance Limits when the Sample is Large" (SRG No. 559, September 24, 1945) A brief account of the application of tolerance limits, including tables, will be published in *Techniques of Statistical Analysis* described in the footnote on page 217.

TABLE 1

Comparative Values of Exact and Approximate λ

β \ N	γ	50			100			160		
		Exact	Approximate	Difference	Exact	Approximate	Difference	Exact	Approximate	Difference
.75	.75	1.25480	1.25147	.00333	1.21808	1.21698	.00110	1.20161	1.20108	.00053
	.95	2.13774	2.13226	.00548	2.07533	2.07349	.00184	2.04728	2.04639	.00089
	.999	3.58821	3.57979	.00842	3.48401	3.48112	.00289	3.43704	3.43563	.00141
.95	.75	1.39621	1.38467	.01154	1.31050	1.30670	.00380	1.27204	1.27022	.00182
	.95	2.37866	2.35921	.01945	2.23279	2.22635	.00644	2.16728	2.16420	.00308
	.999	3.99259	3.96080	.03179	3.74835	3.73776	.01059	3.63850	3.63341	.00509
.99	.75	1.51184	1.48901	.02283	1.38251	1.37511	.00740	1.32566	1.32215	.00351
	.95	2.57565	2.53698	.03867	2.35546	2.34290	.01256	2.25865	2.25268	.00597
	.999	4.32325	4.25926	.06399	3.95429	3.93343	.02086	3.79189	3.78196	.00993

Comparative Values of Exact and Approximate λ*—Continued*

β \ N	γ	500			800			1000		
		Exact	Approximate	Difference	Exact	Approximate	Difference	Exact	Approximate	Difference
.75	.75	1.17733	1.17724	.00009	1.17126	1.17122	.00004	1.16891	1.16888	.00003
	.95	2.00593	2.00578	.00015	1.99559	1.99552	.00007	1.99158	1.99153	.00005
	.999	3.36769	3.36744	.00025	3.35034	3.35022	.00012	3.34361	3.34352	.00009
.95	.75	1.21501	1.21470	.00031	1.20062	1.20047	.00015	1.19502	1.19491	.00011
	.95	2.07013	2.06960	.00053	2.04562	2.04536	.00026	2.03608	2.03589	.00019
	.999	3.47547	3.47459	.00088	3.43433	3.43390	.00043	3.41831	3.41800	.00031
.99	.75	1.24268	1.24208	.00060	1.22198	1.22169	.00029	1.21395	1.21374	.00021
	.95	2.11727	2.11626	.00101	2.08201	2.08152	.00049	2.06832	2.06797	.00035
	.999	3.55462	3.55292	.00170	3.49543	3.49460	.00083	3.47244	3.47186	.00058

tinuous function of x and y, and since $\left.\dfrac{\partial f}{\partial y}\right|_{\substack{x=0\\ y=y_0}} \neq 0$, the function $y(x)$ defined implicitly by (2) is continuous. Since $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = \tanh xy$, the higher derivatives of $y(x)$ exist and are continuous and $y(x)$ permits of a finite Taylor's expansion. The coefficients of odd powers of x drop out and we obtain

$$y = y_0 + \frac{y_0}{2!}x^2 + \frac{3y_0 - 2y_0^3}{4!}x^4 + 0(x^6),$$

or returning to the original notation and retaining terms in $1/N$,

$$r \sim r_\infty\left(1 + \frac{1}{2N}\right). \tag{3}$$

If x_p is defined by $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_p} e^{-t^2/2}\,dt = p$ we know from [3] that

$$\frac{\chi_\beta^2}{n} \sim 1 + \frac{\sqrt{2}\,x_{1-\beta}}{\sqrt{n}} + \frac{2}{3}\frac{x_{1-\beta}^2 - 1}{n}. \tag{4}$$

Proceeding formally and retaining terms in $1/N$ we obtain

$$\left(\frac{n}{\chi_\beta^2}\right)^{\frac{1}{2}} = \left(1 - \frac{x_{1-\beta}}{\sqrt{2N}} + \frac{4 + 5x_{1-\beta}^2}{12N}\right)$$

and multiplying by the expression for r given by equation (3) we find the desired expansion for λ.

$$\lambda \sim r_\infty\left(1 - \frac{x_{1-\beta}}{\sqrt{2N}} + \frac{5x_{1-\beta}^2 + 10}{12N}\right). \tag{5}$$

Recall that both r_∞ and $x_{1-\beta}$ are readily obtainable from tables of the normal curve, in fact, r_∞ is defined by

$\frac{1}{\sqrt{2\pi}}\int_{-r_\infty}^{r_\infty} e^{-t^2/2}\,dt = \gamma$ and $x_{1-\beta}$ is defined by $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_{1-\beta}} e^{-t^2/2}\,dt = 1 - \beta$.

A comparative table of approximate and exact values of λ is given in Table 1. From the table we see that for $N \geqq 800$ the error is less than 1 in the 4th significant figure, and for $N \geqq 160$ the error is less than 1 in the 3rd significant figure within the limits of β and γ considered. The approximation will be less exact for higher values of β and γ.

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THE PROBABILITY DISTRIBUTION OF THE MEASURE OF A RANDOM LINEAR SET

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1. Introduction. Consider a random sample $O_n(x_1, \cdots, x_n)$ of n values of a one-dimensional random variable x having cumulative distribution function $F(x)$. Let there be associated with each x an interval of length D centered at x

(D a positive constant). Let $\bar{S}(0_n)$ denote the random set which is the point-set sum of the n intervals associated with 0_n; $\bar{S}(0_n)$ is a set of one or more intervals. Let S denote the measure of $\bar{S}(0_n)$ (S is the sum of lengths of the intervals composing $\bar{S}(0_n)$). Given F, n and D, what is the probability function of S? This note contains a solution of the problem for $F(x) = x$, $(0 \leq x \leq 1)$; the case of $F(x) = \int_0^x He^{-Ht}\,dt$, $(0 \leq x < \infty; H > 0)$, is also treated.

2. Sampling from a uniform distribution. Let $y = S - D$. The range of y is $0 \leq y \leq m$, where m denotes the minimum of 1 and $(n - 1)D$. Let $x_1, \cdots, x_n$ be the sample values arranged in increasing order of magnitude. Make the transformation

$$
\begin{aligned}
y_0 &= x_1 \\
y_i &= x_{i+1} - x_i, \qquad (i = 1, \cdots, n - 1).
\end{aligned} \tag{2.1}
$$

y can be expressed as $\sum_{i=1}^{n-1} m(y_i, D)$, where $m(y_i, D)$ denotes the minimum of y_i and D. The probability function of $(y_0, y_1, \cdots, y_{n-1})$ is $n! \prod_{u=0}^{n-1} dy_u$, $\left(y_u \geq 0; \sum_{u=0}^{n-1} y_u \leq 1\right)$. If $m = (n - 1)D$, then $y = (n - 1)D$ if and only if $y_i \geq D$, $(i = 1, \cdots, n - 1)$; for a fixed y_0 it can be shown by use of the Dirichlet integral that the volume of the $(n - 1)$ dimensional region in which any point $(y_0, y_1, \cdots, y_{n-1})$ satisfies this condition is $\dfrac{(1 - y_0 - (n - 1)D)^{n-1}}{(n - 1)!}$. It follows that:

$$
\begin{aligned}
\Pr\{y = (n - 1)D\} &= n \int_{y_0=0}^{1-(n-1)D} [1 - y_0 - (n - 1)D]^{n-1}\,dy_0 \\
&= [1 - (n - 1)D]^n, \qquad ((n - 1)D \leq 1).
\end{aligned} \tag{2.2}
$$

The probability that $Y < y < Y + \Delta Y$ (where $Y < m$ and ΔY denotes an arbitrarily small positive increment in Y) can be evaluated by determining volumes of certain regions contained in the tetrahedron defined by $y_u \geq 0$, $\sum_{u=0}^{n-1} y_u \leq 1$. Consider the following conditions:

(a) $qD \leq Y < (q + 1)D$ $\quad$ ($q = 0, 1, \cdots, M$; M denotes the minimum of $(n - 2)$ and the greatest integer less than $\frac{1}{D}$),

(b) $\quad y_u \geq D$ $\quad$ $(u = 1, \cdots, j; j \leq q)$,

(c) $\sum_{u=0}^{j} y_u \leq 1 - y_0 - y + jD$,

(d) $\quad y_v < D$ $\quad$ $(v = j + 1, \cdots, n - 1)$.

The probability that $Y < y < Y + \Delta Y$ and that (b), (c) and (d) are satisfied is:

$$n! \int_{y=Y}^{Y+\Delta Y} B_j(y) \int_{y_0=0}^{1-y} A_j(y, y_0)\, dy_0 \frac{dy}{\sqrt{n-j-1}}, \tag{2.3}$$

where $A_j(y, y_0)$ denotes the j dimensional volume of the region in which any point $(y_1, \cdots, y_j)$ satisfies (b) and (c), and $B_j(y)$ denotes the $(n - j - 2)$ dimensional volume of intersection of the hyperplane $\sum_{v=j+1}^{n-1} y_v = y - jD$ with an $(n - j - 1)$ dimensional cube $(0 \leq y_v \leq D)$. It is clear that if any other of the $\binom{n-1}{j}$ combinations of j y's out of the set of $(n - 1)$ y's had been specified in (b) and the $(n - j - 1)$ complementary y's had been specified in (d), the corresponding $A_j(y, y_0)$ and $B_j(y)$ would be equal to those given in (2.3); hence

$$\begin{aligned} \Pr\{Y < y < Y + \Delta Y\} &= n! \sum_{j=0}^{q} \binom{n-1}{j} \int_{y=Y}^{Y+\Delta Y} B_j(y) \\ &\quad \cdot \int_{y_0=0}^{1-Y} A_j(y, y_0)\, dy_0 \frac{dy}{\sqrt{n-j-1}}, \end{aligned} \tag{2.4}$$

$$qD \leq Y < (q + 1)D, \; Y < m, \qquad (q = 0, 1, \cdots, M).$$

$A_j(y, y_0) = \dfrac{(1 - y_0 - y)^j}{j!}$, and (see [1] and [2])

$$B_j(y) = \frac{\sqrt{n-j-1}}{(n-j-2)!} \sum_{r=0}^{q-j} (-1)^r \binom{n-j-1}{r} [y - D(j + r)]^{n-j-2}. \tag{2.5}$$

From (2.4) and (2.5) it follows that the probability function of y, say $f_n(y)$, is:

$$\begin{aligned} f_n(y) &= n \sum_{j=0}^{q} \sum_{r=0}^{q-j} (-1)^r \binom{n-1}{j} \binom{n-1}{j+1} \\ &\quad \cdot \binom{n-j-1}{r} (1 - y)^{j+1} [y - D(j + r)]^{n-j-2}, \end{aligned} \tag{2.6}$$

$$qD \leq y < (q + 1)D, \quad (q = 0, \cdots, M), \qquad y < m.$$

$f_n(y)$ is not defined at $(n - 1)D$ if $(n - 1)D < 1$ (see (2.2)); if $m = 1$, the range of definition of $f_n(y)$ as given in (2.6) is $y \leq 1$.

The cumulative distribution function of y is continuous with the exception, in the case of $(n - 1)D < 1$, of a saltus of amount $[1 - (n - 1)D]^n$ at $y = (n - 1)D$ (see (2.2)). The probability function $f_n(y)$ is continuous over the range $0 \leq y < m$ with the exception, in the case of $n \geq 3$ and $(n - 2)D < 1$, of a simple discontinuity at $y = (n - 2)D$.

For $n = 2$ and $D < 1$,

$$f_2(y) = 2(1 - y), \qquad (0 \leq y < D),$$

and $\Pr\{y = D\} = (1 - D)^2$.

For $n = 3$ and $2D < 1$,

$$f_3(y) = 6(1 - y)y, \qquad (0 \le y < D),$$

$$f_3(y) = 6(1 - y)y - 12(1 - y)(y - D) + 6(1 - y)^2, \quad (D \le y < 2D),$$

and $\Pr\{y = 2D\} = (1 - 2D)^3$.

The expected value, say $E(y)$, of y is:

$$\begin{aligned} E(y) &= \frac{(n-1)}{(n+1)}[1 - (1 - D)^{n+1}] && (D \le 1); \\ &= \frac{(n-1)}{(n+1)} && (D > 1). \end{aligned} \tag{2.7}$$

The expected value of S is $D + E(y)$. $E(y)$ can be derived by use of (2.6) or by use of a theorem of Robbins [3].

3. Probability that random linear set covers range of variate. Given that $F(x) = x$, $(0 \le x \le 1)$, and $nD > 1$, what is the probability, say ${}_nP_D$, that $\bar{S}(0_n)$ contains the interval $(0 \le x \le 1)$? If $D < 1$, the interval is covered if and only if (i), (ii) and (iii) below are all satisfied:

$$\text{(i)} \qquad y_u \le D, \qquad (u = 1, \cdots, n - 1),$$

$$\text{(ii)} \qquad \sum_{u=1}^{n-1} y_u \ge \left(1 - y_0 - \frac{D}{2}\right),$$

$$\text{(iii)} \qquad y_0 \le \frac{D}{2}.$$

${}_nP_D$ can be expressed as follows:

$$ {}_nP_D = n! \int_{y_0=0}^{D/2} \int_{z=1-y_0-D/2}^{1-y_0} C_{n-1}(z) \frac{dz}{\sqrt{n-1}}\, dy_0, \tag{3.1}$$

where $C_{n-1}(z)$ (see [2]) denotes the $(n - 2)$ dimensional volume of the intersection of the hyperplane $\sum_{u=1}^{n-1} y_u = z$ with an $(n - 1)$ cube $0 \le y_u \le D$. It follows from (2.5) and (3.1) that

$$\begin{aligned} {}_nP_D = \sum_{u=0}^{[1/D]} (-1)^u \binom{n-1}{u}(1 - uD)^n & \\ - 2 \sum_{u=0}^{[(1/D)-1]} (-1)^u \binom{n-1}{u}\left(1 - uD - \frac{D}{2}\right)^n & \\ + \sum_{u=0}^{[(1/D)-1]} (-1)^u \binom{n-1}{u}(1 - uD - D)^n, & \end{aligned} \tag{3.2}$$

where $D < 1$ and $[x]$ denotes the greatest integer less than x. If $1 \le D < 2$, ${}_nP_D = 1 - 2\left(1 - \frac{D}{2}\right)^n$.

4. Sampling from $F(x) = \int_0^x He^{-Ht}\,dt$, $(0 \leq x < \infty;\ H > 0)$. If $F(x) = \int_0^x He^{-Ht}\,dt$, the probability function of S can be determined but is very cumbersome in the form in which it is known to the writer. The characteristic function, say $g(\theta)$, of the probability function of S will be given instead. By use of (2.1) it can be shown that:

$$g(\theta) = e^{iD\theta} \prod_{\lambda=1}^{n-1} \left\{ \frac{i\theta e^{D(i\theta-\lambda H)} - \lambda H}{i\theta - \lambda H} \right\}, \tag{4.1}$$

where $i = \sqrt{-1}$.

The expected value, $E(S)$, and variance, σ_S^2, of S are:

$$\begin{aligned} E(S) &= D + \frac{1}{H} \sum_{\lambda=1}^{n-1} \frac{(1 - e^{-DH\lambda})}{\lambda}, \\ \sigma_S^2 &= \frac{1}{H^2} \sum_{\lambda=1}^{n-1} \frac{(1 - e^{-2DH\lambda})}{\lambda^2} - \frac{2D}{H} \sum_{\lambda=1}^{n-1} \frac{e^{-DH\lambda}}{\lambda}. \end{aligned} \tag{4.2}$$

REFERENCES

[1] P. S. LAPLACE, *Théorie Analytique Des Probabilités*, *Gauthier-Villars*, Paris, Third Edition (1820), Book 2, Paragraph 13.
[2] PHILIP HALL, "The distribution of means for samples of size n drawn from a population in which the variate takes values between 0 and 1, all such values being equally probable," *Biometrika*, Vol. 19 (1927), pp. 240-245.
[3] H. E. ROBBINS, "On the measure of a random set," *Annals of Math. Stat.*, Vol. 15 (1944), p. 72.

INFORMATION GIVEN BY ODD MOMENTS

BY EDMUND CHURCHILL

Rutgers University

The widespread use of the third moment about the mean as a measure of skewness and the belief engendered by this use that a distribution is symmetric if its third moment is zero prompt the question of how much information about a distribution can be deduced from a knowledge of its odd moments. An answer to this question is: *Let* $F(x)$, *a cumulative distribution function;* $\{\mu_{2n-1}\}$, $(n = 1, 2, \cdots)$, *a sequence of real numbers; and* $\epsilon > 0$ *be arbitrary. There exists a c.d.f.,* $F^*(x)$, *having as odd moments the terms of the given sequence and such that*

$$| F(x) - F^*(x) | \leq \epsilon, \ \underline{all}\ x. \tag{1}$$

If the mean of $F(x)$ is equal to μ_1 and the variance of $F(x)$ is not zero, it can be shown that $F^*(x)$ may be chosen so that in addition the variance of $F^*(x)$ is equal to that of $F(x)$.

An immediate consequence of our statement is that a distribution need not be

symmetric even though all its odd moments vanish. Such an asymmetric distribution, due to Stieltjes, is given by:

$$(2)\quad dF(x) = 1/48\, e^{-|x|^{\frac{1}{4}}} \cdot (1 - k \sin |x|^{\frac{1}{4}})\, dx, \; -\infty < x < \infty, \; k = -1 \text{ if } x < 0,$$

$$k = 1 \text{ if } x \geq 0.$$

The proof of our statement will follow easily from the following:

LEMMA. *Let $\{m_{2n-1}\}$, a sequence of real numbers be given. There exists a c.d.f. having as odd moments the given numbers.*

We construct a sequence $\{H_n\}$ of increasing step-functions in such a manner that for every n, the first n moments of H_n are the first of the given numbers, and such that this sequence converges to a monotone function having all the desired moments. A slight modification of this function will give the desired c.d.f.

Let H_0 be identically zero. We form H_1 by adding to H_0 a jump or mass of $\frac{1}{2}$ at $x = 2m_1$. In general, H_k is formed from H_{k-1} by adding to it k masses chosen so that their first $(k-1)$ odd moments are zero and so that the kth odd moment of H_k is m_{2k-1}. This we do by adding the masses $|x_j|$, $(j = 1, 2, \cdots, k)$, at the points $e_j jp$ where the x_j's are the solutions of:

$$\begin{aligned}
px_1 + \quad 2px_2 + \cdots + \quad kpx_k &= 0\\
p^3x_1 + \quad (2p)^3x_2 + \cdots + \quad (kp)^3x_k &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
p^{2k-3}x_1 + (2p)^{2k-3}x_2 + \cdots + (kp)^{2k-3}x_k &= 0\\
p^{2k-1}x_1 + (2p)^{2k-1}x_2 + \cdots + (kp)^{2k-1}x_k &= m_{2k-1} - m(H_{k-1}),
\end{aligned}$$

$m(H_{k-1})$ is the kth odd moment of H_{k-1}, e_j is the sign of x_j and p is a parameter. Since the determinant of this system is a Vandermonde determinant, there exists a unique set of solutions for every non-zero value of the parameter. The masses thus chosen clearly have the specified moments. Eliminating p from the left sides of the equations by division, it is apparent that the x_j's are all linear functions of $p^{-(2k-1)}$. Thus we may choose p so large that the sum of the masses added at this step does not exceed $1/2^k$. The absolute odd moments of orders less than $(2k-1)$ of these k masses are also linear functions of negative powers of p. We may thus insure by further increasing our choice of p that the $(2k - 1 - 2r)$th absolute moment of H_k does not exceed the corresponding moment of H_{k-1} by more than $1/2^r$. For definiteness, we choose p as the smallest number satisfying these requirements.

The first of these restrictions on p insures that for each value of x, the sequence $H_n(x)$ is increasing and bounded from above by one. The sequence of functions thus converges to a monotone function $H^*(x)$ with the property that $H^*(-\infty) = 0, H^*(\infty) \leq 1$. The other restrictions on p insure that the sequences of abso-

lute odd moments of all orders are uniformly bounded, a bound for the absolute moments of order $2k - 1$ being one greater than the absolute moment of this order of H_k. This in turn insures that the odd moments of $H^*(x)$ exist and that they have the desired values. By adding a jump of $1 - H^*(\infty)$ at the origin we obtain $H(x)$, a c.d.f with the given odd moments.

The main statement of this note is an immediate consequence of the lemma. Let the kth odd moment of $F(x)$ be M_{2k-1}, which we assume to be finite, and let the sequence $\{m_{2k-1}\}$ be defined by the relationships:

$$\mu_{2k-1} = (1 - \epsilon)M_{2k-1} + \epsilon m_{2k-1}, \qquad (k = 1,2, \cdots).$$

Let $H(x)$ have the m's as odd moments. The c.d.f. $F^*(x)$ defined by

$$F^*(x) = (1 - \epsilon)F(x) + \epsilon H(x)$$

clearly has the properties stated above, and our statement is proved. If the moments of $F(x)$ are not all finite, the proof will need only minor modifications.

If one asks in addition that F^* have a finite range, F^* will, in general, not exist. If, for example, the range of F is finite and its odd moments are zero, then F must be symmetric about the origin, for F^* defined by $dF^*(x) = dF(-x)$ would have the same moments as F. But a c.d.f. with finite range is determined by its moments; hence $F(x) = F^*(x)$.

SOME ORDER STATISTIC DISTRIBUTIONS FOR SAMPLES OF SIZE FOUR

By John E. Walsh

Princeton University

1. Summary. Let x_1, x_2, x_3, x_4 represent the values of a sample of size four drawn from a normal population. There is no loss of generality in assuming that the distribution function of this population has zero mean and unit variance. Denote it by $N(0, 1)$. Let $x_{(i)}$ be the ith largest of x_1, x_2, x_3, x_4. The purpose of this note is to determine the joint distribution of

$x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)}$, $x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)}$, and $x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)}$, and derive from this joint distribution the joint distributions of these statistics taken in pairs, also the distribution of each statistic itself.

2. Analysis. Consider the joint distribution of

$$r_1 = \tfrac{1}{2}(x_4 + x_3 - x_2 - x_1)$$

$$r_2 = \tfrac{1}{2}(x_4 - x_3 + x_2 - x_1)$$

$$r_3 = \tfrac{1}{2}(x_4 - x_3 - x_2 + x_1).$$

Evidently,

$$E(r_i) = 0, \quad (i = 1, 2, 3). \quad E(r_ir_j) = 0, \quad (i \neq j). \quad E(r_i^2) = 1.$$

Hence the r_i are independently distributed according to $N(0, 1)$.

Let v_j be the jth largest of $|r_1|, |r_2|, |r_3|$. Then by first finding the joint distribution of $|r_1|, |r_2|, |r_3|$ and then applying the distribution for order statistics [1], it is easily seen that the joint distribution element of v_1, v_2, v_3 is

$$48f(v_1)f(v_2)f(v_3)dv_1dv_2dv_3,$$

where

$$f(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}, \qquad 0 \leq v_1 \leq v_2 \leq v_3.$$

Examination shows, however, that

$$\begin{aligned} v_3 &= \tfrac{1}{2}(x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)}) \\ v_2 &= \tfrac{1}{2}(x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)}) \\ v_1 &= \tfrac{1}{2}\,|\,x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)}\,| \end{aligned}$$

Let

$$\begin{aligned} m_3 &= x_{(4)} + x_{(3)} - x_{(2)} - x_{(1)} \\ m_2 &= x_{(4)} - x_{(3)} + x_{(2)} - x_{(1)} \\ m_1 &= x_{(4)} - x_{(3)} - x_{(2)} + x_{(1)}. \end{aligned}$$

Then the joint distribution element of $|m_1|$, m_2 and m_3 is

$$6f(\tfrac{1}{2}\,|\,m_1\,|)f(\tfrac{1}{2}m_2)f(\tfrac{1}{2}m_3)d\,|\,m_1\,|\,dm_2dm_3.$$

Since the function f is symmetrical about the origin, it follows immediately that the joint distribution element of m_1, m_2 and m_3 is

$$3f(\tfrac{1}{2}m_1)f(\tfrac{1}{2}m_2)f(\tfrac{1}{2}m_3)dm_1dm_2dm_3,$$

where $|\,m_1\,| \leq m_2 \leq m_3$.

3. Derived results. By taking marginal distributions it is found that the joint distribution elements of m_1, m_2 and m_3 taken in pairs are

$$g_1(m_1, m_2)dm_1\,dm_2 = 3\left(\int_{m_2}^{\infty} f(\tfrac{1}{2}y)dy\right) f(\tfrac{1}{2}m_1)f(\tfrac{1}{2}m_2)dm_1\,dm_2.$$

$$g_2(m_1, m_3)dm_1\,dm_3 = 3\left(\int_{|m_1|}^{m_3} f(\tfrac{1}{2}y)dy\right) f(\tfrac{1}{2}m_1)f(\tfrac{1}{2}m_3)dm_1\,dm_3.$$

$$g_3(m_2, m_3)dm_2\,dm_3 = 6\left(\int_{0}^{m_2} f(\tfrac{1}{2}y)dy\right) f(\tfrac{1}{2}m_2)f(\tfrac{1}{2}m_3)dm_2\,dm_3.$$

The distribution elements of m_1, m_2 and m_3 are seen to be

$$g_1(m_1)dm_1 = \frac{3}{2}\left(\int_{|m_1|}^{\infty} f(\tfrac{1}{2}y)dy\right)^2 f(\tfrac{1}{2}m_1)dm_1 .$$

$$g_2(m_2)dm_2 = 6\left(\int_0^{m_2} f(\tfrac{1}{2}y)dy\right)\left(\int_{m_2}^{\infty} f(\tfrac{1}{2}y)dy\right)f(\tfrac{1}{2}m_2)dm_2 .$$

$$g_3(m_3)dm_3 = 3\left(\int_0^{m_3} f(\tfrac{1}{2}y)dy\right)^2 f(\tfrac{1}{2}m_3)dm_3 .$$

It is to be noted that if $a > 0$,

$$Pr(0 < m_1 < a) = Pr(-a < m_1 < 0) = \tfrac{1}{2} - 4\left(\int_{a/2}^{\infty} f(y)dy\right)^3 ,$$

$$Pr(0 < m_2 < a) = 12\left(\int_0^{a/2} f(y)dy\right)^2 - 16\left(\int_0^{a/2} f(y)dy\right)^3 ,$$

$$Pr(0 < m_3 < a) = 8\left(\int_0^{a/2} f(y)dy\right)^3 ,$$

so that the probability that any of m_1, m_2, m_3 lie between two given numbers is expressed explicitly and can be calculated with the aid of standard tables for the normal distribution.

4. Generalization of method. The method used to obtain the joint distribution of the order statistics m_1, m_2 and m_3 was to take all possible combinations of 4 variables with two plus and two minus signs (except for factor of -1) and show that these combinations behave as normally distributed independent variables. The question arises as to whether this method of finding order statistic distributions would apply in general to $2n$ variables with n plus and n minus signs. It is easily proved that this will occur only when $n = 2$.

REFERENCES

[1] S. S. Wilks, *Mathematical Statistics*, Princeton Univ. Press, 1943, p. 90.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Institute of Statistics of the University of North Carolina

Announcement of detailed plans for the North Carolina All-University Institute of Statistics has been made by Professor Gertrude M. Cox, Director of the Institute.

To provide graduate-level training for students in statistics and to combine the theoretical or mathematical statistics with applied or experimental statistics, a Graduate Department of Mathematical Statistics is being set up at Chapel Hill with Professor Harold Hotelling as Head. The existing Department of Experimental-Statistics at Raleigh is a part of the Institute, and will be headed by Professor Gertrude M. Cox with Professor W. G. Cochran as Director of Research. Professors Hotelling and Cochran will be Associate Directors of the Institute.

Professor Hotelling, who will head the Department at Chapel Hill comes to North Carolina from Columbia University, where he has been directing its graduate mathematical statistics program. Previously, he had held positions with the University of Washington, Princeton University and Stanford University. His undergraduate training was taken at the University of Washington where he majored in journalism; his Master of Science degree was awarded by the same institution in mathematics; and his doctorate by Princeton University, also in mathematics. In addition, he has done some graduate work at the University of Chicago. Professor Hotelling's publications in mathematical statistics are numerous and well known. Among the members of his staff will be a visiting professor, M. S. Bartlett, on leave of absence from Cambridge University. A graduate of Cambridge and native of England, Bartlett has also held positions with the University of London and the Imperial Chemical Industries, and during the war was engaged in war research in London.

In addition, P. L Hsu, William Madow, and Herbert Robbins, will be members of the Department at Chapel Hill as associate professors. Hsu, a native of China, has held teaching positions with the University of Peking and the University of London. He received his degrees from Tsinghua University and the University of London.

Madow is now in Brazil, where he is serving as a visiting professor of statistics at the University of São Paulo. He received his training, both undergraduate and graduate, from Columbia University, and has worked with the Department of Agriculture Graduate School and the Bureau of the Census in Washington.

Robbins will come to the University of North Carolina from New York University where he has been serving as an assistant professor. Prior to that, he was a staff member of the postgraduate school of the U S. Naval Academy, and an instructor in mathematics at New York University and at Harvard University. He holds A.B., A.M. and Ph.D. degrees from Harvard University.

The appointment of Edward Paulson as an instructor completes the initial Department staff at Chapel Hill. A graduate of Brooklyn College and holder of an M.A. degree from Columbia University, Paulson has been more recently studying mathematical statistics at Columbia under a pre-doctoral fellowship of the National Research Council.

Professor Cochran came to North Carolina in March from Ames, Iowa, where he had been serving as professor in the statistical laboratory of Iowa State College. During the war years he was sent to England, Germany, and Austria on special work for the War Department, after spending a year at Princeton University where he served as research statistician on war work. A native of Glasgow, Scotland, Cochran has been in the United States since 1939, and is a naturalized citizen. Before coming to America, he was employed as statistician with the Rothamsted Experimental Station in England. Cochran's publications in both the theory of statistics and applied statistics are well known, as is his experience with practical research problems. He is serving this year as president of the Institute of Mathematical Statistics. He is a fellow of the American Statistical Association and a fellow of the Royal Statistical Society of England.

Under the plans of the Institute, students who are preparing to teach statistics or to develop statistical theory will take most of their training at Chapel Hill. However, work between the two branches will be so coordinated as to include instruction in the application of statistics as taught in Raleigh.

For students who intend to become statistical consultants in various other fields, basic training will be taken in mathematical statistics, with the main part of the advanced applied training at Raleigh.

For research students, on both campuses, who are working in other sciences, including agriculture, biology, medicine, psychology, sociology, economics, industry, and textiles, training in both basic and applied statistics will be given.

Working with Cochran in Raleigh are Professor J. A. Rigney; Associate Professors R. L. Anderson, J M. Clarkson, H. L. Lucas, and Paul Peach; Assistant Professor H. F. Robinson; Instructors Margaret Fleming, R. J. Monroe and Sarah Porter

Collaborators working with the Raleigh unit are A. L. Finkner, W. A. Hendricks and F. E. McVay of the Bureau of Agricultural Economics; C. E. Lamoureaux and G. P. Weber of the Weather Bureau; and D. D. Mason of the Bureau of Plant Industry.

Joint Session of the Institute and Section A of the AAAS

A joint session of the Institute of Mathematical Statistics and Section A of the American Association for the Advancement of Science was held in the Municipal Auditorium at St. Louis on Saturday, March 30, 1946 at 2:00 P M. At this session invited addresses were given by Lieutenant Commander John H.

Curtiss on *Statistical Inference and its Engineering Applications*, and by Mr. Morris H. Hansen on *Some Sampling Problems in Surveys of Business and Population*.

Personal Items

Dr. Paul H. Anderson is at present Economic Analyst with the War Assets Corporation at Washington. He is also teaching mathematics in the evening school of American University.

Assistant Professor T. A. Bancroft has returned from a teaching position at the University Study Center at Florence, Italy, to his position at Iowa State College.

Associate Dean Walter Bartky of the University of Chicago has been appointed Dean of the Division of Physical Sciences.

Mr. Gordon L. Beckstead in working toward his doctorate in statistics at the University of California.

Mr. Donald Cody has returned to his position as Assistant Actuary at the Equitable Life Assurance Society after spending three years in war research with the NDRC, the Naval Ordnance Station at Indianapolis, and the Naval Ordnance Station at Inyokern, California.

Professor Allen T. Craig, after war service at the Postgraduate School of the U. S. Naval Academy at Annapolis, has returned to his position at the University of Iowa.

Mr. James H. Davidson is studying for his doctorate in chemistry at Princeton University.

Associate Professor J. L. Doob of the University of Illinois has been promoted to a professorship.

Assistant Professor Churchill Eisenhart of the University of Wisconsin has been promoted to an associate professorship.

Dr. Wayne Gutzman recently discharged from the Navy as Lieutenant, has assumed his new duties as Assistant Professor of Mathematics at the Postgraduate School, Naval Academy, Annapolis, Maryland.

Mr. Bernard Hecht has been discharged from the Army and is now Chief Quality Control Engineer with the International Resistance Company at Philadelphia.

Dr. D. G. Humm has been elected president of the Southern California Academy of Criminology.

Mr. Amrom H. Katz is in charge of a group of physicists, engineers, and aerial photographers representing the Aerial Photographic Laboratory at Wright field, which will record photographically various aspects of the forthcoming atomic bomb test at Bikini Island.

Mr. Edward A. Lew has ben released from active duty and has returned to his former position as Assistant Actuary of the Metropolitan Life Insurance Company.

Dr. E. V. Lewis is Junior Research Associate with E. I. duPont de Nemours at the Nylon Research Laboratory at Wilmington.

Associate Professor M. C. MacPhail of Acadia University, Wolfville, Nova Scotia, has been promoted to a professorship.

Mr. C. J. Maloney has been appointed to an instructorship in the department of mathematics at Iowa State College.

Dr. Edward B. Olds is director of the Research Bureau of the Social Planning Council of St. Louis and St. Louis County.

Dr. A. M. Peiser has been appointed head of the Statistics Research Group at the Langley Field Laboratory of the National Advisory Committee for Aeronautics.

Mr. Robert J. Saunders has been released from the Army and is now connected with Mohawk Carper Mills at Amsterdam N. Y.

Mr. Benjamin Stauber is now Chief of the Relocation Planning Division, War Relocation Authority. He has transferred from the Department of Agriculture for this work.

Mr. Arthur I. Sternhell returned from the Army to his position as general staff assistant in the Field Management Division of the Metropolitan Life Insurance Company.

Mr. Harry Weingarten has been appointed Tutor of Mathematics at the College of the City of New York.

Assistant Professor J. R. Vatnsdal has finished his army service and has returned to the State College of Washington where he was promoted to an associate professorship.

Mr. Bertram Yood has completed his duty in the navy and is now at Yale Station, Connecticut.

A symposium on mathematical statistics and probability was held at the University of California at Berkeley, January 28–30, 1946.

New Members

The following persons have been elected to membership in the Institute:

Alchian, Prof. Armen A., Ph D. (Stanford) Univ of Oregon, *Capt. (A.C.) Hq. AAF Training Command, Ft Worth, Texas*

Bingham, M.D. 1920 S St., N. W., Washington, D. C.

Cannon, Edward W., Ph.D. (Johns Hopkins) Comdr., US Navy, Research and Standards Branch of Bureau of Ships, Cannon, Delaware

Carvalho, Prof. Pedro Egydio, Ph.D (São Paulo) Univ. de São Paulo, Faculadade de Higiene, *Avenida Dr. Arnaldo 85, Caixa postal 99-B, Sao Paulo, Brazil*

Delsa, Alexis, A. I. Lg. (Liege) Mgr. Basic Bessemer Steelworks, Société Anonyme John Cockerill, Seraing, Belgium

Duncan, David Beattie, B.SC. (Sydney) Graduate Student, Iowa State, Statistical Laboratory, Ames, Iowa

Froelich, Kathryn, B.A. (Evansville) Statistician, US Dept. of Agriculture, Bureau of Human Nutrition and Home Economics, *1806 Monroe St , N. W., Washington 10, D. C.*

Goldstine, Herman H. Ph D. (Chicago) Institute for Advanced Study, Princeton, N J.

Hammond, Edward Cuyler, Sc D (Johns Hopkins) Major A.C , US AAF, Chief, Statistics of Flying Personnel Branch, Office of the Air Surgeon, *4700 Connecticut Ave , Washington, D. C*

Hsu, Prof. Pao-Lu, Ph D. (London) Columbia University, *1027 John Jay Hall, Columbia Univ , New York City*

Kyle, Garland Dean, M.S. (Michigan) Spectroanalyst, Physicist (US Navy)*5848 Filbert, Philadelphia 39, Penn.*

Leibler, Richard A., Ph.D. (Illinois) Instructor, Purdue Univ , Math Dept , Lafayette, Indiana

Lessard, Prof. Roger, C E. (Montreal) Hull Technical School, Hull, Quebec, Canada

Mosimann, Thomas F., A.B. (Charleston) US Bur Labor Statistics, Regional Employment Analyst, *4216 Western Ave., Dallas 11, Texas*

Patte, W. Edmund, B.A.Sc. (Toronto) Stat. Eng , Canadian Industries Ltd , Shawinigan Falls, P.Q. Canada, *550—16th St , Almaville*

Piza, Prof. Affonso P. de Toledo, Ph.D. (São Paulo) Escola Politechnica, São Paulo, Brazil, *Rua Ministro Godoy, 1123*

Rozen, Daniel I., A.B. (Columbia) Stat , Medical Statistics Div., Office of the Surgeon General, War Department, *Rm 317-1, 3415 38th St , N. W., Washington, D C*

Saidel, Frank, M.A. (Michigan State) Instructor in Math , Michigan State, East Lansing, Michigan

Schmalz, William Herbert, B Sc.A. (Toronto) Technical Superintendent, Dominion Rubber Company Limited, Merchants Rubber Factory, 51 Breithaupt St., Kitchener, Ont.

Stehn, John R., Ph D. (Wisconsin) Physicist, Research Division, Winchester Repeating Arms Co , New Haven, Conn.

Tsao, Prof. Fei, Ph D. (Minnesota) National Central University, Chungking, China

Weaver, Chalmers L., B S. (Kent State) Asst. Actuary, New England Mutual Life Ins. Co , *501 Boylston St , Boston, Mass.*

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REPORT ON THE APRIL MEETING OF THE WASHINGTON CHAPTER OF THE INSTITUTE

A meeting of the Washington Chapter of the Institute of Mathematical Statistics was held at George Washington University, Washington, D. C. on Friday and Saturday, April 12 and 13, 1946, in conjunction with a meeting of the Washington Chapter of the American Statistical Association.

More than 100 people attended the meetings including the following 51 members of the Institute:

Theodore W. Anderson, Jr., Richard O. Been, Archie Blake, David Blackwell, J. B. Boddie, Glenn W. Brier, William Cohen, Jerome Cornfield, John H. Curtiss, Bessie B. Day, Robert Dorfman, Thomas I. Edwards, Andrew Fraser, Meyer A. Girshick, Clyde H. Graves, Margaret J. Hagood, Major Edward C. Hammond, Morris H. Hansen, Alston S. Householder, Leonid Hurwicz, Irwin E. Jackson, Jr., Walter Jacobs, Hyman B. Kaitz, H. S. Konji, Lila F. Knudsen, Colonel S. Kullback, R. B. Ladd, H. G. Landen, Walter Leighton, Gerson Levin, Jacob E. Lieberman, Sophie Marcuse, Ethelyne L. McBee, William J. McCabe, Francis McIntyre, Dorothy Morrow, H. W. Norton, W. R. Pabst, Carl J. Rees, David Rosenblatt, M. Sandomire, Edward M. Schrock, L. W. Shaw, John H. Smith, Frederick F. Stephan, F. M. Wadley, A. Wald, F. M. Weida, Samuel Weiss, S. S. Wilks, C. P. Young.

The session Friday evening was devoted to the following contributed papers:

1. *Estimation of the Parameters of a Single Stochastic Difference Equation in a Complete System.*
   T. W. Anderson and H. Rubin, Cowles Commission for Economic Research
   M. A. Girshick, Bureau of Agricultural Economics
   Presented by T. W. Anderson
2. *Estimation of Linear Functions of Cell Proportions.*
   J H. Smith, Bureau of Labor Statistics
3. *On Functions of Sequences of Independent Chance Vectors with Applications to the Random Walk Problem in k dimensions.*
   D. Blackwell, Howard University
   M. A. Girshick, Bureau of Agricultural Economics
   Presented by D. Blackwell
4. *The Exact Power Curve and Distribution of n for the Sequential Binomial Probability Ratio Test.*
   M. A. Girshick, Bureau of Agricultural Economics

At a business meeting following the session of contributed papers, Professor F. M. Weida and Dr. John H. Smith were elected to succeed Colonel Kullback and Dr Madow as members of the Program Committee.

The program for Saturday morning was devoted to the following invited lectures:

1. *Recent Developments in the Measurement of Simultaneous Economic Relations.*
   T. Koopmans, Cowles Commission for Economic Research
2. *Structural Estimation versus Regressions· use for Policy and Prediction.*
   Leonid Hurwicz, Cowles Commission for Economic Research

The program for Saturday afternoon was devoted to the following:

1. *Basic Concepts Underlying Sequential Analysis with Applications.*
   A. Wald, Columbia University.
2. *Applications of Sequential Analysis to Acceptance Inspection.*
   W. R. Pabst, Navy Department

Irving Siegel, Veterans Administration, was chairman for the morning session and Professor F. M. Weida, George Washington University, for the afternoon session.

A lively discussion followed the presentation of the papers.

S. Kullback,
Secretary, Washington Chapter.

SAMPLE CRITERIA FOR TESTING EQUALITY OF MEANS, EQUALITY OF VARIANCES, AND EQUALITY OF COVARIANCES IN A NORMAL MULTIVARIATE DISTRIBUTION

By S. S. Wilks

Princeton University

Summary. In this paper statistical test criteria are developed for testing equality of means, equality of variances and equality of covariances in a normal multivariate population of k variables on the basis of a sample. More specifically, three statistical hypotheses are considered: (i) H_{mvc}, the hypothesis that the means are equal, the variances are equal, and the covariances are equal, (ii) H_{vc}, the hypothesis that variances are equal and covariances are equal, irrespective of the values of the means, and (iii) H_m, the hypothesis of equal means, assuming variances are equal and covariances are equal.

Test criteria L_{mvc}, L_{vc}, and L_m are developed by the Neyman-Pearson method of likelihood ratios for testing H_{mvc}, H_{vc} and H_m respectively. The exact moments of each of the three test criteria when the three corresponding hypotheses are true are determined for any number k of variables and for any size, n, of the sample for which the distributions exist The exact distributions of L_{mvc} and L_{vc} are determined for $k = 2$ and $k = 3$, and the exact distribution of L_m is found for any k; these are all beta (Pearson Type I) distributions Tables of 5% and 1% points of L_{mvc}, L_{vc} and L_m, based on Thompson's tables of percentage points of the Incomplete Beta Function, are given for certain values of k and n (Tables I and II). Also tables of values of approximate 5% and 1% points of $-n \ln L_{mvc}$, $-n \ln L_{vc}$ and $-n(k-1) \ln L_m$ for large values of n are given (Table III), based on the fact that these three quantities are approximately distributed according to chi-square laws for large values of n with $\frac{1}{2}k(k+3)-3$, $\frac{1}{2}k(k+1) - 2$, and $k - 1$ degrees of freedom respectively. A table (Table IV) is given which shows how accurate the resulting approximate 5% and 1% points of L_{mvc}, L_c and L_m are

The paper is written in two parts. In Part I the problem of testing the three hypotheses is discussed and the mathematical results are presented together with an illustrative example. Part II is given for the reader who wishes to study the mathematical derivation of the results.

I. The Problem and a Statement of Results

1.1. Introduction. Situations occasionally arise, in which it may be desired to test the hypothesis that the means are equal, the variances are equal and the covariances are equal in a multivariate population in which the variables are correlated, the test to be made on the basis of a sample from such a population. In the case of a normal multivariate distribution this means testing the hypothesis that the distribution is symmetric with respect to the variables.

As an example[1] suppose three "parallel forms" of a test are constructed and all are given to a group of n college entrance students. On the basis of the scores of the n students on the three tests, how could one test the hypothesis that the three tests are really parallel forms, as far as means, variances and covariances are concerned? In other words, how could one test the hypothesis that the scores can be regarded as being from a sample of individuals from a college entrance population of individuals in which the distribution function of the three variables is such that the means of the three variables are all equal, the variances are equal and the covariances are equal? Actually, as far as practical considerations are concerned in testing work, it is frequently sufficient to consider only normally distributed populations So therefore one may raise the question as to how to test the hypothesis that the three-variable sample can be considered as having come from a normal three-variable population which is symmetrical in the three variables, i.e. a normal population in which the means are equal, the variances are equal, and the covariances are equal. Or more generally, one may raise the analogous question for the case of k variables.

Similarly, one could mention biological examples which have been treated by intra-class correlation methods and raise the question as to whether the underlying multivariate distribution can be judged to be symmetric in the variables on the basis of information supplied by the sample.

To attempt to deal with this problem by comparing means, or variances or covariances two at a time or performing what might appear to be extensions of existing tests for two or more *independent* samples of one variable leads to complications because of correlation among the variables in the original population. What is needed is some kind of a comprehensive test which will take into account all means, variances and covariances at one time. If it turns out that the hypothesis of equal means, equal variances and equal covariances is not supported by the sample, then one can raise the question as to whether the sample supports the hypothesis that the variances are equal and covariances are equal irrespective of means. If the answer is yes here, one can ask the further question as to whether the sample supports the hypothesis of equal means. Such tests will be developed in this paper for samples from a normal multivariate population. More specifically three tests are developed. (i) Test L_{mvc} for testing the hypothesis H_{mvc} that all means are equal, all variances are equal and all covariances are equal, (ii) test L_{vc} for the hypothesis H_{vc} that all variances are equal and all covariances are equal, irrespective of the values of the means, and (iii) test

[1] The problem treated in this paper arose from discussions with Professor Harold O. Gulliksen, of the Psychology Department of Princeton University, in connection with the problem of testing whether two or more forms of an examination can be considered as "parallel forms". The author would like to take this opportunity to acknowledge various helpful discussions he has also had with his colleague Professor John W. Tukey in connection with this paper

L_m for the hypothesis H_m that the means are equal, assuming that H_{vc} is true, i.e. that the variances are equal and the covariances equal.

There are rather obvious extensions of the hypotheses H_{mvc}, H_{vc} and H_m and their corresponding test criteria. For example, one could divide the variables in the multivariate population into two sets, and consider the hypothesis $H_{mvc}^{(2)}$ (say), analogous to H_{mvc}, that the means are equal, the variances are equal and the covariances are equal within each of the two sets and that the covariances of variables between the two sets are all equal. Similarly, $H_{vc}^{(2)}$ and $H_m^{(2)}$ could be defined so as to be analogous to H_{vc} and H_m. However, these extensions will not be considered in this paper

In Part I of this paper we shall discuss the problem of testing hypotheses regarding equality of means, equality of variances, and equality of covariances in a normal multivariate population, and summarize the mathematical results which have been obtained. An illustrative example will also be given. The derivation of the test criteria and their sampling theory is presented in Part II of the paper.

1.2. The hypotheses to be tested. We assume that there is a k-variate population Π in which the variables $x_1, x_2, \cdots, x_k$ are distributed according to a normal k-variate probability density function such that the mean value of x_i is a_i $(i = 1, 2, \cdots, k)$ and the variance-covariance matrix of $x_1, x_2, \cdots, x_k$ is $\|\rho_{ij}\sigma_i\sigma_j\|$, ρ_{ij} being the correlation coefficient between x_i and $x_j (i \neq j)$, and σ_i being the standard deviation of x_i.

In specifying the hypotheses to be considered it will be convenient to define three conditions on the parameters of population Π:

Condition C_m: that the means of the x_i are all equal.

Condition C_v: that the variances of the x_i are all equal.

Condition C_c: that the covariances of the x_i and x_j $(i \neq j)$ are all equal

The hypotheses regarding Π to be tested are as follows:

H_{mvc}: that conditions C_m, C_v, and C_c hold

H_{vc}: that conditions C_v and C_c hold

H_m: that condition C_m holds, assuming that H_{vc} is true.

A precise statement of these hypotheses in terms of Neyman-Pearson likelihood ratio terminology will be found in Part II.

It should be noted that H_{mvc} is a comprehensive hypothesis which specifies equality of means, equality of variances and equality of covariances and would be tested if one is interested in all of these quantities as a system. On the other hand H_{vc} refers only to equality of variances and equality of covariances regardless of what values the means may have. H_{vc} would be tested if one is only concerned with equality of variances and equality of covariances. H_m is a more restrictive hypothesis than either H_{mvc} or H_{vc}, for it refers to equality of means *under the assumption* that H_{vc} is true. In other words, H_m can only be tested accurately when H_{vc} is true, H_m would be a generalization of the Behrens-Fisher problem [1] when H_{vc} is false.

1.3. The sample test criteria. The three hypotheses H_{mvc}, H_{vc} and H_m are to be tested on the basis of a sample O_n from Π consisting of the following values of the x's: $x_{i\alpha}$, $i = 1, 2, \cdots, k$, $\alpha = 1, 2, \cdots, n$.

The criteria for testing H_{mvc}, H_{vc}, and H_m depend on the following quantities to be determined from the sample:

$$\bar{x}_i = \frac{1}{n}\sum_{\alpha=1}^{n} x_{i\alpha}, \qquad \bar{x} = \frac{1}{k}\sum_{i=1}^{k} \bar{x}_i, \tag{1.1}$$

$$s_{ij} = \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j) = \frac{1}{n}\sum_{\alpha=1}^{n} x_{i\alpha}x_{j\alpha} - \bar{x}_i\bar{x}_j \tag{1.2}$$

$$s^2 = \frac{1}{k}\sum_{i=1}^{k} s_{ii}, \qquad s^2 r = \frac{1}{k(k-1)}\sum_{i \neq j=1}^{k} s_{ij}. \tag{1.3}$$

The sample criteria, based on the method of likelihood ratios, for testing H_{mvc}, H_{vc} and H_m are respectively, as follows:

$$L_{mvc} = L_{vc} \cdot L_m^{k-1} \tag{1.4}$$

$$L_{vc} = \frac{|s_{ij}|}{(s^2)^k(1-r)^{k-1}(1+(k-1)r)} \tag{1.5}$$

$$L_m = \frac{s^2(1-r)}{s^2(1-r) + \frac{1}{k-1}\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2} \tag{1.6}$$

where $|s_{ij}|$ is the determinant of sample variances and covariances.

The range of values of each of the three criteria is from 0 to 1. A necessary and sufficient condition for each criterion to have the value 1 is that the hypothesis for which the criterion is a test be (accidentally) identically supported by the sample. If the hypothesis (any one of the three being considered) is true, the average value of the corresponding criterion will be less than 1, but this average value will be nearer 1 than when the hypothesis is false.

If H_{mvc} is true (i.e., found to be supported by the sample on the basis of the test L_{mvc}) then there will be three parameters which characterize Π, namely, a (the common mean), σ^2 (the common variance), and ρ (the common correlation coefficient). The best estimates of these three parameters are, respectively:

$$\begin{aligned} \bar{x} &= \frac{1}{k}\sum_{i=1}^{k}\bar{x}_i, \\ s_0^2 &= s^2 + \frac{1}{k}\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2 \\ r_0 &= \left[s^2 r - \frac{1}{k(k-1)}\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2\right] \Big/ s_0^2. \end{aligned} \tag{1.7}$$

If H_{vc} is true (i.e., found to be supported by the sample on the basis of the test L_{vc}) there will be $k + 2$ parameters which characterize Π, namely the means

$a_1, a_2, \quad , a_k, \sigma^2$ (the common variance) and ρ (the common correlation coefficient). The best estimates of these parameters are, respectively

$$\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k, s^2, \text{ and } r. \tag{1.8}$$

In order to be able to use the three sample criteria L_{mvc}, L_{vc} and L_m for testing the hypotheses H_{mvc}, H_{vc}, H_m, it is necessary to have their distribution functions under the assumptions that the respective hypotheses H_{mvc}, H_{vc} and H_m are true.

1.4. Sampling theory of the test criteria. The moments of the exact sampling distributions of L_{mvc} and L_{vc} when H_{mvc} and H_{vc} are true respectively, have been determined for all values of k (number of variables) and all values of n (sample size) for which such distributions exist; i.e., for $k \geq 2$ and $n > k$. The g-th moments of the distributions of the two criteria are as follows:

$$M_g(L_{mvc}) = (k-1)^{g(k-1)} \prod_{i=2}^{k} \frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))} \cdot \frac{\Gamma(\frac{1}{2}(k-1)n)}{\Gamma(\frac{1}{2}(k-1)(n-1)+g(k-1))} \tag{1.9}$$

and

$$M_g(L_{vc}) = (k-1)^{g(k-1)} \prod_{i=2}^{k} \frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))} \cdot \frac{\Gamma(\frac{1}{2}(k-1)(n-1))}{\Gamma(\frac{1}{2}(k-1)(n-1)+g(k-1))}. \tag{1.10}$$

For the cases of $k = 2$ and $k = 3$, these moments simplify so that the distribution functions of L_{mvc} and L_{vc} can be readily inferred. They turn out to be as follows:
For $k = 2$:

$$dF(L_{mvc}) = \tfrac{1}{2}(n-2)(L_{mvc})^{\frac{1}{2}(n-4)} d\, L_{mvc} \tag{1.11}$$

$$dF(L_{vc}) = \frac{\Gamma(\frac{1}{2}(n-1))}{\sqrt{\pi}\,\Gamma(\frac{1}{2}(n-2))} L_{vc}^{\frac{1}{2}(n-4)} (1-L_{vc})^{-\frac{1}{2}}\, dL_{vc}. \tag{1.12}$$

For $k = 3$:

$$dF(L_{mvc}) = \frac{\Gamma(n)}{2\Gamma(n-3)} (\sqrt{L_{mvc}})^{n-4} (1-\sqrt{L_{mvc}})^2\, d\sqrt{L_{mvc}} \tag{1.13}$$

$$dF(L_{vc}) = \frac{\Gamma(n-1)}{\Gamma(n-3)} (\sqrt{L_{vc}})^{n-4} (1-\sqrt{L_{vc}})\, d\sqrt{L_{vc}}. \tag{1.14}$$

The distribution function of L_m when the hypothesis H_m is true has been found to be

$$(1.15)\quad dF(L_m) = \frac{\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1))\Gamma(\frac{1}{2}(k-1))} \cdot L_m^{\frac{1}{2}(n-1)(k-1)-1}(1-L_m)^{\frac{1}{2}(k-1)-1}\,dL_m.$$

Details of the derivation of these distribution functions will be found in Part II.

In a paper published elsewhere in the present issue of the *Annals of Mathematical Statistics*, Tukey and Wilks [2] show how the probability integrals of L_{mvc} and L_{vc} and of other statistical criteria having moments of a rather general class can be fitted by Incomplete Beta Functions in such a way that all moments of the fitted distribution agree with those of the actual distribution up to and including terms of order $\frac{1}{n}$.

It will be noted that the probability integrals of L_{mvc} and L_{vc} for $k = 2$, those of $\sqrt{L_{mvc}}$ and $\sqrt{L_{vc}}$ for $k = 3$, and that of L_m for any value of k, are Incomplete Beta Functions [3], with the following values of p and q:

k	criterion	p	q
2	L_{mvc}	$\frac{1}{2}(n-2)$	1
2	L_{vc}	$\frac{1}{2}(n-2)$	$\frac{1}{2}$
3	$\sqrt{L_{mvc}}$	$n-3$	3
3	$\sqrt{L_{vc}}$	$n-3$	2
k	L_m	$\frac{1}{2}(n-1)(k-1)$	$\frac{1}{2}(k-1)$

Percentage points[2] of the distributions of these criteria for the cases mentioned in this table can therefore be read from Thompson's [4] tables of per cent points for the Incomplete Beta Function. 5% and 1% points for L_{mvc} and L_{vc} for $k = 2$ and 3 are given in Table I for certain values of n. Table II shows 5% and 1% points of L_m for certain values of n for $k = 2, 3, 4, 5$ and 6.

1.5. The equivalence of L_m and an analysis of variance test for a k by n layout. One can set up a Snedecor F ratio for testing hypothesis H_m by setting

$$(1.16)\quad F = \frac{\frac{1}{2}(n-1)(k-1)(1-L_m)}{\frac{1}{2}(k-1)L_m}$$

and entering the F tables with $n_1 = k - 1$ and $n_2 = (n-1)(k-1)$ degrees of

[2] The $100\epsilon\%$ point, say L_ϵ, of a given criterion L (any of those being considered) having distribution $dF(L)$ is given by $\int_0^{L_\epsilon} dF(L) = \epsilon$

TABLE I

5% and 1% points of L_{mvc} and L_{vc} for $k = 2$ and $k = 3$

	$k = 2$					$k = 3$			
n	L_{mvc}		L_{vc}		n	L_{mvc}		L_{vc}	
	5%	1%	5%	1%		5%	1%	5%	1%
3	0 0025	.0001	0.0062	.0002	4	0.00029	0.00001	0 00064	0 00003
4	0500	0100	.0975	0199	5	.0095	.0018	0183	.0035
5	.1357	.0464	.2285	0808	6	.0358	.0112	0618	.0198
6	.2236	.1000	.3416	.1588	7	0736	0300	.1174	0493
7	3017	.1585	4307	.2352	8	.1165	.0559	.1749	.0866
8	.3684	2154	.5005	.3039	9	1603	0860	2297	.1272
9	.4249	.2683	5559	.3637	10	.2028	1181	2802	.1682
10	.4729	.3162	.6007	4154	11	.2432	.1508	.3259	.2079
11	5139	.3594	6375	4601	12	.2808	.1829	3670	2457
12	.5493	.3981	6682	.4989	13	3157	.2141	.4040	.2811
13	.5800	4329	.6943	.5328	14	3480	2439	.4373	.3141
14	6070	.4642	7165	.5626	15	.3778	.2722	.4674	3448
15	6307	4924	7358	.5889	16	4052	.2990	.4946	3732
16	6518	5180	.7528	6124	17	.4306	.3243	.5193	.3996
17	.6707	5411	7675	6334	18	.4540	.3482	.5418	.4240
18	,6877	.5623	.7807	.6522	23	.5484	.4482	.6293	.5230
19	.7030	.5817	7925	.6693	33	.6660	.5811	7326	.6470
20	.7169	.5995	.8031	.6848	63	.8135	.7591	.8549	.8029
21	.7294	.6159	8126	6989	∞	1.0000	1 0000	1 0000	1.0000
22	.7411	.6310	.8213	.7119					
23	.7518	.6450	8292	7237					
24	7616	.6579	8365	7347					
25	7707	.6700	8431	.7448					
26	7791	.6813	8493	.7542					
27	7869	.6918	8549	7629					
28	.7942	.7017	.8602	7710					
29	.8010	.7110	.8651	7786					
30	8074	7197	8697	.7857					
31	.8133	.7279	.8739	.7924					
32	8190	7356	8779	7987					
42	8609	7943	.9073	.8454					
62	.9050	.8577	9375	.8945					
122	.9513	.9261	9684	9460					
∞	1.0000	1.0000	1 0000	1 0000					

TABLE II

5% and 1% points of L_m

n	$k = 2$		$k = 3$			$k = 4$			$k = 5$		
	5%	1%	n	5%	1%	n	5%	1%	n	5%	1%
2	0.0062	0.0002	2	.0500	0.0100	2	.0973	.0328	2	.1354	.0589
3	.0975	0199	3	.2236	.1000	3	2960	.1698	3	.3426	.2221
4	.2285	.0808	4	.3684	.2154	4	.4372	.3002	4	.4793	.3566
5	.3416	.1588	5	.4729	.3162	5	.5340	.4019	5	.5709	4560
6	.4307	.2352	6	.5493	.6033	6	.6033	.4800	6	.6356	.5302
7	.5005	.3039	7	.6070	.4642	7	.6550	.5409	7	.6837	.5872
8	.5559	3637	8	.6518	5180	8	.6950	.5895	8	.7206	.6321
9	.6007	4154	9	.6877	.5623	9	.7267	.6290	11	.7933	.7232
10	.6375	.4601	10	.7169	.5995	10	.7525	.6617	16	.8559	.8043
11	.6682	4989	11	.7411	.6310	11	.7730	.6892	31	.9246	.8961
12	6943	.5328	12	.7616	.6579	21	.8788	.8290	∞	1.0000	1.0000
13	.7165	.5626	13	.7791	.6813	41	.9372	.9101			
14	7358	.5889	14	.7942	.7017	∞	1.0000	1.0000			
15	.7527	.6124	15	.8074	7197						
16	.7675	.6334	16	.8190	.7356						
17	.7807	.6522	21	.8609	.7943						
18	.7925	.6693	31	.9050	.8577						
19	.8031	.6848	61	.9513	.9261						
20	.8126	.6989	∞	1.0000	1.0000						
21	.8213	.7119									
22	.8292	.7237									
23	.8365	.7347									
24	.8431	.7448									
25	.8493	.7542									
26	8549	.7629									
27	.8602	7710									
28	.8651	.7786									
29	.8697	.7857									
30	.8739	7924									
31	.8779	.7987									
41	.9073	8454									
61	.9375	.8945									
121	.9684	.9460									
∞	1.0000	1.0000									

freedom. Making use of the definition of s^2, s_0^2, r and r_0 in L_m, one finds that F can be written as

$$(1.17) \qquad F = \frac{S_1}{(k-1)} \Big/ \frac{S_2}{(n-1)(k-1)}$$

where $S_1 = n \sum_{i=1}^{k} (\bar{x}_i - \bar{x})^2$, and $S_2 = \sum_{\alpha=1}^{n} \sum_{i=1}^{k} (x_{i\alpha} - \bar{x}'_\alpha - \bar{x}_i + \bar{x})^2$ and $\bar{x}'_\alpha = \frac{1}{k} \sum_{\alpha=1}^{n} x_{i\alpha}$. Thus, the use of L_m as a criterion for testing H_m is equivalent to an analysis of variance test for testing "row" effects in a k by n rectangular layout when rows are associated with the k variables in the multivariate population and columns are associated with the n individuals in the sample.

1.6. Approximate sampling theory of the test criteria for large samples. In the case of large samples, it follows from a theorem [5] concerning the distribution of likelihood ratio criteria for large samples that $-n \ln L_{mvc}$, $-n \ln L_{vc}$, and $-n(k-1) \ln L_m$ are approximately distributed according to chi-square distributions with $\frac{1}{2}k(k+3) - 3$, $\frac{1}{2}k(k+1) - 2$, and $k - 1$ degrees of freedom respectively. Approximate 5% and 1% points of these three quantities taken from Thompson's [6] tables of the percentage points of the chi-square distribution are given in Table III

Table IV is given in order to furnish some idea of how the accuracy of the approximations provided by Table III depend on n. It will be noted that the approximate values exceed the exact values in every case, differences occuring in the third decimal place in almost every case in which n exceeds 60. The approximate percentages to which the approximate per cent points correspond are given by the numbers in the parentheses in Table IV These numbers in each case were obtained by linear interpolation from the exact 5% and 1% points

1.7. Comparison of L_{vc} with Mauchly's "sphericity" test. The criterion L_{vc} for testing hypothesis H_{vc} is, in a sense, an extension of a test developed by Mauchly [7] for testing the hypothesis of "sphericity" of a normal multivariate distribution Mauchly's test was designed for testing the hypothesis that all variances are equal, and that all covariances are equal to *zero* irrespective of the values of the population means The likelihood criterion for testing this hypothesis of "sphericity" is

$$(1.18) \qquad L_s = \frac{|s_{ij}|}{(s^2)^k}$$

which should be compared with L_{vc}. Actually, Mauchly used $\sqrt{L_s}$ as the test criterion, which, of course, is equivalent to using L_s. The g-th moment of L_s when the hypothesis of sphericity is true is given by

$$(1.19) \qquad k^{gk} \cdot \prod_{i=1}^{k} \left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))} \right] \cdot \frac{\Gamma(\frac{1}{2}k(n-1))}{\Gamma(\frac{1}{2}k(n-1)+gk)}$$

TABLE III

Approximate 5% and 1% points for $-n \ln L_{mvc}$, $-n \ln L_{vc}$, and $-n(k-1) \ln L_m$ for $k = 2, 3, 4, 5, 6$.

k	$-n \ln L_{mvc}$ d.f.	5%	1%	$-n \ln L_{vc}$ d.f.	5%	5%	$-n(k-1) \ln L_m$ d.f.	5%	1%
2	2	5.99147	9.21034	1	3.84146	6.63490	1	3.84146	6.63490
3	6	12.5916	16.8119	4	9.48773	13.2767	2	5.99147	9.21034
4	11	19.6751	24.7250	8	15.5073	20.0902	3	7.81473	11.3449
5	17	27.5871	33.4087	13	22.3621	27.6883	4	9.48773	13.2767
6	24	36.4151	42.9798	19	30.1435	36.1908	5	11.0705	15.0863

TABLE IV

Table indicating the accuracy of the approximate 5% and 1% points of L_{mvc}, L_{vc} and L_m provided by Table III

criterion	k	n	5% exact	5% approx.	1% exact	1% approx.
L_{mvc}	2	30	0.8074	0.8190 (5.53)*	0.7197	0.7357 (1.73)*
L_{mvc}	2	62	.9050	.9079 (5.25)	.8577	.8619 (1.36)
L_{mvc}	2	122	.9513	.9521 (5.13)	.9261	.9273 (1.19)
L_{mvc}	3	33	.6660	.6828 (5.79)	.5811	.6008 (1.88)
L_{mvc}	3	63	.8135	.8188 (5.40)	.7591	.7658 (1.49)
L_{vc}	2	30	.8697	.8799 (5.49)	.7857	.8016 (1.76)
L_{vc}	2	62	9375	.9399 (5.22)	.8945	.8985 (1.37)
L_{vc}	2	122	.9684	.9690 (5.11)	.9460	.9471 (1.20)
L_{vc}	3	33	.7326	.7501 (5.82)	.6470	.6688 (2.01)
L_{vc}	3	63	.8549	.8602 (5.41)	.8029	.8100 (1.55)
L_m	2	31	.8779	.8835 (5.28)	.7987	.8073 (1.43)
L_m	2	61	.9375	.9389 (5.13)	.8945	.8969 (1.20)
L_m	2	121	.9684	.9688 (5.07)	.9460	.9467 (1.13)
L_m	3	31	.9050	.9079 (5.25)	.8577	.8619 (1.36)
L_m	3	61	.9513	.9521 (5.10)	.9261	.9273 (1.14)
L_m	4	41	9372	.9385 (5.19)	.9101	.9119 (1.26)
L_m	5	31	.9246	.9264 (5.25)	.8961	.8984 (1.32)

*The numbers in the parentheses are approximate percentages (obtained by linear interpolation) to which the approximate percent points correspond.

which should be compared with the g-th moment of L_{vc}. Stated in other words, Mauchly's criterion L_s is a test for the hypothesis that contours of equal proba-

bility density in the multivariate normal population distribution are spheres, while L_{vc} is a test for the hypothesis that the contours of equal probability are k-dimensional ellipsoids with $k - 1$ equal axes in general shorter than the k-th axis which is equally inclined to the k coordinate axes of the distribution function.

1.8. Illustrative Example. As an example to illustrate the use of the test criteria L_{mvc}, L_{vc}, L_m, we shall consider data on three forms of a subtest in verbal aptitude, and inquire as to whether the data are consistent with the hypothesis of the three forms being "parallel forms".

A procedure[3] was used for partitioning the first 60 of an entire test of 80 items into three sets of 20 items each by using only a "difficulty' and a "validity" index on each of the items. A random sample of 100 test booklets was selected from those in which the first 60 items had been attempted. Total scores were obtained on each of the three subtests selected in this manner. The question is this: Does this procedure of selecting items produce "parallel" subtests? In other words considering the three scores on the three subtests in each of the 100 test booklets as a sample of 100 items from a trivariate normal population is the sample consistent with the hypothesis H_{mvc} of equal means, equal variances and equal covariances? If not, is the sample consistent with the hypothesis H_{vc} of equal variances and equal covariances irrespective of means? If the answer to this question is no, then the failure of the tests to be parallel is at least partially attributable to differences in variances and/or differences in covariances. If the answer to the question is yes, we test H_m, the hypothesis of equal means, assuming equal variances and equal covariances. If the sample is not consistent with H_m, then the subtests fail to be parallel because of significant differences in means.

If we denote the three subtests by T_1, T_2, T_3, and the scores on the α-th individual in the sample on the three tests by $x_{1\alpha}$, $x_{2\alpha}$, $x_{3\alpha}$ respectively ($\alpha = 1, 2, \cdots, 100$), the information in the sample needed for computing L_{mvc}, L_{vc} and L_m and testing H_{mvc}, H_{vc} and H_m is as follows:

$$
\begin{array}{rclcrcl}
\bar{x}_1 &=& 10\ 9900 & \qquad & s^2 &=& 17.5558 \\
\bar{x}_2 &=& 10\ 9300 & & s_0^2 &=& 17.5764 \\
\bar{x}_3 &=& 11\ 2600 & & r &=& 7963 \\
s_{11} &=& 16.8451 & & r_0 &=& .7948 \\
s_{22} &=& 18.1099 & & |s_{ij}| &=& 545.5308 \\
s_{33} &=& 17.7124 & & & & \\
s_{12} &=& 13.5493 & & & & \\
s_{13} &=& 14.5826 & & & & \\
s_{23} &=& 13\ 8056 & & & &
\end{array}
$$

[3] Devised by Mr L. R Tucker of the College Entrance Examination Board. The author is indebted to Mr Tucker for the data used in the illustrative example.

Using formulas (1.4), (1.5), and (1.6), for $k = 3$, for calculating the values of L_{mvc}, L_{vc} and L_m, we find

$$\begin{aligned} L_{mvc} &= .9209 \\ L_{vc} &= .9370 \\ L_m &= .9914 \end{aligned}$$

It will be seen from Table III that the 5 % point of $-n \ln L_{mvc}$ for $k = 3$ is 12.5912. Setting $-100 \ln L_{mvc} = 12.5912$ and solving we find the approximate 5% point of L_{mvc} to be .8817 which is considerably less than the observed value of L_{mvc}, namely .9209. Hence, the sample is consistent with H_{mvc}. As a matter of fact the observed value .9209 lies at approximately the 25% point of L_{mvc}.

In practice, there would be no point in proceeding to test H_{vc} or H_m, because if L_{mvc} is non-significant there is a high probability (not certainty) that both L_{vc} and L_m will be non-significant. But for illustrative purposes, it is perhaps useful to consider L_{vc} and L_m anyway.

The 5% point of $-n \ln L_{vc}$ for $k = 3$ is 9.48773 (See Table III). Setting $-100 \ln L_{vc} = 9.48773$ and solving, we get .9095 as the approximate 5% point of L_{vc}, which is considerably less than the observed value .9370, thus indicating that L_{vc} is not significant at the 5% level. In fact the observed value .9370 lies between the 25% and 10% point of L_{vc}.

The 5% point of $-n(k - 1) \ln L_m$ for $k = 3$ is 5.99147. Setting $-200 \ln L_m = 5.99147$ and solving we get .9704 as the approximate 5% point. Since the observed value of L_m exceeds .9704, we find L_m not significant at the 5% level. In fact, the observed value .9914 lies between the 50% and 25% points.

II. Derivation of Results

In this part we shall derive the criteria L_{mvc}, L_{vc} and L_m for testing H_{mvc}, H_{vc} and H_m by the Neyman-Pearson method of likelihood ratios, and determine the distribution theory of the criteria.

2.1. The test L_{mvc} for H_{mvc}, the hypothesis of equality of means, equality of variances and equality of covariances.

2.1.1 *Derivation of the criterion* L_{mvc}. Let Π be a normal k-variate population, in which $x_1, x_2, \cdots, x_k$ are variables, such that a_i is the mean of x_i, σ_i^2 the variance of x_i and $\rho_{ij}\sigma_i\sigma_j$ the covariance (ρ_{ij} the correlation coefficient) between x_i and x_j. The distribution law of $x_1, x_2, \cdots, x_k$ in the population, is

$$\frac{|A_{ij}|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}k}} \exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{k} A_{ij}(x_i - a_i)(x_j - a_j)\right] \tag{2.1}$$

where $\| A_{ij} \|$ is symmetric and is the inverse of the variance-covariance matrix, i.e. $\| A_{ij} \|^{-1} = \| \rho_{ij}\sigma_i\sigma_j \|$, $(\rho_{ii} = 1)$.

Now suppose O_n is a random sample of n individuals from population Π,

and let $x_{i\alpha}$ be the value of the x_i for the αth individual in the sample. Then, the probability function for the entire sample (likelihood function) is

$$P = \frac{|A_{ij}|^{\frac{1}{2}n}}{(2\pi)^{\frac{1}{2}nk}} \exp\left[-\tfrac{1}{2}\sum_{\alpha=1}^{n}\sum_{i,j=1}^{k} A_{ij}(x_{i\alpha}-a_i)(x_{j\alpha}-a_j)\right]. \tag{2.2}$$

The hypothesis which we wish to test is that the population means a_1, a_2, $\cdots$, a_k are equal, the variances σ_1^2, σ_2^2, $\cdots$, σ_k^2 are all equal and the covariances $\rho_{12}\sigma_1\sigma_2$, $\rho_{13}\sigma_1\sigma_3$, $\cdots$, $\rho_{k-1,k}\sigma_{k-1}\sigma_k$ are all equal, the test to be made on the basis of the sample of values $x_{i\alpha}$. In other words, we wish to test the hypothesis that

$$\begin{cases} a_1 = a_2 = \cdots = a_k = a \\[1em] \begin{Vmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1k}\sigma_1\sigma_k \\ \rho_{21}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2k}\sigma_2\sigma_k \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot \\ \rho_{k1}\sigma_1\sigma_k & \rho_{k2}\sigma_2\sigma_k & \cdots & \sigma_k^2 \end{Vmatrix} = \begin{Vmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{Vmatrix}. \end{cases} \tag{2.3}$$

Testing the hypothesis that (2.3) holds is equivalent to testing the hypothesis that

$$\begin{cases} a_1 = a_2 = \cdots = a_k = a \\[1em] \begin{Vmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & & \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ A_{k1} & & & A_{kk} \end{Vmatrix} = \begin{Vmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ B & B & \cdots & A \end{Vmatrix} \end{cases} \tag{2.4}$$

where

$$A = \frac{1 + (k-2)\rho}{\sigma^2(1-\rho)(1+(k-1)\rho)}, \qquad B = \frac{-\rho}{\sigma^2(1-\rho)(1+(k-1)\rho)}. \tag{2.5}$$

To obtain the likelihood criterion L_{mvc} for testing the hypothesis H_{mvc} we maximize the likelihood (2.2) under two conditions, for the given sample O_n, and take the ratio of the two resulting maxima. First, we maximize (2.2) over the set Ω of admissible values of the parameters, i e with respect to all means a_i and all variances and covariances $\rho_{ij}\sigma_i\sigma_j$, denoting the resulting maximum of (2.2) by P_Ω. Secondly, we maximize (2.2) over the set of values ω of the parameters which satisfy the hypothesis H_{mvc}; that is, we replace in (2.2) each mean a_1, a_2, $\cdots$, a_k by a, and each of the variances σ_1^2, σ_2^2, $\cdots$, σ_k^2 by σ^2 and each of the covariances $\rho_{ij}\sigma_i\sigma_j$, $(i \neq j)$, by $\rho\sigma^2$ and then maximize (2.2) with respect to a, σ^2, and ρ, denoting the resulting maximum by P_ω.

Maximizing (2.2) under the first set of conditions is equivalent to maximizing it with respect to the a_i, and the A_{ij}, while maximizing (2.2) under the second set of conditions is equivalent to imposing condition (2.4) and maximizing it with respect to a, A and B.

The values of the a_i and A_{ij} which maximize (2.2) under the first set of conditions are given by solving the following $(k^2 + 3k)/2$ equations.

$$(2.6) \qquad \frac{\partial P}{\partial a_i} = 0, \qquad i = 1, 2, \cdots, k$$

$$(2.7) \qquad \frac{\partial P}{\partial A_{ij}} = 0, \qquad i, j = 1, 2, \cdots, k, \quad (i \leq j).$$

Expressions for these equations are

$$(2.8) \qquad \left[n\sum_{j=1}^{k} A_{ij}(\bar{x}_j - a_j)\right] P = 0, \qquad i = 1, 2, \cdots, k$$

$$(2.9) \quad \left[\frac{n}{2} A^{ij} - \tfrac{1}{2}\sum_{\alpha=1}^{n}(x_{i\alpha} - a_i)(x_{j\alpha} - a_j)\right] P = 0, \quad i, j, = 1, 2, \cdots, k, (i \leq j),$$

where A^{ij} is the element in the ith row and jth column of $\| A_{ij} \|^{-1}$, i.e. $A^{ij} = \rho_{ij}\sigma_i\sigma_j$, and $\bar{x}_j = \frac{1}{n}\sum_{\alpha=1}^{n} x_{j\alpha}$

The solution of (2.8) and (2.9) is

$$(2.10) \qquad a_j = \bar{x}_j, \qquad j = 1, 2, \cdots, k$$

$$A^{ij} = s_{ij}, \text{ or } A_{ij} = s^{ij}, \qquad i, j = 1, 2, \cdots, k, \quad (i \leq j)$$

where $s_{ij} = \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j)$, and where $\| s^{ij} \| = \| s_{ij} \|^{-1}$. Inserting the values of (2.10) in (2.2) and noting that the exponent in (2.2) reduces to $-\frac{n}{2}\sum_{i,j=1}^{k} s^{ij}s_{ij}$, which in turn reduces to $-\frac{1}{2}kn$, since $\sum_{i=1}^{k} s^{ij}s_{ij} = 1$ for each value of j, we obtain

$$(2.11) \qquad P_\Omega = \frac{e^{-\frac{1}{2}kn}}{| s_{ij} |^{\frac{1}{2}n}(2\pi)^{\frac{1}{2}kn}}.$$

In order to obtain P_ω, we specialize the a_i and the matrix $\| A_{ij} \|$ in (2.2) in accordance with (2.4), noting that the determinant $| A_{ij} |$ reduces to $(A - B)^{k-1}(A + (k - 1)B)$, thus obtaining the following specialized form of (2.2)

$$(2.12) \quad P' = \frac{[(A - B)^{k-1}(A + (k - 1)B)]^{\frac{1}{2}n}}{(2\pi)^{\frac{1}{2}nk}}$$
$$\exp\left\{-\tfrac{1}{2}\left[A\sum_{\alpha=1}^{n}\sum_{i=1}^{k}(x_{i\alpha} - a)^2 + B\sum_{\alpha=1}^{n}\sum_{i\neq j=1}^{k}(x_{i\alpha} - a)(x_{j\alpha} - a)\right]\right\}.$$

The values of a, A and B which maximize P' are given by solving the following three equations

$$\frac{\partial P'}{\partial a} = 0, \qquad \frac{\partial P'}{\partial A} = 0, \qquad \frac{\partial P'}{\partial B} = 0. \tag{2.13}$$

These equations are respectively

$$\left\{\begin{aligned} &\left[(A - B)\sum_{\alpha=1}^{n}\sum_{i=1}^{k}(x_{i\alpha} - a) + B\sum_{\alpha=1}^{n}\left(\sum_{i=1}^{k}(x_{i\alpha} - a)\right)\right]P' = 0 \\ &\left[\frac{\frac{1}{2}n(k-1)}{A - B} + \frac{\frac{1}{2}n}{A + (k-1)B} - \tfrac{1}{2}\sum_{\alpha=1}^{n}\sum_{i=1}^{k}(x_{i\alpha} - a)^2\right]P' = 0 \\ &\left[\frac{-\frac{1}{2}n(k-1)}{A - B} + \frac{\frac{1}{2}n(k-1)}{A + (k-1)B} - \sum_{\alpha=i}^{n}\sum_{i\neq j=1}^{k}(x_{i\alpha} - a)(x_{j\alpha} - a)\right]P' = 0. \end{aligned}\right. \tag{2.14}$$

Replacing a_i by $\bar{x}_i$ in (2.15) putting $\frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j) = s_{ij}$, and setting

$$\left\{\begin{aligned} \bar{x} &= \frac{1}{nk}\sum_{\alpha=1}^{n}\sum_{i=1}^{k}x_{i\alpha} \\ s_{0ij} &= \frac{1}{n}\sum_{\alpha=1}^{n}(x_{i\alpha} - \bar{x})(x_{j\alpha} - \bar{x}) = s_{ij} + (\bar{x}_i - \bar{x})(\bar{x}_j - \bar{x}) \\ r_0 &= \sum_{i\neq j=1}^{k} s_{0ij}\Big/(k-1)\sum_{i=1}^{k}s_{0ii} \\ &= \left[\sum_{i\neq j=1}^{k}s_{ij} - \sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2\right]\Big/(k-1)\left[\sum_{i=1}^{k}s_{ii} + \sum_{i=1}^{k}(\bar{x}_i - x)^2\right] \\ s_0^2 &= \sum_{i=1}^{k}s_{0ii}/k = \frac{1}{k}\left[\sum_{i=1}^{k}s_{ii} + \sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2\right] \end{aligned}\right. \tag{2.15}$$

we obtain as solutions of (2.14)

$$\begin{aligned} a &= \bar{x} \\ A &= \frac{1 + (k-2)r_0}{s_0^2(1 - r)(1 + (k-1)r_0)} \\ B &= \frac{-r_0}{s_0^2(1 - r_0)(1 + (k-1)r_0)}. \end{aligned} \tag{2.16}$$

Substituting these in (2.12) we obtain

$$P_\omega = \frac{e^{-\frac{1}{2}kn}}{[(s_0^2)^k(1 - r_0)^{k-1}(1 + (k-1)r_0)]^{\frac{1}{2}n}(2\pi)^{\frac{1}{2}kn}}. \tag{2.17}$$

The likelihood ratio λ_{mvc} for testing hypothesis H_{mvc} is given by

$$\lambda_{mvc} = \frac{P_\omega}{P_\Omega}.$$

It will be convenient to use the $\frac{2}{n}$ th root of λ_{mvc} as the test criterion for H_{mvc}. Denoting this criterion by L_{mvc}, we have

$$L_{mvc} = \frac{| s_{ij} |}{(s_0^2)^k(1 - r_0)^{k-1}(1 + (k - 1)r_0)}. \tag{2.18}$$

The use of L_{mvc} as a test criterion is obviously equivalent to the use of λ_{mvc}.

It will be seen that L_{mvc} is equal to unity when and only when the sample means $\bar{x}_i$ are all equal, the sample variances s_{ii} are all equal and when the sample covariances s_{ij}, $(i \neq j)$, are all equal. The greater the departure of sample means from equality, sample variances from equality and sample covariances from equality, the smaller will be the value of L_{mvc}, its value, of course, always remaining between 0 and 1.

2.1.2. *Approximate distribution of $-n$ ln L_{mvc} in large samples.* In order to make use of L_{mvc} as a criterion for testing hypothesis H_{mvc} we must find its sampling distribution under the assumption that H_{mvc} is true, i.e. that our sample has, in fact, been drawn from a k-variate normal population having equal means, equal variances and equal covariances. In the case of large samples, it follows from a theorem on asymptotic distributions of likelihood ratios [5] that $-2 \ln \lambda_{mvc}$ (i.e. $-n \ln L_{mvc}$) is approximately distributed according to the chi-square law with $\frac{1}{2}k(k + 3) - 3$ degrees of freedom (obtained by taking the difference between the number of parameters used in maximizing P to obtain P_Ω and that used in maximizing P' to obtain P_ω).

Thus, to apply the test, one computes the value of $-n \ln L_{mvc}$ for the given sample, and sees whether the obtained value is significant at the given probability level (5% or 1%) using the chi-square table for $\frac{1}{2}k(k + 3) - 3$ degrees of freedom.

To make a study of how closely the chi-square distribution approximates the exact distribution of $-n \ln L_{mvc}$ for various values of k and n would be an arduous task in computation. But existing experience with approximations to large sample distributions indicates that the approximation in the present problem would be satisfactory for small values of k (say not more than 5) and values of n not less than 50 Some light is thrown on this question for $k = 2$ and 3 by Table IV

2.1 3. *Moments of the exact distribution of L_{mvc}.* In Section 2.1.2 an approximation is given to the distribution of $-n \ln L_{mvc}$ for large samples. As a matter of fact, one can find expressions for the moments of the exact distribution of L_{mvc}, which for the cases of $k = 2$ and $k = 3$ yield simple expressions for the exact distribution of L_{mvc}

To find the moments of L_{mvc} it will be noted that if one sets

$$ns_{ij} = a_{ij}$$

$$ns_{0ij} = a_{0ij}$$

in expression (2.18) for L_{mvc}, the following expression is obtained for L_{mvc}.

$$L_{mvc} = \left[\frac{|a_{ij}|}{R_0^{k-1}S_0}\right] \tag{2.19}$$

where

$$\begin{aligned} R_0 &= \frac{1}{k}\sum_{i=1}^{k} a_{0ii} - \frac{1}{k(k-1)}\sum_{i\neq j=1}^{k} a_{0ij} \\ S_0 &= \frac{1}{k}\left(\sum_{i=1}^{k} a_{0ii} + \sum_{i\neq j=1}^{k} a_{0ij}\right). \end{aligned} \tag{2.20}$$

It will be seen that L_{mvc} depends on the $\bar{x}_i$ and the a_{ij}. In the case of a sample from a general normal multivariate population, we know the a_{ij} to be distributed according to the Wishart [8] distribution function

$$W_{n-1,k}(a_{ij};A_{ij}) = \frac{|A_{ij}|^{\frac{1}{2}(n-1)}|a_{ij}|^{\frac{1}{2}(n-k-2)}\exp\left[-\frac{1}{2}\sum_{i,j=1}^{k}A_{ij}a_{ij}\right]}{2^{\frac{1}{2}k(n-1)}\pi^{\frac{1}{4}k(k-1)}\prod_{i=1}^{k}\Gamma(\frac{1}{2}(n-i))} \tag{2.21}$$

and the means $\bar{x}_i$ to be independently distributed according to the normal distribution

$$f(\bar{x}_i) = \frac{n^{\frac{1}{2}k}|A_{ij}|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}k}}\exp\left[-\frac{n}{2}\sum_{i,j=1}^{k}A_{ij}(\bar{x}_i - a_i)(\bar{x}_j - a_j)\right] \tag{2.22}$$

where the A_{ij} and a_i were defined in (2.1).

We now define a function $\varphi(g, u, v)$ as the mean value of $|a_{ij}|^g e^{uR_0+vS_0}$ when H_{mvc} is true, i.e.,

$$\varphi(g, u, v) = E(|a_{ij}|^g e^{uR_0+vS_0}) \tag{2.23}$$

where the right hand side denotes multiplication of (2.21) by (2.22) (after imposing condition (2.4)) by $|a_{ij}|^g e^{uR_0+vS_0}$ and then integration with respect to the a_{ij} and $\bar{x}_i$. This yields

$$\begin{aligned} \varphi(g, u, v) &= 2^{gk}\prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))}\right] \\ &\times \frac{(A-B)^{\frac{1}{2}n(k-1)}(A+(k-1)B)^{\frac{1}{2}(n-1)}}{\left(A-B-\frac{2u}{k-1}\right)^{\frac{1}{2}(k-1)(n+g)}(A+(k-1)B-2v)^{\frac{1}{2}(n-1)+g}}. \end{aligned} \tag{2.24}$$

Now the gth moment $M_g(L_{mvc})$ of L_{mvc} is defined by

$$M_g(L_{mvc} = E[(L_{mvc})^g] \tag{2.25}$$

and is obtained by evaluating the partial derivative

$$\frac{\partial^{r(k-1)+s}}{\partial u^{r(k-1)}\partial v^s}(\varphi) \tag{2.26}$$

at $u = v = 0$, and then putting $r = -g$ and $s = -g$. The validity of this operation for the range of values of g in which we are interested can be established by an argument based on analytic continuation. Alternatively, the same result can be achieved by taking the indefinite integral of φ $r(k-1)$ times successively with respect to n, and s times successively with respect to v (the lower limit of integration being $-\infty$ in every case) and then evaluating the final result at $u = v = 0$. Accordingly, we obtain for the gth moment of L_{mvc}, when hypothesis H_{mvc} is true, the following expression

$$\begin{aligned} Mg(L_{mvc}) &= \prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma(\frac{1}{2}(n-i))}\right] \\ &\times (k-1)^{g(k-1)}\frac{\Gamma(\frac{1}{2}(n-1)\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)+g)\Gamma(\frac{1}{2}n(k-1)+g(k-1))}. \end{aligned} \tag{2.27}$$

2.1.4. *Distribution of L_{mvc} for $k = 2$ and 3.* For $k = 2$, the criterion L_{mvc} can be expressed as

$$L_{mvc} = \frac{\begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}}{\begin{vmatrix} \frac{1}{2}(s_{11}+s_{22})+\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 & s_{12}-\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 \\ s_{21}-\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 & \frac{1}{2}(s_{11}+s_{22})+\frac{1}{4}(\bar{x}_1-\bar{x}_2)^2 \end{vmatrix}}. \tag{2.28}$$

The gth moment of L_{mvc} for $k = 2$ (obtained by putting $k = 2$ in (2.26) is

$$M_g(L_{mvc}) = \frac{\Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}(n-2)+g)}{\Gamma(\frac{1}{2}n+g)\Gamma(\frac{1}{2}(n-2))} = \frac{(\frac{1}{2}(n-2))}{(\frac{1}{2}(n-2)+g)}, \tag{2.29}$$

and the distribution function of L_{mvc} is found to be

$$dF(L_{mvc}) = \tfrac{1}{2}(n-2)L_{mvc}^{\frac{1}{2}(n-4)}\,dL_{mvc}, \qquad (0 \le L_{mvc} \le 1). \tag{2.30}$$

For $k = 3$, L_{mvc} can be written as

$$L_{mvc} = \frac{\begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix}}{(s_0^2)^3(1-r_0)^2(1+2r_0)} \tag{2.31}$$

where s_0^2 and r_0 are defined in (2.15) for $k = 3$. Putting $k = 3$ in (2.26) we find the gth moment of L_{mvc} for this case to be

$$(2.32) \qquad M_g(L_{mvc}) = 2^{2g} \frac{\Gamma(\frac{1}{2}(n-2)+g)\Gamma(\frac{1}{2}(n-3)+g)\Gamma(n)}{\Gamma(\frac{1}{2}(n-2)\Gamma(\frac{1}{2}(n-3)\Gamma(n+2g)}.$$

By using the fact that

$$\Gamma(t+\tfrac{1}{2})\Gamma(t+1) = \frac{\sqrt{\pi}\,\Gamma(2t+1)}{2^{2t}},$$

it is seen that $M_g(L_{mvc})$ reduces to

$$(2.33) \qquad M_g(L_{mvc}) = \frac{\Gamma(n)\Gamma(n-3+2g)}{\Gamma(n+2g)\Gamma(n-3)},$$

from which we deduce the distribution of L_{mvc} to be

$$(2.34) \qquad dF(L_{mvc}) = \frac{\Gamma(n)}{\Gamma(3)\Gamma(n-3)} (\sqrt{L_{mvc}})^{n-4}(1-\sqrt{L_{mvc}})^2\, d\sqrt{L_{mvc}}, \qquad (0 \leq L_{mvc} \leq 1).$$

For values of $k > 3$, the exact distribution of L_{mvc} seems to be too complicated to lend itself to ready computation.

Thus, relatively simple exact tests of significance of L_{mvc} can be set up for $k = 2$ and $k = 3$ by using distribution functions (2.30) and (2.34) respectively. For large values of n we have pointed out that the significance of L_{mvc} can be tested by making use of the fact that $-n \ln L_{mvc}$ is approximately distributed according to a chi-square law with $\frac{1}{2}k(k+3) - 3$ degrees of freedom when H_{mvc} is true

For $k = 2$, L_{mvc} is essentially a criterion for simultaneously testing, on the basis of a sample, the hypothesis of equality of means and equality of variances of a normal bivariate population

It should be noted that if H_{mvc} is true, or more realistically, is supported by the sample as a result of applying test L_{mvc}, then population II is characterized by the three parameters a, σ^2 and ρ in (2.3). The likelihood estimates of these parameters are $\bar{x}$, s_0^2 and r_0.

2.2. The test L_{vc} for H_{vc}, the hypothesis of equality of variances and equality of covariances, irrespective of the values of the means.

2.2.1 *Derivation of the criterion L_{vc}.* If, in testing hypothesis H_{mvc} by means of the criterion L_{mvc}, at a given level of significance, say ϵ, a non-significant value of L_{mvc} is obtained, one states that the sample is consistent with the hypothesis H_{mvc} that all the population means are equal, the variances are equal and the covariances are equal. Consideration of the Neyman-Pearson Type II error involved in this statement would be very arduous and involved and will not be attempted. On the other hand, if a significant value of L_{mvc} is obtained, one

states that the sample contradicts the hypothesis H_{mvc} with probability ϵ of making a Neyman-Pearson Type I error. In this case it may be reasonable to inquire whether the sample would support the hypothesis if the variability due to the means were eliminated. In other words, we may inquire whether the sample supports the hypothesis H_{vc} of equal variances and equal covariances, irrespective of what values the population means may have. To obtain the likelihood ratio criterion L_{vc} for testing H_{vc} we maximize the likelihood (2.2) under the following two sets of conditions: First, with respect to the means a_i and the variances and covariances $\rho_{ij}\sigma_i\sigma_j$; and Secondly, with respect to the means a_i and A and B, where A and B are obtained by imposing the condition on the matrix $\| A_{ij} \|$ specified in (2 14). The maximum of (2.2) under the first condition is given by (2.11). Denoting the maximum of (2.2) under the second set of conditions by $P_{\omega'}$, it is found, by a procedure similar to that used in finding P_ω (given by (2.17), that $P_{\omega'}$ is given by

$$P_{\omega'} = \frac{e^{-\frac{1}{2}kn}}{[(s^2)^k(1-r)^{k-1}(1+(k-1)r)]^{\frac{1}{2}n}(2\pi)^{\frac{1}{2}kn}} \tag{2.35}$$

where

$$\begin{aligned} r &= \sum_{i\neq j=1}^{k} s_{ij} \Big/ (k-1)\sum_{i=1}^{k} s_{ii} \\ s^2 &= \sum_{i=1}^{k} s_{ii} \Big/ k. \end{aligned} \tag{2.36}$$

The likelihood ratio λ_{vc} for testing H_{vc} is given by

$$\lambda_{vc} = \left[\frac{|s_{ij}|}{(s^2)^k(1-r)^{k-1}(1+(k-1)r)}\right]^{\frac{1}{2}n}.$$

The test criterion which will be used for testing H_{vc} is L_{vc}, the $\frac{2}{n}$th root of λ_{vc}, i e.,

$$L_{vc} = \frac{|s_{ij}|}{(s^2)^k(1-r)^{k-1}(1+(k-1)r)}. \tag{2.37}$$

2.2.2. *Approximate distribution of $-n \ln L_{vc}$ in large samples.*

In the case of large samples $-n \ln L_{vc}$ is approximately distributed according to the chi-square law with $\frac{1}{2}k(k+1) - 2$ degrees of freedom when hypothesis H_{vc} is true.

2.2.3. *Moments of the exact distribution of L_{vc}.* The moments of L_{vc} when H_{vc} is true can be found by a method similar to that used in Section 2.1.3 for determining the moments of L_{mvc}. For it will be noted that L_{vc} can be written as

$$L_{vc} = \left[\frac{|a_{ij}|}{R^{k-1}S}\right] \tag{2.38}$$

where

$$
R = \frac{1}{k}\sum_{i=1}^{k} a_{ii} - \frac{1}{k(k-1)}\sum_{i\neq j=1}^{k} a_{ij}
\tag{2.39}
$$
$$
S = \frac{1}{k}\left[\sum_{i=1}^{k} a_{ii} + \sum_{i\neq j=1}^{k} a_{ij}\right],
$$

from which it is evident that L_{vc} depends only on the a_{ij}, whose distribution in the case of a general normal multivariate population is given by (2.21). We now define a function $\theta(g, y, z)$ as the mean value of $| a_{ij} |^g e^{yR+zS}$ under the assumption that H_{vc} is true, i.e.,

$$
\theta(g, y, z) = E(| a_{ij} |^g e^{yR+zS})
\tag{2.40}
$$

where the value of the right hand side is obtained by multiplying (2.21) by $| a_{ij} |^g e^{yR+zS}$, then imposing the condition on $|| A_{ij} ||$ stated in (2.4) and integrating with respect to the a_{ij}. Accordingly, we find

$$
\theta(g, y, z) = 2^{gk}\prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma\frac{1}{2}(n-i)}\right]
\times \frac{(A-B)^{\frac{1}{2}(k-1)(n-1)}(A+(k-1)B)^{\frac{1}{2}(n-1)}}{\left(A-B-\frac{2y}{k-1}\right)^{\frac{1}{2}(k-1)(n-1+2g)}(A+(k-1)B-2z)^{\frac{1}{2}(n-1)+g}}.
\tag{2.41}
$$

The gth moment $M_g(L_{vc})$ of L_{vc} is obtained by evaluating the partial derivative

$$
\frac{\partial^{r(k-1)+s}}{\partial y^{r(k-1)}\,\partial z^{s}}\,\theta
\tag{2.42}
$$

at $y = z = 0$, and then setting, $r = -g$ and $s = -g$. These operations yield

$$
M_g(L_{vc}) = \prod_{i=1}^{k}\left[\frac{\Gamma(\frac{1}{2}(n-i)+g)}{\Gamma\frac{1}{2}(n-i)}\right]
\times (k-1)^{g(k-1)}\frac{\Gamma(\frac{1}{2}(n-1))\Gamma(\frac{1}{2}(k-1)(n-1))}{\Gamma(\frac{1}{2}(n-1)+g)\Gamma(\frac{1}{2}(k-1)(n-1)+g(k-1))}.
\tag{2.43}
$$

2.2.4. *Distribution of L_{vc} for $k = 2$ and 3.* For $k = 2$, L_{vc} can be expressed as follows:

$$
L_{vc} = \frac{\begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}}{\begin{vmatrix} \frac{1}{2}(s_{11}+s_{22}) & s_{12} \\ s_{21} & \frac{1}{2}(s_{11}+s_{22}) \end{vmatrix}}
\tag{2.44}
$$

and the gth moment of L_{vc} is given by

$$M_g(L_{vc}) = \frac{\Gamma(\frac{1}{2}(n-1))\Gamma(\frac{1}{2}(n-2)+g)}{\Gamma(\frac{1}{2}(n-1)+g)\Gamma(\frac{1}{2}(n-2))} \tag{2.45}$$

from which the distribution of L_{vc} is deduced to be

$$dF(L_{vc}) = \frac{\Gamma(\frac{1}{2}(n-1))}{\sqrt{\pi}\Gamma(\frac{1}{2}(n-2))} L_{vc}^{\frac{1}{2}(n-4)}(1-L_{vc})^{-\frac{1}{2}}\, dL_{vc}, \qquad (0 \leq L_{vc} \leq 1). \tag{2.46}$$

For $k = 3$, L_{vc} can be expressed as

$$L_{vc} = \frac{\begin{vmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{vmatrix}}{(s^2)^3(1-r)^2(1+2r)} \tag{2.47}$$

where s^2 and r are defined in (2.36) by setting $k = 3$. Setting $k = 3$ in (2.43), we find as the gth moment of L_{vc}

$$M_g(L_{vc}) = 2^{2g}\frac{\Gamma(\frac{1}{2}(n-2)+g)\Gamma(\frac{1}{2}(n-3)+g)\Gamma(n-1)}{\Gamma(\frac{1}{2}(n-2))\Gamma(\frac{1}{2}(n-3))\Gamma(n-1+2g)}. \tag{2.48}$$

Following the method by which (2.32) was reduced to (2.33), we find that the gth moment of L_{vc} reduces to

$$M_g(L_{vc}) = \frac{\Gamma(n-1)\Gamma(n-3+2g)}{\Gamma(n-1+2g)\Gamma(n-3)}, \tag{2.49}$$

and hence the distribution function of L_{vc} for $k = 3$ is

$$dF(L_{vc}) = \frac{\Gamma(n-1)}{\Gamma(n-3)}(\sqrt{L_{vc}})^{n-4}(1-\sqrt{L_{vc}})\, d\sqrt{L_{vc}}, \quad (0 \leq L_{vc} \leq 1). \tag{2.50}$$

For higher values of k the distribution of L_{vc} is apparently too complicated for ready computation. But distributions (2.46) and (2.50) provide relatively simple significance tests for the cases $k = 2$ and 3, respectively. For large samples, we remark again that a significance test for L_{vc} is provided by the fact $-2 \ln \lambda_{vc}$ (i.e., $-n \ln L_{vc}$) is approximately distributed according to the chi-square law with $\frac{1}{2}k(k+1) - 2$ degrees of freedom when H_{vc} is true.

For $k = 2$, L_{vc} is essentially a criterion for testing, on the basis of a sample, the hypothesis of equality of variances of a normal bivariate population.

If H_{vc} is true, Π will be characterized by the parameters $a_1, a_2, \cdots, a_k, \sigma^2$ and ρ. The maximum likelihood estimates of these parameters are $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k, s^2$ and r, respectively.

2.3. The test L_m for H_m, the hypothesis of equality of means, when the variances are equal and covariances are equal.

2.3.1. *Derivation of the criterion L_m.* Suppose L_{vc}, described in Section 2.2.1 for testing H_{vc}, the hypothesis of equal variances and equal covariances, does not have a significantly small value, thus indicating that the sample does not contradict the hypothesis H_{vc}. Then, assuming that the original test L_{mvc} of H_{mvc} turned out to have a significantly small value, we may inquire as to whether the significance of L_{mvc} is due to the inequality of the population means a_i. In this section we shall consider a criterion L_m for testing the hypothesis H_m that the means a_i are equal, assuming that the variances are equal and that the covariances are equal. In this hypothesis we maximize the likelihood (2.2) under the following two sets of conditions: First, with respect to the a_i, A and B, where A and B are defined by the condition on $\| A_{ij} \|$ given in (2.4); secondly, with respect to a, A and B where these parameters are specified by (2.4). The maxima of the likelihood (2.2) under these two conditions are $P_{\omega'}$, and P_ω, given by (2.35) and (2.17) respectively. The likelihood ratio λ_m is therefore

$$\lambda_m = \frac{P_\omega}{P_{\omega'}} = \left[\frac{(s^2)^k(1 - r)^{k-1}(1 + (k - 1)r)}{(s_0^2)^k(1 - r_0)^{k-1}(1 + (k - 1)r_0)}\right]^{\frac{1}{2}n}. \tag{2.51}$$

Now it follows from the definitions of s^2, , s_0^2 and r_0, (2.15) and (2.36) that

$$s^2(1 + (k - 1)r) \equiv s_0^2(1 + (k - 1)r_0)$$

and hence we may write

$$\lambda_m^{2/n} = \left[\frac{s^2(1 - r)}{s_0^2(1 - r_0)}\right]^{k-1} \tag{2.52}$$

We can also express $\lambda_m^{2/n}$ as

$$\lambda_m^{2/n} = \left(\frac{R}{R_0}\right)^{k-1} \tag{2.53}$$

where R_0 and R are defined by (2.20) and (2.39) respectively.

It will be most convenient for our purposes to use L_m, the $[2/n(k - 1)]$-th root of λ_m. as the criterion for testing H_m, i.e.

$$L_m = R/R_0 = \frac{s^2(1 - r)}{s_0^2(1 - r_0)} = \frac{s^2(1 - r)}{s^2(1 - r) + \dfrac{1}{k - 1}\sum_{i=1}^{k}(\bar{x}_i - \bar{x})^2}. \tag{2.54}$$

2.3.2. *Approximate distribution of $-n(k - 1) \ln L_m$ in large samples.* In large samples $-2 \ln \lambda_m$ (i.e., $-n(k - 1) \ln L_m$) is approximately distributed according to the chi-square law with $k - 1$ degrees of freedom. However, the exact distribution of L_m is relatively simple and will be derived.

2.3.3. *Exact distribution of L_m when H_m is true.* We shall determine the distribution of L_m by first finding the gth moment of L_m when H_m is true. For this purpose we set up the function

$$\psi(p, q) = E(e^{pR+qR_0}) \tag{2.55}$$

where the mean value is taken when H_m is true, i.e., when the a_i and $\| A_{ij} \|$ satisfactory conditions (2.4). Now R and R_0 are functions of the a_{ij} and $\bar{x}_i$. Hence, to find $E(e^{pR+qR_0})$ we multiply (2.21) by (2.22) by e^{pR+qR_0} and impose conditions (2.4), then take the integral over the entire space of the a_{ij} and $\bar{x}_i$. These operations yield

$$\psi(p, q) = \frac{(A - B)^{\frac{1}{2}n(k-1)}}{\left(A - B - \frac{2(p+q)}{k-1}\right)^{\frac{1}{2}(n-1)(k-1)} \left(A - B - \frac{2q}{k-1}\right)^{\frac{1}{2}(k-1)}}. \tag{2.56}$$

The gth moment of L_m is obtained by performing the following differentiations

$$\left[\frac{\partial^h}{\partial q^h}\left\{\frac{\partial^g \psi}{\partial p^g}\right\}_{p=0}\right]_{q=0} \tag{2.57}$$

and then putting $h = -g$. These operations yield

$$M_g(L_m) = \frac{\Gamma(\frac{1}{2}(n-1)(k-1) + g)\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1))\Gamma(\frac{1}{2}n(k-1) + g)} \tag{2.58}$$

from which the distribution of L_m (when H_m is true) is found to be

$$dF(L_m) = \frac{\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1))\Gamma(\frac{1}{2}(k-1))} L_m^{\frac{1}{2}(n-1)(k-1)-1} \cdot (1 - L_m)^{\frac{1}{2}(k-1)-1} \, dL_m, \quad (0 \leq L_m \leq 1). \tag{2.59}$$

Thus, we are able to make an exact test of significance of L_m on the basis of the function (2.59)

2.4. Relations between L_{mvc}, L_{vc} and L_m.

It will be seen from the definitions of L_{mvc}, L_{vc} and L_m in (2.18), (2.37) and (2.54) (noting that $s^2(1 + (k-1)r) \equiv s_0^2(1 + (k-1)r_0)$) that

$$L_{mvc} = L_{vc} \cdot L_m^{k-1}.$$

Furthermore, it will be noted that when H_{mvc} is true, the gth moment of L_{mvc} given by (2.27) is equal to the product of the gth moment of L_{vc} given by (2.43) and the gth moment of L_m^{k-1} (obtained by replacing g by $g(k-1)$ in (2.58). Thus, when H_{mvc} is true λ_{mvc} is composed of the product of two independently distributed quantities, namely L_{vc} and L_m^{k-1}.

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CONTRIBUTIONS TO THE THEORY OF SEQUENTIAL ANALYSIS, II, III

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Summary. This is a continuation of a paper Part I of which was published in the June, 1946 issue of the *Annals of Mathematical Statistics*. The present paper is divided into two parts, Parts II and III, which are summarized as follows:

Part II. The Exact Power Curve and the Distribution of n for Sequential Tests Where z Takes on a Finite Number of Integral Values.

Consider a sequential test defined by a decision function $Z_n = \sum_{\alpha=1}^{n} z_\alpha$ with boundaries $-b$ and a where a and b are positive integers and z_α is the αth observation of a variate z which takes on a finite number of integral values ranging from the negative integer $-r$ to the positive integer m with respective probabilities $p_{-r}, \cdots, p_m$. Let $\xi_{ai} = P[Z_n = (a + i)]$, $(i = 1, 2, \cdots, m - 1)$, and $\xi_{bj} = P[Z_n = -(b + j)]$, $(j = 1, 2, \cdots, r - 1)$. Furthermore, let A be a square matrix of $a + b - 1$ rows and columns with elements defined by: $a_{ii} = 1 - p_0$ for all i; $a_{i,i+k} = -p_k$ for $k = 1, 2, \cdots, m$; $a_{i,i-j} = -p_{-j}$ for $j = 1, 2, \cdots, r$, and $a_{ij} = 0$ otherwise.

It is proved that

$$\text{(i)}\quad \xi_{bj} = \sum_{i=0}^{r-j-1} p_{i-r} A_{r-j-i,b}, \qquad (j = 0, 1, \cdots, r-1)$$

$$\text{(ii)}\quad \xi_{aj} = \sum_{i=0}^{m-j-1} p_{i+j+1} A_{a+b-i-1,b}, \qquad (j = 0, 1, \cdots, m-1),$$

where A_{kb} is the element of the kth row and bth column in A^{-1}. Let $E_{aj}\tau^n$ and $E_{bj}\tau^n$ be the conditional generating function of n under the restriction that $Z_n = (a + j)$ and $Z_n = -(b + j)$ respectively. Then $\xi_{bj}E_{bj}\tau^n$ is obtained by substituting τp_j for each p_j occurring in equation (i) and $\xi_{aj}E_{aj}\tau^n$ is obtained by substituting τp_j for each p_j occurring in equation (ii). The probability that $Z_n = a + j$ in exactly n steps is given by the coefficient of τ^n in the expansion of $\xi_{aj}E_{aj}\tau^n$ in a power series in τ. The probability that $Z_n = -(b + j)$ in exactly n steps is similarly obtained.

This method is applied to the derivation of the exact power function and the distribution of n for the sequential binomial probability ratio test.

Part III. On Conjugate Distributions.

Consider a random variable X with a distribution density $f(x, \theta)$ which satisfies certain specified conditions. Let θ_1 and θ_2 be two values of θ and let $z = \log(f(x, \theta_2)/f(x, \theta_1))$. For any hypothesis $\theta = \theta'$, let $\varphi(t \mid \theta')$ be the moment

generating function of z and h the non-zero value of t for which $\varphi(t \mid \theta') = 1$. We set $F(x) = e^{hz}f(x, \theta')$. Then f and F are conjugate distributions. If $F = f(x, \theta'')$, then θ' and θ'' are defined as conjugate pairs.

A method is given for obtaining the totality of conjugate pairs for the general class of distributions which admit a sufficient statistic. It is then shown that the power of the sequential probability ratio test based on such distributions is given explicitly in terms of these pairs. It is proven that within the approximation obtained by neglecting the excess of $|Z_n|$ over a and b at a decision point the following relationship holds:

$$P_b(n \mid F) = e^{-hb}P_b(n \mid f)$$
$$P_a(n \mid F) = e^{ha}P_a(n \mid f)$$

where $P_b(n \mid g)$ and $P_a(n \mid g)$ stand for the probability that $Z_n \geq a$ and $Z_n \leq -b$ respectively in exactly n steps under the hypothesis g.

II The Exact Power Curve and the Distribution of n for Sequential Tests Where z Takes on a Finite Number of Integral Values

2.1. General discussions. Let a sequential test be defined by a decision function $Z_n = \sum_{\alpha=1}^{n} z_\alpha$ with boundaries $-b$ and a where a and b are positive and z_α is the αth observation of a variate z which takes on a finite number of integral values, $-r, r + 1, \cdots, -1, 0, 1, 2, \cdots, m$. Let $P(z = i) = p_i$ where $P(z = i)$ stands for the probability that z takes on the value i. We shall assume without any loss of generality that a and b are integers.

When the sequential test terminates with $Z_n \geq a$, the possible values that Z_n can take on are: $a, a + 1, \cdots, a + m - 1$. Similarly, when the sequential test terminates with $Z_n \leq -b$, the possible values which Z_n can take on are: $-b, -(b + 1), \cdots, -(b + r - 1)$. Let $\xi_{ai} = P[Z_n = (a + i)], i = 0, 1, \cdots, m - 1$, and $\xi_{bi} = P[Z_n = -(b + i)], i = 0, 1, \cdots, r - 1$.

For any variate u, let the symbol $E_{bi}(u)$ stand for the expected value of u under the restriction that $Z_n = -(b + i)$, and the symbol $E_{ai}(u)$ stand for the expected value of u under the restriction that $Z_n = a + i$. Let $\phi(t)$ be the generating function of z. Then

$$\phi(t) = \sum_{i=-r}^{m} p_i t^i. \tag{2.101}$$

In terms of the generating function, the Fundamental Identity (see section 2.32 in [6]) can be written as

$$\sum_{i=0}^{r-1} \xi_{bi} t^{-(b+i)} E_{bi}[\varphi(t)]^{-n} + \sum_{i=0}^{m-1} \xi_{ai} t^{a+i} E_{ai}[\varphi(t)]^{-n} = 1. \tag{2.102}$$

It follows from (2.102) that for all values of t for which

$$\phi(t) = \sum_{i=-r}^{m} p_i t^i = 1, \tag{2.103}$$

$$\psi(t) = \sum_{i=0}^{r-1} \xi_{bi} t^{-(b+i)} + \sum_{i=0}^{m-1} \xi_{ai} t^{a+i} = 1 \tag{2.104}$$

where $\psi(t)$ is the generating function of Z_n.

In the paper "The cumulative sums of random variables" [2] Wald has given the following method for obtaining the probabilities ξ_{ai} and ξ_{bi}. Let $t_1, t_2, \cdots, t_{r+m}$ be the $r + m$ roots of (2.103). Substituting these in (2.104) we get $r + m$ linear equations in the $r + m$ unknowns, ξ_{ai} and ξ_{bi}. Thus, if the determinant of these equations is different from zero, the unknowns can be solved in terms of the roots of (2.103). In a similar manner, the characteristic function of n under the restriction that $Z_n = i$ can also be obtained.

The above method has two disadvantages. First, it involves solving for all the roots of a polynomial which will often be of a high degree and second, it involves solving a set of linear equations with coefficients which are powers of complex numbers.

The method outlined below is in many respects much simpler. It requires only the evaluation of one column of the inverse of a matrix of $a + b - 1$ rows and columns. The elements of the matrix are given explicitly and are either 0, 1 or p_i. This permits obtaining general solutions for special classes of sequential tests.

2.2. Derivation of the exact power functions. We multiply $\phi(t) - 1$ by t^r and $\psi(t) - 1$ by t^{b+r-1} and obtain two polynomials.

$$f(t) = \sum_{j=0}^{m+r} (p_{j-r} - \delta_{jr}) t^j \tag{2.201}$$

and

$$g(t) = \sum_{j=0}^{r-1} \xi_{bj} t^{r-j-1} - t^{b+r-1} + \sum_{j=0}^{m-1} \xi_{aj} t^{a+b+r+j-1} \tag{2.202}$$

where $\delta_{ik} = 1$ when $i = k$ and zero otherwise.

By the Fundamental Identity, every root of $f(t)$ is also a root of $g(t)$. Since $f(t)$ is of degree $m + r$ and $g(t)$ is of degree $a + b + m + r - 2$, it must follow that $g(t)$ equals $f(t)$ times a polynomial of degree $a + b - 2$.[1] That is,

$$g(t) = f(t) \sum_{i=0}^{a+b-2} c_i t^i \tag{2.203}$$

where the c's are undetermined constants. Substituting from (2.201) in (2.203) we obtain

$$g(t) = \sum_{j=0}^{a+b+m+r-2} Q_j t^j \tag{2.204}$$

[1] It is assumed here that $f(t)$ has no multiple roots. The author conjectures that this is true for the polynomial under consideration for all values of p

where

$$Q_j = \sum_{i=0}^{j} (p_{i-r} - \delta_{ir})c_{j-i} . \tag{2.205}$$

Comparing the coefficients of (2.204) with those of (2.202) and taking into account the fact that $p_k = 0$ when $k > m$ and $c_k = 0$ when $k > a + b - 2$, we get

$$\xi_{bj} = \sum_{i=0}^{r-j-1} p_{i-r}c_{r-j-i-1} , \qquad (j = 0, 1, \cdots , r - 1), \tag{2.206}$$

and

$$\xi_{aj} = \sum_{i=0}^{m-j-1} p_{i+j+1}c_{a+b-i-2} , \qquad (j = 0, 1, \cdots , m - 1). \tag{2.207}$$

Thus, if the c's (we require only the first r and the last m) are determined, the probabilities ξ_{ai} and ξ_{bi} are also determined from (2.206) and (2.207). But, if we examine the structure of $g(t)$ in (2.202) we see that the coefficients of all the powers of t from r to $(a + b + r - 2)$ inclusive are zero except for the coefficient of t^{b+r-1} which is equal to -1 Consequently, if in (2.204) we set $Q_j = -\delta_{j,b+r-1}$, for all $j = r, r + 1, \cdots, a + b + r - 2$, we shall have the required number of equations to solve for the $a + b - 1$ unknown c's.

In view of (2.205) these equations can be written as

$$\sum_{i=0}^{j} (\delta_{ir} - p_{i-r})c_{j-i} = \delta_{j,b+r-1} , \qquad (j = r, \cdots , a + b + r - 2) \tag{2.208}$$

Changing the range of the subscript j, we get

$$\sum_{i=0}^{j+r-1} (\delta_{ir} - p_{i-r})c_{j+r-i-1} = \delta_{jb} , \qquad (j = 1, 2, \cdots , a + b - 1), \tag{2.209}$$

with the understanding that $p_k = 0$ when $k > m$ and $c_k = 0$ when $k > a + b - 2$.

Let A be the matrix of the equations in (2.209). Then A is of the following form. The elements in the main diagonal are $(1 - p_0)$. In the diagonals to the right of and parallel to the main diagonal, the elements are $-p_{-1}, -p_{-2}, \cdots, -p_{-r}, 0, \cdots, 0$ successively, in the diagonals to the left of and parallel to the main diagonal, the elements are $-p_1, -p_2, \cdots, -p_m, 0, \cdots, 0$ successively. Assume that the determinant of A is different from zero[2] and let A^{-1} be the inverse of A. Let the elements of A^{-1} be designated by A_{ij}, $(i, j = 1, 2, \cdots, a + b - 1)$. Then, in view of (2.209) we get

$$c_j = A_{j+1,b} , \qquad (j = 0, 1, 2, \cdots, a + b - 2). \tag{2.210}$$

Finally, from (2.206) and (2.207), we have,

$$\xi_{bj} = \sum_{i=0}^{r-j-1} p_{i-r}A_{r-j-i,b} , \qquad (j = 0, 1, 2, \cdots , r - 1), \tag{2.211}$$

[2] P. L. Hsu has submitted a simple proof to the author that A is non-singular.

and

$$\xi_{aj} = \sum_{i=0}^{m-j-1} p_{i+j+1} A_{a+b-i-1,b}, \qquad (j = 0, 1, 2, \cdots, m-1) \tag{2.212}$$

where, as before, it is understood that $p_k = 0$ when $k > m$ and $A_{kb} = 0$ when, $k > a + b - 1$.

From (2.211) and (2.212) we can obtain the probability that $Z_n \leq -b$ and the probability that $Z_n \geq a$ since these are given by

$$\sum_{j=0}^{r-1} \xi_{bj} \quad \text{and} \quad \sum_{j=0}^{m-1} \xi_{aj} \left(= 1 - \sum_{j=0}^{r-1} \xi_{bj}\right)$$

respectively We can also obtain En, the average number of steps required to reach a decision For, if we differentiate (2.102) with respect to t and set $t = 1$, we get

$$E(n) = \frac{EZ_n}{Ez} = \frac{\sum_{i=0}^{m-1} \xi_{ai}(a+i) - \sum_{i=0}^{r-1} \xi_{bi}(b+i)}{\sum_{i=-r}^{m} ip_i} \tag{2.213}$$

2.3. Derivation of the probability that the sequential test will terminate in exactly n steps. Let $\phi(t)$ be the generating function of z and $\psi(t, \tau)$ the joint generating function of Z_n and n. Then

$$\phi(t) = \sum_{i=-r}^{m} p_i t^i \tag{2.301}$$

and

$$\psi(t, \tau) = \sum_{i=0}^{r-1} \xi_{bi} t^{-(b+i)} E_{bi} \tau^n + \sum_{i=0}^{m-1} \xi_{ai} t^{a+i} E_{ai} \tau^n. \tag{2.302}$$

Furthermore, let $\phi_1(t, \tau) = \tau\phi(t) - 1$ and $\psi_1(t, \tau) = \psi(t, \tau) - 1$ In terms of these functions, the Fundamental Identity can be stated as follows: For a fixed τ, every root of $\phi_1(t, \tau)$ is also a root of $\psi_1(t, \tau)$. Let $f(t, \tau) = t^r\phi_1(t, \tau)$ and $g(t, \tau) = t^{b+r-1}\psi_1(t, \tau)$. Then

$$f(t, \tau) = \sum_{j=0}^{m+r} (\tau p_{j-r} - \delta_{jr}) t^j \tag{2.303}$$

and

$$g(t, \tau) = \sum_{j=0}^{r-1} (\xi_{bj} E_{bj} \tau^n) t^{r-j-1} - t^{b+r-1} + \sum_{j=0}^{m-1} (\xi_{aj} E_{aj} \tau^n) t^{a+b+r+j-1}. \tag{2.304}$$

Since for a fixed τ, every root of $f(t, \tau)$ is a root of $g(t, \tau)$, and since $f(t, \tau)$ is a polynomial in t of degree $m + r$ and $g(t, \tau)$ is a polynomial in t of degree $a + b + m - 2$, it must follow that[3]

[3] See footnote 1, section 2.2

(2.305) $$g(t, \tau) = f(t, \tau) \sum_{i=0}^{a+b-2} d_i t^i.$$

The rest of the argument is identical with that of section 2.2 except that the unknowns in this case are $\xi_{bj}E_{bj}\tau^n$ and $\xi_{aj}E_{aj}\tau^n$ and are given by

(2.306) $$\xi_{bj}E_{bj}\tau^n = \sum_{i=0}^{r-j-1} \tau p_{i-r} d_{r-j-i-1}, \qquad (j = 0, 1, \cdots, r-1),$$

and

(2.307) $$\xi_{aj}E_{aj}\tau^n = \sum_{i=0}^{m-j-1} \tau p_{i+j+1} d_{a+b-i-2}, \qquad (j = 0, 1, \cdots, m-1),$$

(see (2.206), and (2.207)) where the d's are obtained by solving the linear equations:

(2.308) $$\sum_{i=0}^{j+r-1} (\delta_{i-r} - \tau p_{i-r}) d_{j+r-i-1} = \delta_{jb}, \qquad (j = 1, 2, \cdot\ , a+b-1),$$

(see (2 209)). Thus, we see that the solution for $\xi_{bi}E_{bi}\tau^n$ is obtainable from the solution given in 2.2 for ξ_{bi} by substituting τp_i for every p_i appearing in the expression (2.211). Similarly, the solution for $\xi_{ai}E_{ai}\tau^n$ is obtainable from the solution given for ξ_{ai} by substituting τp_i for every p_i appearing in the expression (2.212).

Let $p(Z_n = k \mid n)$ stand for the probability that $Z_n = k$ in exactly n steps and let $p_{ai}(n) = p[Z_n = (a + i) \mid n]$ and $p_{bi}(n) = p[Z_n = -(b + i) \mid n]$. Then $p_{ai}(n)$ and $p_{bi}(n)$ are given by the coefficient of τ^n in the expansion of $\xi_{ai}E_{ai}\tau^n$ and $\xi_{bi}E_{bi}\tau^n$ respectively in a power series in τ. That the expansions are valid can be seen from the following considerations: If we examine the solutions given for $\xi_{ai}E_{ai}\tau^n$ $(i = 0, 1, \cdots, m-1)$, and $\xi_{bi}E_{bi}\tau^n$ $(i = 0, 1, \cdots, r-1)$, we see that each is a ratio of two polynomials in τ, the polynomial in the denominator is, in each case, the determinant of the linear equations (2.308). Now, it is easy to see that this determinant eqals 1 when $\tau = 0$. Hence the expansions are valid in a neighborhood of $\tau = 0$.[4]

Let $p_{an} = p[Z_n \geq a \mid n]$ and $p_{bn} = p[Z_n \leq -b \mid n]$; then

(2.309) $$p_{an} = \sum_{i=0}^{m-1} p_{ai}(n)$$

and

(2.310) $$p_{bn} = \sum_{i=0}^{r-1} p_{bi}(n).$$

We have also:

(2.311) $$\sum_{m_1}^{\infty} p_{an} = \sum_{i=0}^{m-1} \xi_{ai} = p(Z_n \geq a)$$

[4] It can be seen from (2.303) that for a fixed $\tau, f(t, \tau) = 0$ implies that $\varphi(t) = 1/\tau$. Hence if $\tau \leq 1$, $\varphi(\tau) \geq 1$. Thus, the Fundamental Identity is valid in the neighborhood of $\tau = 0$.

and

$$\sum_{m_2}^{\infty} p_{bn} = \sum_{i=0}^{r-1} \xi_{bi} = p(Z_n \leq -b) \tag{2.312}$$

where m_1 is the smallest integer greater than or equal to a/m, and m_2 is the smallest integer greater than or equal to b/r.

2.4. Application of the method to the binomial distribution. We shall consider the binomial in terms of acceptance inspection although the results are general.

Let a sequential acceptance inspection plan be defined by p_1, p_2, α and β where p_1 is the fraction defective which can be tolerated in the lot, p_2 is the fraction defective which cannot be tolerated, α is the maximum probability that the lot will be rejected when the fraction defective is p_1 or less and β is the maximum probability that the lot will be accepted when the faction defective is p_2 or greater. Then the sequential criterion is given by two parallel lines ([1] and [3]).

$$d_1 = -h_1 + sn \tag{2.401}$$

$$d_2 = h_2 + sn \tag{2.402}$$

where

$$h_1 = \frac{\log \dfrac{1-\alpha}{\beta}}{\log \dfrac{p_2(1-p_1)}{p_1(1-p_2)}} \tag{2.403}$$

$$h_2 = \frac{\log \dfrac{1-\beta}{\alpha}}{\log \dfrac{p_2(1-p_1)}{p_1(1-p_2)}} \tag{2.404}$$

$$s = \frac{\log \dfrac{1-p_1}{1-p_2}}{\log \dfrac{p_2(1-p_1)}{p_1(1-p_2)}} \tag{2.405}$$

and n is the number of observations taken sequentially. We assume that $\alpha + \beta < 1$ and $p_1 < p_2$. Then h_1 and h_2 are positive and s lies between 0 and 1.

The sequential procedure is as follows: Items are examined one at a time in sequence. If at any stage, the cumulative number of defectives found in the sample thus far taken is less than or equal to d_1 given by (2.401), the lot is accepted; if the cumulative number of defectives is greater than d_2 given by (2.402), the lot is rejected; if neither holds then another observation is taken and the process continued.

It is easy to show that the sequential test described above is equivalent to the following: A variate z takes on the values $-s$ and $1 - s$ with respective

probabilities q and p. A sequential test is defined by the two boundaries $-h_1$ and h_2 and by the decision function $Z_n = \sum_{\alpha=1}^{n} z_\alpha$ where z_α is the αth observation on z. The sequential test terminates if and only if $Z_n \leq -h_1$ or $Z_n \geq h_2$.

As was mentioned above, s lies between 0 and 1.[5] We shall derive the exact power and the distribution of n for this sequential test by assuming that $s = u/v$ where u and v are integers and $u < v$. This restriction is not serious since every value of s can be approximated to any degree of accuracy by a rational fraction, and, moreover, when the sequential test is applied in practice, s is always taken as rational.

Suppose $s = u/v$. Then the sequential test is equivalent to a test in which the variate z takes on the values $-u$ and $v - u$ with probabilities q and p, respectively, and the boundaries are given by $-h_1v$ and h_2v. Let b be the smallest integer greater than or equal to h_1v and a be the smallest integer greater than or equal to h_2v. Then, since u and v are integers, there is no loss in generality in assuming that the boundaries are $-b$ and a. We shall also assume that u and v are prime to each other (i.e. the fraction u/v is reduced to lowest terms) so that the interval $(-b, a)$ is the shortest possible for this test.

The above discussion shows that a sequential test based on the binomial can be considered as a special case of the class of tests treated in this section. Since z takes only on two values, the linear equations (2.209) assume the simple form:

$$(2.401)\qquad -pC_{j+u-v-1} + C_{j-1} - qC_{j+u-1} = \delta_{bj}, \qquad (j = 1, 2, \cdots, a + b - 1)$$

where $C_k = 0$ when k is negative or greater than $a + b - 2$. In terms of the C's, the ξ_{bj} and E_{aj} are given by

$$(2.402)\qquad \xi_{bj} = qC_{u-j-1}, \qquad (j = 0, 1, \cdots, u - 1),$$

$$(2.403)\qquad \xi_{aj} = qC_{a+b+u-v+j-1}, \quad (j = 0, 1, \cdots, v - u - 1)$$

The conditional generating functions of n are obtained by solving (2.401) with τp substituted for p and τq substituted for q.

Since the first $v - u$ and the last u equations in (2.401) contain only two terms and all the other equations contain only three terms, the C's can be obtained without too much difficulty by direct substitution provided $a + b$ is not very large. When $a + b$ is sizeable, a general solution is called for. So far, the author has been able to obtain this only for the case $u = 1$. This special case also has been considered by Walter Bartky [4].

Setting $u = 1$ in (2.401) we get

$$(2.407)\qquad -pC_{j-v} + C_{j-1} - qC_j = \delta_{bj}, \qquad (j = 1, 2, \cdots, a + b - 1),$$

where $C_k = 0$ when k is negative or greater than $a + b - 2$.

[5] In fact, it follows from Theorem 1, section 3.2 below that $p_1 \leq s \leq p_2$.

Consider a general set of equations of the form (2.407) with the subscripts ranging from 1 to an arbitrary integer k. Let the determinant of these equations be designated by Δ_k. Then by direct expansion it can be shown that Δ_k satisfies the difference equation.

$$\Delta_k = \Delta_{k-1} - pq^{v-1}\Delta_{k-v} \tag{2.408}$$

with the initial conditions

$$\Delta_i = 1, \qquad i = 1, 2, \cdots, v-1; \qquad \Delta_v = 1 - pq^{v-1}. \tag{2.409}$$

The difference equation (2.408) can be solved by well known methods. We set

$$\phi(x) = \sum_{j=1}^{\infty} \Delta_j x^{j-1} \tag{2.410}$$

and then multiply each side of (2.410) by $1 - x + pq^{v-1}x^v$. This yields

$$(1 - x + pq^{v-1}x^v)\phi(x) = \sum_{j=1}^{\infty} [\Delta_j - \Delta_{j-1} + pq^{v-1}\Delta_{j-v}]x^{j-1}. \tag{2.411}$$

But by (2.408) and (2.409) we find that the right-hand side of (2.411) equals $1 - pq^{v-1}x^{v-1}$. Therefore,

$$\phi(x) = \frac{1 - pq^{v-1}x^{v-1}}{1 - x + pq^{v-1}x^v} \tag{2.412}$$

If we expand (2.412) in a power series in x, the coefficient of x^k will be Δ_{k+1}. This expansion can be performed readily and we get:

$$\Delta_{k+1} = \sum_{j=0}^{m_1} (-1)^j C_j^{k-j(v-1)}(pq^{v-1})^j - \sum_{j=0}^{m_2} (-1)^j C_j^{k-v-j(v-1)+1}(pq^{v-1})^{j+1} \tag{2.413}$$

where m_1 stands for the largest integer less than or equal to, k/v m_2 stands for the largest integer less than or equal to $k - v + 1/v$ and $C_t^r = r!/t!(r-t)!$.

Let us define $\Delta_0 = 1$ and $\Delta_k = 0$ when $k < 0$. Then, in terms of the extended definition of Δ_k, C_j is given by

$$C_j = \frac{\Delta_j \Delta_{a-1} - \Delta_{j-b}\Delta_{a+b-1}}{q^{j-b+1}\Delta_{a+b-1}} \tag{2.414}$$

for $j = 0, 1, \cdots, a + b - 2$. To prove this, we substitute in the left-hand member of (2.407) the expression for C_k given in (2.414) and get

$$\frac{\Delta_{a+b-1}(\Delta_{j-b} - \Delta_{j-b-1} + pq^{v-1}\Delta_{j-v-b}) - \Delta_{a-1}(\Delta_j \Delta_{j-1} + pq^{v-1}\Delta_{j-v})}{q^{j-b}\Delta_{a+b-1}}. \tag{2.415}$$

But in view of (2.408), (2.409) and the extended definition of Δ_k, the expression in (2.415) vanishes for all $j \neq b$. When $j = b$, the expression equals 1. Hence, it follows that (2.414) is the desired solution.

Let $L_p = p[Z_n \leq -b]$. Then L_p, when plotted against p, gives the operating characteristic curve for this sequential test. But $L_p = qC_0$. Hence, we have

$$L_p = q^b \frac{\Delta_{a-1}}{\Delta_{a+b-1}}. \tag{2 416}$$

As a final remark, we wish to point out that the solution to the sequential problem presented in this section, where taken in conjunction with Wald's solution, is of mathematical interest, since it relates each element of the inverse of a square matrix (designated by A in this section) with the roots of a polynomial $f(t)$ given by (2.201).

III. Conjugate Distributions

3.1. General discussion. Consider a random variable X with a distribution density $f(x, \theta)$.[6] Let θ_1 and θ_2 be two specified values of θ and let

$$z = \log \frac{f(x, \theta_2)}{f(x, \theta_1)}. \tag{3.101}$$

For any hypothesis $\theta = \theta'$, let $\phi(t \mid \theta')$ be the moment generating function of z. That is,

$$\phi(t \mid \theta') = \int_{-\infty}^{\infty} e^{tz} f(x, \theta')\, dx. \tag{3.102}$$

Let h be the real non-zero value of t for which $\phi(t \mid \theta') = 1$[7] and let

$$F(x) = e^{hz} f(x, \theta'). \tag{3.103}$$

Then $F(x)$ is a distribution density. Following Wald [5], we shall call $F(x)$ and $f(x, \theta')$ conjugate distributions.

The distribution density $F(x)$ depends on θ_1, θ_2, and θ'. In some instances $F(x)$ will be a member of the class of distributions $f(x, \theta)$. This is the case, for example, when z is a discrete variate. It is the case also if $\theta' = \theta_1$. For then $h = 1$ and $F(x) = f(x, \theta_2)$. If $F(x)$ belongs to the class of distributions $f(x, \theta)$, we shall designate $F(x)$ by $f(x, \theta'')$ and call θ' and θ'' a conjugate pair.

3.2. Conjugate pairs and the power curve for sequential probability ratio tests in which the underlying distributions admit a sufficient statistic. Let $f(x, \theta)$ admit a sufficient statistic and let a sequential test be defined in terms of the probability ratio z given by (3.101) for some specified hypothesis θ_1 and alternative hypothesis θ_2 with $\theta_1 < \theta_2$. Let the boundaries be given by $-b$ and a where a and b are positive. Since $f(x, \theta)$ admits a sufficient statistic, it can be written in the form

$$f(x, \theta) = e^{u(x)v(\theta)+r(x)+w(\theta)}. \tag{3.201}$$

The probability ratio z is then given by the simple expression

$$z = u(x)[v(\theta_2) - v(\theta_1)] + w(\theta_2) - w(\theta_1). \tag{3.202}$$

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[6] If X is discrete, then $f(x, \theta)$ stands for the probability that $X = x$ when θ is true

[7] See section 2.31 and Lemma II, section 2 32 in [6]

Let

$$b^* = \frac{b}{v(\theta_2) - v(\theta_1)} \tag{3.203}$$

$$a^* = \frac{a}{v(\theta_2) - v(\theta_1)} \tag{3.204}$$

$$s = \frac{w(\theta_1) - w(\theta_2)}{v(\theta_2) - v(\theta_1)}. \tag{3.205}$$

In terms of b^*, a^* and s, the sequential criterion is defined by two parallel lines[8]

$$A_n = -b^* + sn \tag{3.206}$$

$$R_n = a^* + sn \tag{3.207}$$

and the decision functions $\sum_{\alpha=1}^{n} u(x_\alpha)$. The hypothesis $\theta = \theta_1$ is accepted whenever $\sum_{\alpha=1}^{n} u(x_\alpha) \leq A_n$ and rejected whenever $\sum_{\alpha=1}^{n} u(x_\alpha) \geq R_n$. If $A_n < \sum_{\alpha=1}^{n} u(x_\alpha) < R_n$, another observation is taken. This process is continued until one or the other decision is reached.

In what follows, we shall restrict ourselves to the general class of functions $f(x, \theta)$ for which the differentiations under the integral sign indicated below are permissible and $v(\theta)$ is a monotonic function of θ.

Consider the function

$$\psi(\theta) = sv(\theta) + w(\theta). \tag{3.208}$$

We shall show that $\psi(\theta) =$ constant has exactly two roots in θ. To this end, we prove the following theorems.

THEOREM 1. *Let $Eu(x) \mid \theta$ be the expected value of $u(x)$ under the assumption that θ is true. Then there exists a value of $\theta = \theta_0$ such that (a) $Eu(x) \mid \theta_0 = s$; (b) $\theta_1 \leq \theta_0 \leq \theta_2$ and $Eu(x) \mid \theta_1 \leq s \leq Eu(x) \mid \theta_2$ if $v(\theta)$ is an increasing function of θ, and the inequalities are reversed if $v(\theta)$ is a decreasing function of θ.*

PROOF: Assume that $v(\theta)$ is an increasing function of θ. Let $z^* = u(x) - s$ and let $\phi(t) \mid \theta$ be the moment generating function of z^* under the hypothesis that θ is true. Then, it is easy to see that $\phi(h \mid \theta_1) = 1$ and $\phi(-h \mid \theta_2) = 1$ where $h = v(\theta_2) - v(\theta_1)$. Since h is positive, it follows by Lemma 1, section 2.6 of [6], that $Ez^* \mid \theta_1 < 0$ and $Ez^* \mid \theta_2 > 0$. Therefore, $Eu(x) \mid \theta_1 < s$ and $Eu(x) \mid \theta_2 > s$. Moreover, as we shall see in the proof of Theorem 2 below, $Eu(x) \mid \theta$ is assumed to be a continuous function of θ and proved to be mono-

[8] It is here assumed that $v(\theta_2) - v(\theta_1) > 0$. If this is not the case, then a^* and b^* have to be interchanged.

tonically increasing. Hence it must follow that there exists a $\theta = \theta_0$ such that $Eu(x) \mid \theta_0 = s$ and $\theta_1 \leq \theta_0 \leq \theta_2$. This proves the theorem in case $v(\theta)$ is monotonically increasing. However, the argument is identically the same in case $v(\theta)$ is monotonically decreasing.

THEOREM 2. *Let $\psi(\theta)$ be defined as in* (3 208). *Then $\psi(\theta)$ is a monotonically increasing function of θ in the interval $\theta < \theta_0$; assumes a maximum at $\theta = \theta_0$; and is a monotonically decreasing function of θ in the interval $\theta > \theta_0$.*

PROOF. If we differentiate twice the identity

$$\int_{-\infty}^{\infty} e^{u(x)v(\theta)+r(x)+w(\theta)}\, dx = 1 \tag{3.209}$$

with respect to θ we get

$$v'(\theta)Eu(x) \mid \theta + w'(\theta) = 0 \tag{3.210}$$

and

$$v''(\theta)Eu(x) \mid \theta + w''(\theta) = [v'(\theta)]^2 \sigma^2_{u(x)} \tag{3.211}$$

where $\sigma^2_{u(x)}$ is the variance of $u(x)$. Also, if we differentiate under the integral sign the function $Eu(x) \mid \theta$ with respect to θ, we get

$$\frac{dEu(x) \mid \theta}{d\theta} = v'(\theta)\sigma^2_{u(x)}. \tag{3.212}$$

Now by hypothesis, $v(\theta)$ is monotonic in θ. Hence from (3.212) we see that $Eu(x) \mid \theta$ is also monotonic. Moreover, if $v(\theta)$ is an increasing function of θ, so is $Eu(x) \mid \theta$, and conversely. Let us assume that $v(\theta)$ increases with θ. Then for all $\theta < \theta_0$, $Eu(x) \mid \theta < s$ and for all $\theta > \theta_0$, $Eu(x) \mid \theta > s$. Consequently, we have

$$\psi'(\theta) > v'(\theta)Eu(x) \mid \theta + w'(\theta) \tag{3.213}$$

for all $\theta < \theta_0$ and

$$\psi'(\theta) < v'(\theta)Eu(x) \mid \theta + w'(\theta) \tag{3.214}$$

for all $\theta > \theta_0$. But by (3.210) the right-hand side of these inequalities is equal to zero for all θ. Hence $\psi'(\theta) > 0$ for $\theta < \theta_0$ and $\psi'(\theta) < 0$ for $\theta > \theta_0$. The same argument holds when $v(\theta)$ is a decreasing function of θ. Now let $\theta = \theta_0$. Then by (3.210), we see that $\psi'(\theta_0) = 0$. Hence, $\psi(\theta)$ is a maximum at $\theta = \theta_0$. This proves the theorem.

Let c be any constant $< \psi(\theta_0)$ within the domain of $\psi(\theta)$. Then by Theorem 2, the equation $\psi(\theta) = c$ has two roots in θ. Let these roots be designated by θ' and θ''. We now prove the following theorem.

THEOREM 3. *Let z^* and $\phi(t \mid \theta)$ be defined as above. Then* (*a*) *$\phi(t \mid \theta') = 1$ for $t = v(\theta'') - v(\theta')$*; (*b*) *$\phi(t \mid \theta'') = 1$ for $t = v(\theta') - v(\theta'')$; and* (*c*) *$\theta'$ and θ'' from a conjugate pair with respect to z^*.*

PROOF: By definition

$$\phi(t \mid \theta') = \int_{-\infty}^{\infty} e^{u(x)[v(\theta')+t)]+r(x)+w(\theta')-ts} dx. \tag{3.215}$$

Now let $t = v(\theta'') - v(\theta') = h$. Then, in view of the fact that $\psi(\theta') = \psi(\theta'')$, we get

$$\phi(h \mid \theta') = \int_{-\infty}^{\infty} e^{u(x)v(\theta'')+r(x)+w(\theta'')} dx = 1. \tag{3.216}$$

In a similar manner, it can be shown that $\phi(-h \mid \theta'') = 1$. Moreover, the same argument also shows that $f(x, \theta'') = e^{hz^*}f(x, \theta')$. This proves the theorem.

Turning now to the sequential test defined by (3.206) and (3.207), we see that it is equivalent to a test with the decision function $Z_n^* = \sum_{\alpha=1}^{n} z_\alpha^*$ and the two boundaries $-b^*$ and a^*. Let L_θ be the probability that the sequential test will terminate and $Z_n^* \leq -b^*$ (i.e. the hypothesis θ_1 is accepted) when θ is true. Then (neglecting the fact that at a decision point Z_n^* might exceed a^* or fall short of $-b_1^*$), $L_{\theta'}$ and $L_{\theta''}$ are given by (see for example (2.406) in [6]).

$$L_{\theta'} = \frac{e^{(a^*+b^*)h} - e^{b^*h}}{e^{(a^*+b^*)h} - 1} \tag{3.217}$$

and

$$L_{\theta''} = \frac{e^{-h(a^*+b^*)} - e^{-hb^*}}{e^{-h(a^*+b^*)} - 1} = e^{-b^*h} L_{\theta'} \tag{3.218}$$

where $h = v(\theta'') - v(\theta')$. Thus, we see that the two roots of the equation $\psi(\theta) = c$ determine two points on the power curve for the sequential test. By assigning various values to c we obtain as many pairs of points as desired.

The above results show that for the class of distributions under consideration, the real non-zero roots of $\varphi(t \mid \theta) = 1$ are obtainable from the roots of $\psi(\theta) =$ constant. Since $\psi(\theta)$ is completely defined by the form of the distribution $f(x, \theta)$, the power curve of the sequential test can be obtained without a knowledge of the moment generating function of z^*. This might be advantageous in some cases.

3.3. The distribution of n under conjugate hypotheses. Let $P_b(n \mid g)$ stand for the probability that a sequential test will terminate with $Z_n \leq -b$ in exactly n steps when the distribution density of x is g. Let $P_a(n \mid g)$ be similarly defined.

THEOREM 1. *If we neglect the excess of Z_n over a and $-b$ at a decision point,*

$$P_b(n \mid F) = e^{-hb}P_b(n \mid f) \tag{3.301}$$

$$P_a(n \mid F) = e^{ha}P_a(n \mid f) \tag{3.302}$$

where f and F are conjugate distributions as defined in (3.103) *and h is the non-zero*

real value of t for which $\phi(t \mid f)$, the characteristic function of $z = \log f(x, \theta_2)/f(x, \theta_1)$ under the hypothesis f, equals 1.

PROOF: Since, by definition, $F = e^{zh}f$, it follows that $\psi(t - h \mid F) = \phi(t \mid f)$ where $\psi(t \mid F)$ is the characteristic function of z under the hypothesis F. Let

$$\phi(t \mid f) = e^{-\tau} \tag{3.303}$$

where τ is a pure imaginary. Furthermore, let $t_1(\tau)$ and $t_2(\tau)$ be the roots of (3.303) such that $\lim_{\tau \to 0} t_1(\tau) = 0$ and $\lim_{\tau \to 0} t_2(\tau) = h$ (see [2], page 289). Then $t_1(\tau) - h$, and $t_2(\tau) - h$ will be the corresponding roots of

$$\psi(t \mid F) = e^{-\tau}. \tag{3.304}$$

Now by the Fundamental Identity we have

$$L_f e^{-bt_1(\tau)} E_{bf} e^{\tau n} + (1 - L_f) e^{at_1(\tau)} E_{af} e^{\tau n} = 1 \tag{3.305}$$

$$L_f e^{-bt_2(\tau)} E_{bf} e^{\tau n} + (1 - L_f) e^{at_2(\tau)} E_{af} e^{\tau n} = 1 \tag{3.306}$$

and

$$L_F e^{-b[t_1(\tau)-h]} E_{bF} e^{\tau n} + (1 - L_F) e^{a[t_1(\tau)-h]} E_{aF} e^{\tau n} = 1 \tag{3.307}$$

$$L_F e^{-b[t_2(\tau)-h]} E_{bF} e^{\tau n} + (1 - L_F) e^{a[t_2(\tau)-h]} E_{aF} e^{\tau n} = 1 \tag{3.308}$$

where $L_f = P[Z_n \leq -b \mid f]$, E_{bf} stands for the expected value of $e^{\tau n}$ under the hypothesis f and the restriction $Z_n \leq -b$; E_{af} stands for the expected value of $e^{\tau n}$ under the hypothesis f and the restriction $Z_n \geq a$; and the symbols L_F, E_{bF} and E_{aF} are similarly defined.

By comparing equations (3.305) and (3.306) with (3.307) and (3.308) we see that

$$L_F E_{bF} e^{\tau n} = e^{-hb} L_f E_{bf} e^{\tau n} \tag{3.309}$$

and

$$(1 - L_F) E_{aF} e^{\tau n} = e^{ha}(1 - L_f) E_{af} e^{\tau n}. \tag{3.310}$$

Since the above relationships hold for the characteristic functions of n, they must also hold for the distribution of n. This proves the theorem.

If we set $\tau = 0$ in (3.309) and (3.310) we also get

$$L_F = e^{-hb} L_f \tag{3.311}$$

and

$$1 - L_F = e^{ha}(1 - L_f). \tag{3.312}$$

In view of (3.311) and (3.312) we see from (3.309) and (3.310) that

$$E_{bF} e^{\tau n} = E_{bf} e^{\tau n} \tag{3.313}$$

and

$$E_{aF} e^{\tau n} = E_{af} e^{\tau n}. \tag{3.314}$$

From (3.313) and (3.314) we obtain the following rather surprising theorem.

THEOREM 4. *Except for the approximation indicated in Theorem 1, the conditional distribution of n under the restriction that $Z_n \leq -b$ as well as the restriction that $Z_n \geq a$ is identical for the two hypotheses F and f.*

The above theorems are of particular interest when F is a member of the class of distributions f. In any given sequential test the results of Theorem 1 can be used to facilitate the computation of the probabilities of making a decision. Furthermore, the results of Theorem 4 show that the conditional distribution of n throws no light on the parameter θ involved in the distribution of z. This follows since the conditional distribution of n is identical for the conjugate pair θ' and θ'', and, in any practical problem, θ' and θ'' will represent opposing hypotheses.

We shall now establish exact relationships of the type considered above when the variate z takes on a finite number of integral values.

Let z take on the values $-r, -r+1, \cdots, -1, 0, 1, 2, \cdots, m$ with $P(z = i) = p_i$. Furthermore, let $P_i = e^{hi} p_i$ where h is the real non-zero root of

$$\sum_{i=-r}^{m} p_i e^{hi} = 1. \tag{3.315}$$

Then the probabilities P_i and p_i are conjugate. We set $e^t = u$ and define $\phi(u \mid \theta)$ to be the generating function of z under the hypothesis $p(z = i) = \theta_i$. Then

$$\phi(u \mid p) = \sum_{i=-r}^{m} p_i u^i \tag{3.316}$$

and

$$\phi(u \mid P) = \sum_{i=-r}^{m} P_i u^i = \sum_{i=-r}^{m} p_i (e^h u)^i \tag{3.317}$$

Consider a sequential test defined by two boundaries $-b$ and a and a decision function $Z_n = \sum_{\alpha=1}^{n} z_\alpha$. Let ξ_{bi}^{θ} and ξ_{ai}^{θ} stand for the probabilities that $Z_n = -(b+i)$ and $Z_n = a + i$ respectively under the hypothesis that $\theta_i = P(z = i)$. Furthermore, let $P_{bi}(n \mid \theta)$ and $P_{ai}(n \mid \theta)$ stand for the probabilities that $Z_n = -(b+i)$ and $Z_n = (a+i)$ respectively in exactly n steps, under the hypothesis $\theta_i = P(z = i)$. Also, let the symbols E_{bi}^{θ} and E_{ai}^{θ} stand for conditional expectations under the hypothesis $\theta_i = P(z = i)$ and under the restriction that $Z_n = -(b+i)$ and $Z_n = a + i$ respectively.

Since z takes on a finite number of integral values, the Fundamental Identity for the two conjugate hypotheses, p and P can be written as:

$$\sum_{i=0}^{r-1} \xi_{bi}^{p} u^{-(b+i)} E_{bi}^{p}[\phi(u \mid p)]^{-n} + \sum_{i=0}^{m-1} \xi_{ai}^{p} u^{a+i} E_{ai}^{p}[\phi(u \mid p)]^{-n} = 1 \tag{3.318}$$

and

$$\sum_{i=0}^{r-1} \xi_{bi}^{P} u^{-(b+i)} E_{bi}^{P}[\phi(u \mid P)]^{-n} + \sum_{i=0}^{m-1} \xi_{ai}^{P} u^{a+i} E_{ai}^{P}[\phi(u \mid P)]^{-n} = 1. \tag{3.319}$$

For any real number τ let $u_1(\tau), u_2(\tau), \cdots, u_{r+m}(\tau)$ be the $r + m$ roots of the equation:

$$\phi(u \mid p) = \sum_{i=-r}^{m} p_i u^i = \frac{1}{\tau} \tag{3.320}$$

Then, in view of (3.317) the corresponding roots of

$$\phi(u \mid P) = \sum_{i=-r}^{m} P_i u^i = \frac{1}{\tau} \tag{3.321}$$

are given by $u_1(\tau)e^{-h}, u_2(\tau)e^{-h}, \cdots, u_{r+m}(\tau)e^{-h}$ Substituting these roots in (3.318) and (3.319) successively, we get

$$\sum_{i=0}^{r-1} \xi_{bi}^{p} u_j(\tau)^{-(b+i)} E_{bi}^{p} \tau^n + \sum_{i=0}^{m-1} \xi_{ai}^{p} u_j(\tau)^{a+i} E_{ai}^{p} \tau^n = 1 \tag{3.322}$$

and

$$\sum_{i=0}^{r-1} \xi_{bi}^{P} [u_j(\tau)e^{-h}]^{-(b+i)} E_{bi}^{P} \tau^n + \sum_{i=0}^{m-1} \xi_{ai}^{P} [u_j(\tau)e^{-h}]^{a+i} E_{ai}^{P} \tau^n = 1 \tag{3.323}$$

for $j = 1, 2, \quad , r + m$. Since the roots $u_j(\tau)$ are assumed to be known, the unknowns in (3.322) and (3.323) can be solved in terms of these roots provided the determinant of the equations is different from zero. But in section 2, we have indirectly shown that for a sufficiently small τ, the determinant is different from zero. Thus, assuming that the solution has been obtained we see from (3.322) and (3.323) that

$$\xi_{bi}^{P} E_{bi}^{P} \tau^n = e^{-h(b+i)} \xi_{bi}^{p} E_{bi}^{p} \tau^n \tag{3.324}$$

and

$$\xi_{ai}^{P} E_{ai}^{P} \tau^n = e^{h(a+i)} \xi_{ai}^{p} E_{ai}^{p} \tau^n. \tag{3.325}$$

Setting $\tau = 1$, we get

$$\xi_{bi}^{P} = e^{-h(b+i)} \xi_{bi}^{p} \tag{3.326}$$

and

$$\xi_{ai}^{P} = e^{h(a+i)} \xi_{ai}^{p}. \tag{3.327}$$

Moreover, if we expand the expressions in (3.324) and (3.325) in a power series in τ (which by section 2 is permissible), and compare coefficients of τ^n we get

$$P_{b_i}(n \mid P) = e^{-h(b+i)} P_{b_i}(n \mid p) \tag{3.328}$$

and

$$P_{a_i}(n \mid P) = e^{h(a+i)} P_{a_i}(n \mid p). \tag{3.329}$$

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SUFFICIENT STATISTICAL ESTIMATION FUNCTIONS FOR THE PARAMETERS OF THE DISTRIBUTION OF MAXIMUM VALUES

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1. Summary. The problem of estimating from a sample a confidence region for the parameters of the distribution of maximum values is treated by setting up what are called "statistical estimation functions" suggested by the functional form of the probability distribution of the sample, and finding the moment generating function of the probability distribution of these estimation "functions. Such an estimate by the method of maximum likelihood is also treated.

A definition of "sufficiency" is proposed for "statistical estimation functions" analogous to that which applies to "statistics." Also the concept of "stable statistical estimation functions" is introduced.

By means of a numerical illustration, four methods are discussed for setting up an approximate confidence interval for the estimated value of x of the universe of maximum values which corresponds to a given cumulative frequency .99, for confidence level .95 Two procedures for solving this problem are recommended as practicable

2. Introduction. If the universe comprises a set of maximum values of a large number of quantities, it has been shown that in many cases the probability density function of such a set of values of x is given approximately by

$$(2.1) \qquad f(x) = \alpha e^{-t}e^{-e^{-t}}, \qquad t = \alpha(x - u), \qquad -\infty < x < +\infty,$$

where α and u denote parameters, usually unknown [1].

This paper is concerned with the problem of estimation of the parameters α and u on the basis of sample data.

The notion of "sufficiency" is fundamental in the problem of estimation, since it means that the necessary elements of the sample have been used which will result in complete determination of that part of the sample probability distribution function involving the unknown parameters to be estimated. Unfortunately it does not seem to be possible to set up "sufficient statistics" within the usual definition of "statistic" for the above distribution. In this investigation the writer was struck by the fact that certain functions of the data involving one of the parameters could be used to play a very similar role to a set of *sufficient statistics* for determining α and u, in spite of the fact that one function involved the value of α, and hence was not directly determined by the data,—and hence not a "statistic."

Various statistics have been used in the past to estimate the parameters α and u, such as the sample mean, variance, mean deviation and an adjusted modal value (see [2] and [3]). For the reason noted above, sufficient statistics

have not been developed. In order to bridge this impasse and meet the *essentials* of the condition of sufficiency, the writer believes that a broader definition of sufficiency is needed. Such a definition is developed in the following section.

3. A broader definition of sufficiency. If the reader reexamines the process of estimating the two parameters of the normal distribution, and the determination of the two parameter confidence region for them from the statistics consisting of the sample mean, and the mean square deviation of the sample values from their mean, he will find that the separate determination of $\bar{x}$ and s^2 is not inherently necessary. The mean a and the variance σ^2 of the universe, are usually estimated from the *pair* of equations

$$E(\bar{x}) = a, \qquad E(s^2) = (n-1)\sigma^2/n$$

and the boundary of the confidence region is determined from knowledge of the bivariate distribution of $\bar{x}$ and s, which involves the four variables $\bar{x}$, s, a, and σ. The equation of the bounding curve is most easily set up in terms of transformed variables such as

$$U = \sqrt{n}\,(\bar{x} - a)/\sigma, \qquad V = \sqrt{n}\,s/\sigma. \tag{3.1}$$

Then the probability density of U and V is given by

$$f(U,V) = (\text{const.})V^{n-2}e^{-(U^2+V^2)/2}$$

and with confidence coefficient β; a bounding curve may be defined implicitly by the two equations

$$\iint f(U, V)\, dU dV = \beta,$$

$$f(U_1, V_1) = \text{constant}$$

where the above integral is taken over the region of the $V \geqq 0$ half of U,V plane bounded by the curve $f(U_1, V_1) = \text{constant}$.

A range of estimate of the parameters a and σ is offered by this confidence region by virtue of the fact that each point of the region corresponds to a *unique* pair of values of a and σ for a given set of sample values $0_n(x_i)$, and the fact that the equation of the bounding curve does not involve the parameters a and σ. Thus one arrives at a determinate range of estimate of a and σ, after the sample values have been observed. In this paper such functions will be referred to as *statistical estimation functions* (see [4]).

The classical idea of sufficiency implies (a) that the estimate be adequate for unique determination of the parameters, and (b) that *all* the sample information pertinent to such estimation be used. In the case of "statistics" the second requirement has been simply and elegantly formulated by the requirement that the probability density function of the sample distribution

factor in such a way that one factor be *completely* determined by the statistical estimates and the parameters of the distribution, and that the remaining factor be *independent* of the parameters to be estimated (see [7], or [5] p. 135).

It seems to be possible to carry over this formulation to statistical estimation functions (denoted by T_i). Since one or more of the parameters to be estimated, denoted by $(a_1, a_2, \cdots, a_r)$, are involved in these functions, a requirement that they be adequate for unique determination of these parameters is obviously that there be a *one-to-one correspondence* between the parameter set $(a_1, a_2, \cdots, a_r)$ and the set of estimation functions $(T_1, T_2, \cdots, T_r)$ in the region of estimate. This requirement will be referred to as Requirement (1).

It has been pointed out by a referee that some further requirement as to the independence of the probability density function of $(T_1, T_2, \cdots, T_r)$ relative to the parameters to be estimated is needed.

If one requires that the p. d. f. of $(T_1, T_2, \cdots, T_r)$ be entirely independent of the parameters $(a_1, a_2, \cdots, a_r)$ the estimation functions will furnish "confidence regions" for estimates of the parameters;—see example noted above for the normal distribution.

However, in some cases the mean values $E(T_i)$ may be independent of the parameters, while the p. d f. may not be; for example, —estimation functions for the two parameters of the Pearson Type III distribution formed from the maximum likelihood functions of that distribution. In such cases, a *point estimation* of the parameters is still possible, and would seem to satisfy the classical requirements of sufficiency.

The author accordingly makes the following proposals:

(a) Statistical estimation functions that satisfy the first two requirements—that of one-to-one correspondence with the parameters to be estimated, and the factoribility condition—be termed *sufficient* for estimation of the parameters. The reasonableness of such a definition is strengthened by the observation that given a set of "sufficient statistics" in the classical sense, statistical estimation functions that satisfy the factoribility condition can always be formed from them, and hence they are subject further only to Requirement (1) to make them *sufficient statistical estimation functions* under the proposed definition.

(b) Statistical estimation functions that satisfy Requirement (1) and also have a p. d. f. which is independent of the parameters to be estimated shall be called *stable* —a term suggested to the author by a referee.

(c) Statistical estimation functions T_i that satisfy Requirement (1) and are such that $E(T_i)$, $(i = 1, 2, \cdots, r)$, be independent of the parameters to be estimated, be called *stable in mean*, and that similarly, if the modal or median values of T_i be independent of these parameters, they be called *stable in mode*, *stable in median*, etc

Thus a definition of sufficiency applicable to statistical estimation functions is formulated as follows:

The term "statistical estimation function" will be used to denote a function of the sample values and one or more population parameters, used for purposes of statistical estimation

Given a universe with probability density function involving m parameters $a_1, a_2, \cdots, a_m$ in an admissible region R, and a set of r statistical estimation functions $T_i(0_n; a_1, a_2, \cdots, a_m)$ to be used for estimating the r parameters $a_1, a_2, \cdots, a_r$ relative to the information in a given sample 0_n. Consider the conditions:

(1) The functional form T_i insures a one-to-one correspondence between the points of the r-parameter space $(a_1, a_2, \cdots, a_r)$ contained in R and the points of the r-space defined by $(T_1, T_2, \cdots, T_r)$ for fixed $0_n(x_i)$ and fixed parameter values $a_{r+1}, a_{r+2}, \cdots, a_m$.

(2a) It shall be possible to express the probability density function of the sample 0_n as

$$P(0_n) = g_1(T_1, T_2, \cdots, T_r; a_1, a_2, \cdots, a_m)\cdot g_2(0_n; a_{r+1}, a_{r+2}, \cdots, a_m),$$

where the first factor is uniquely determinable for fixed $(a_1, a_2, \cdots, a_m)$ from the corresponding values of the functions T_i, and the second factor is independent of the parameters to be estimated.

(2b) It shall be possible to express the probability density function of the sample 0_n as

$$P(0_n) = G(T_1, T_2, \cdots, T_r; a_1, a_2, \cdots, a_m)g_2(0_n; a_{r+1}, a_{r+2}, \cdots, a_m),$$

where $G(\,,\,, \cdots, ; a_1, a_2, \cdots, a_m)$ is a functional, depending on $a_1, a_2, \cdots, a_m$, which in general involves the values of the T_i for values of $a_1, a_2, \cdots, a_m$ different from those appearing in the rest of the identity. (For example,

$$G(T, a) = \exp \int_0^a T(0_n; a')da'.)$$

(3) The r-variate probability density function of T_i based on $P(0_n; a_1, a_2, \cdots, a_m)$ shall exist.

Definition A. A set of statistical estimation functions T_i which satisfies conditions (1) and (2a) will be said to be a *sufficient* set of estimation functions for estimating the parameters a_i, $(i = 1, 2, \cdots, r)$, relative to the sample 0_n.

Definition B. A set of statistical estimation functions T_i which satisfies conditions (1) and (2b) will be said to be a *functionally sufficient* set of estimation functions for estimating the parameters a_i $(i = 1, 2, \cdots, r)$, relative to the sample O_n.

Definition C. If the conditions (1) and (3) are satisfied, and the p.d.f. of $(T_1, T_2, \cdots, T_r)$ is independent of the parameters a_i, $(i = 1, 2, \cdots, r)$, the functions T_i will be said to be *stable* relative to estimation of these parameters.

Definition D. If the conditions (1) and (3) are met, and $E(T_i)$, $(i = 1, 2, \cdots, r)$ are independent of the parameters to be estimated, the functions T_i will be said to be *stable-in-mean*; and similarly if modal or median values of T_i are independent of these parameters, the estimation functions will be said to be *stable-in-mode*, *stable-in-median*, etc.

It is not difficult to prove that a set of maximum likelihood functions

$$L_\alpha = \partial[\log P(0_n ; \alpha, \beta)]/\partial\alpha, \qquad L_\beta = \partial[\log P(0_n ; \alpha, \beta)]/\partial\beta$$

under the condition that the second order determinant

$$\begin{vmatrix} L_{\alpha\alpha} & L_{\alpha\beta} \\ L_{\beta\alpha} & L_{\beta\beta} \end{vmatrix}$$

exists and does not vanish over the admissible range of α and β, constitutes a set of estimation functions for α and β that are *functionally sufficient* and *stable-in-mean* under the definition given above. The meeting of Condition (2b) is demonstrated by the relation

$$\log P(0_n ; \alpha, \beta) = \int_{\alpha_0}^{\alpha} L_\alpha(\alpha, \beta_0)\, d\alpha + \int_{\beta_0}^{\beta} L_\beta(\alpha, \beta)\, d\beta + \log P(0_n ; \alpha_0, \beta_0)$$

since the first two terms on the right depend entirely upon the functions L_α and L_β, and the third term on the right becomes independent of α and β, if α_0 and β_0 are arbitrarily chosen, once for all, in the admissable region R.

In general the maximum likelihood functions are not *stable* estimation functions, but in many cases by the introduction of suitable factors which appear in the variance-covariance matrix (see (5.3) and (5.4)) estimation functions may be formed which satisfy Definition C.

4. Sufficient statistical estimation functions for the distribution of maximum values. The probability density function for the sample $0_n(x_i)$ drawn from a universe of maximum values is

(4.1) $$P(0_n) = \alpha^n e^{-\Sigma e^{-\alpha(x_i - u)}} e^{-\alpha \Sigma (x_i - u)}$$

where the summation sign used here and hereinafter refers to summation over all indices from 1 to n. Let $\bar{x}$ denote the sample mean, and define a new set of variables z_i by

(4.2) $$z_i = e^{-\alpha x_i}, \qquad (i = 1, 2, \cdots, n),$$

with mean $\bar{z}$. Also set

$$z_0 = e^{-\alpha u}.$$

Recognizing that the variables $2z_i/z_0$ are independently distributed like χ^2 on two degrees of freedom, the probability density function of $\bar{z}$ is given by

(4.3) $$P(\bar{z})\, d\bar{z} = [1/\Gamma(n)] e^{-n\bar{z}/z_0} (n\bar{z}/z_0)^{n-1} n\, d\bar{z}/z_0$$

with mean equal to z_0 and variance equal to z_0^2/n.

The mean value of t of the original distribution (2.1) is known to be Euler's constant, which will be denoted by C. Thus

(4.4) $$E[\alpha(\bar{x} - u)] = C = .5772157.$$

The above considerations point to a set of statistical estimation functions defined as follows

$$X = \sqrt{n}\,[\alpha(\bar{x} - u) - C],$$
$$(4.5) \qquad Y = \sqrt{n}\,[\bar{z}/z_0 - 1].$$

The author was not able to determine the explicit bivariate probability density function of X and Y, but the moment generating function G may be found with some degree of facility if the variables z_i are used in (4.1). Using simplified functions $n\alpha(\bar{x} - u)$ and $n\bar{z}/z_0$,

$$(4.6) \qquad G(\theta_1, \theta_2) = E[e^{\theta_1 n\alpha(\bar{x}-u)}e^{\theta_2 n\bar{z}/z_0}] = (1 - \theta_2)^{n(\theta_1 - 1)}\Gamma^n(1 - \theta_1).$$

Clearly $\bar{x}$ and $\bar{z}$ are not statistically independent The first and second partial derivatives give

$$(4.7) \qquad \begin{gathered} G_{\theta_1}(0, 0) = nC, \qquad G_{\theta_2}(0, 0) = n, \qquad G_{\theta_1\theta_1}(0, 0) = n\pi^2/6 + n^2C^2, \\ G_{\theta_2\theta_2}(0, 0) = n^2 + n, \qquad G_{\theta_1\theta_2}(0, 0) = n^2C - n. \end{gathered}$$

Hence the variances of the marginal distributions are

$$(4.8) \qquad \sigma^2[n\alpha(\bar{x} - u)] = n\pi^2/6, \quad \sigma^2(n\bar{z}/z_0) = n,$$

and the covariance is equal to $-n$.

Now the marginal distributions rapidly approach normality with increasing n. The question arises whether the bivariate distribution approaches normality. One way to prove this is as follows: Consider the moment-generating function G_2 of the statistical functions X and Y defined by (4.5). Following methods outlined above, with $\theta_3 = \sqrt{n}\theta_1$, $\theta_4 = \sqrt{n}\,\theta_2$, it is not difficult to show that the logarithm of the moment generating function $G_2(\theta_3, \theta_4)$ is given by

$$\log G_2$$
$$= (\sqrt{n}\,\theta_3 - n)\log(1 - \theta_4/\sqrt{n}) - \sqrt{n}\,\theta_4 + n\log\Gamma(1 - \theta_3/\sqrt{n}) - \sqrt{n}\,C.$$

As $n \to \infty$, one notes the relations

$$(4.9) \qquad \begin{gathered} -n\log(1 - \theta_4/\sqrt{n}) - \sqrt{n}\,\theta_4 = \theta_4^2/2 + o_1(\sqrt{n}), \\ n\log\Gamma(1 - \theta_3/\sqrt{n}) - \sqrt{n}\,C\theta_3 = (\theta_3^2/2)(\pi^2/6) + o_2(\sqrt{n}), \\ \sqrt{n}\,\theta_3\log(1 - \theta_4/\sqrt{n}) = -\theta_3\theta_4 + o_3(\sqrt{n}), \end{gathered}$$

where $o_i(\sqrt{n})$ denote functions that approach zero as $\sqrt{n} \to \infty$, uniformly for θ_3 and θ_4 in the neighborhood of zero. The limit

$$\lim_{n\to\infty} \log G_2 = \tfrac{1}{2}[\theta_4^2 - 2\theta_3\theta_4 + (\pi^2/6)\theta_3^2]$$

is recognized as the logarithm of the moment generating function of a normal bivariate distribution

Thus *the bivariate probability distribution function of the estimation functions X and Y approaches the normal bivariate distribution with zero means and variance-covariance matrix*

$$\left\| \begin{matrix} \pi^2/6 & -1 \\ -1 & 1 \end{matrix} \right\| \tag{4.10}$$

as n increases without limit, and the means and second order moments thus indicated, hold precisely for all values of n.

The functions X and Y satisfy Condition (1) for sufficiency relative to estimation of the parameters α and u provided α and u can be expressed as single valued functions of X and Y. A condition for this is that the Jacobian of the transformation shall not vanish. This Jacobian may be reduced to

$$[(n\alpha\bar{z})/z_0][\bar{x} - (\Sigma x_i e^{-\alpha x_i})/(\Sigma e^{-\alpha x_i})].$$

Let x_i be ordered so that $x_i \leqq x_{i+1}$. Then for $\alpha > 0$, the second term constitutes a weighted mean with positive weights which monotonically decrease as i increases, when the inequality $x_i < x_{i+1}$ holds. Hence unless all x_i are equal, this weighted mean is less algebraically than $\bar{x}$. Condition (2a) for sufficiency is clearly met by these functions. Thus one concludes that *for $\alpha > 0$, and the case that not all x_i are equal, the estimation functions X and Y constitute a sufficient set of estimation functions for the parameters α and u of distribution* (2.1). Since the moment generating function (see (4.6)) is independent of α and u, these functions are also *stable* estimation functions

5. Maximum likelihood estimation functions. General theory points to the use of the method of maximum likelihood as giving the most efficient solution (see [5]) With

$$f(x) = \alpha e^{-e^{-\alpha(x-u)}} e^{-\alpha(x-u)} \tag{5.1}$$

the maximum likelihood estimation functions are

$$\begin{aligned} L_u &= -n\alpha(\bar{z}/z_0 - 1) \\ L_\alpha &= n[1/\alpha - (\bar{x} - u) + \partial(\bar{z}/z_0)/\partial\alpha] \end{aligned} \tag{5.2}$$

with variance-covariance matrix

$$\left\| \begin{matrix} n\alpha^2 & n(1 - C) \\ n(1 - C) & (n/\alpha^2)[\pi^2/6 + (1 - C)^2] \end{matrix} \right\|. \tag{5.3}$$

Thus with

$$\begin{aligned} &X = \sqrt{n}\,(\bar{z}/z_0 - 1], \; Y = \sqrt{n}\,[\alpha(u - \bar{z}e^{-\alpha u}(u + \bar{z}_\alpha/\bar{z})) - (\alpha\bar{x} - 1)]/B \\ &B = \sqrt{\pi^2/6 + (1 - C)^2}, \end{aligned} \tag{5.4}$$

where

$$z_\alpha = \partial[\Sigma e^{-\alpha x_i}/n]/\partial\alpha,$$

the bivariate distribution of X and Y rapidly approaches normality with increasing n, with zero means, unit variances, and correlation coefficient given by (negative, since sign of L_u has been reversed)

$$(5.5) \qquad r = -(1 - C)/(\sqrt{\pi^2/6 + (1 - C)^2}).$$

With non-vanishing Jacobian, X and Y constitute a sufficient set of estimation functions for the parameters α and u (see (3.2) above). Furthermore the unit variances and correlation value given above are exact for all values of n. By setting up the moment generating function it is not difficult to show that these functions are also *stable* estimation functions for all values of n.

The theory of maximum likelihood further shows that if $\hat{u}$ and $\hat{\alpha}$ are defined as the u and α solutions of the equations

$$(5.6) \qquad L_u = 0, \qquad L_\alpha = 0$$

the distribution of $\sqrt{n}\,(\hat{u} - u)$ and $\sqrt{n}\,(\hat{\alpha} - \alpha)$ will approach normality asymptotically with zero means and variance-covariance matrix which is the reciprocal of the above matrix (multiplied by n); namely,

$$(5.7) \qquad \begin{Vmatrix} (1/\alpha^2)[1 + (1 - C)^2/(\pi^2/6)] & -(1 - C)/(\pi^2/6) \\ -(1 - C)/(\pi^2/6) & \alpha^2/(\pi^2/6) \end{Vmatrix}$$

6. Numerical applications. As an illustration of the application of the methods outlined above for determining the parameters of the distribution of maximum values from an observed sample, data is taken from the 57 year record of annual maximum flood flows previously used as an illustration by the author ([6] p. 324). There is some evidence to indicate that such a series follows approximately the distribution of maximum values. At any rate the series serves pretty well as a numerical illustration.

Confidence regions for u and α can be determined by four methods based upon the preceding theory. In order to make the numerical illustration more cogent, we shall answer the following question by each of the methods. What is the confidence interval (with confidence level .95) for annual flood x corresponding to a cumulated frequency of .99 (often referred to as a 100 yr. flood) based upon our observed 57 yr. sample, under the assumption that the distribution of maximum values (2.1) applies to this data?

Method 1. (Based on estimation functions of section 4.) In this case the statistical estimation functions X_1 and Y_1 defined from (4.5) by $X_1 = X\sqrt{6}/\pi$, $Y_1 = Y$, are used. The "best values" of u and α are taken as the solutions of $X_1 = 0$, $Y_1 = 0$, found by trial and error. As a starting point values of u and α may be estimated from $X_1 = 0$ and the standard deviation of x, (see

[2] or [6]), the mean deviation of x_i, or an adjusted modal value (see [3]). A few trials gives

$$\hat{u} = 179.7, \qquad \hat{\alpha} = 01998.$$

Approximating the distribution function of X_1 and Y_1 by the limiting normal bivariate distribution (4.10), with confidence level of .95 the equation of the bounding constant probability ellipse is found to be

$$X_1^2 + (1.5594)X_1Y_1 + Y_1^2 = 2.3491 \tag{6 1}$$

where the constants are independent of the sample values. This ellipse, by virtue of the one-to-one correspondence between (X_1, Y_1) and (u, α) bounds u and α based upon the observed sample (see [4]).

For cumulated frequency .99, the distribution of maximum values (2.1) yields

$$t = \alpha(x - u) = 4.60015$$

Thus the analytic problem is that of determining the maximum and minimum value of

$$x = g(u, \alpha) = 4.60015/\alpha + u \tag{6.2}$$

which occurs on the ellipse (6.1).[1]

The writer solved this graphically. It was found necessary to compute three values of $\bar{z}$,—at $\alpha = .01$, .015 and .025, in addition to the value of $\bar{z}$ at $\alpha = .01998$ previously found. From these computations the curves $\alpha = .01$, $\alpha = 015$, $\alpha = .01998$ and $\alpha = .025$ were drawn on the chart of the ellipse (6.1). The $u =$ const. curves were quite easily determined by points on the $\alpha =$ const. curves found from their X_1 coordinates which are linear functions of u (see (4.5)). The extreme values of $g(u, \alpha)$ will be found to occur near the extreme values of α on the ellipse. A construction of several $u =$ const. curves near these extremes enables one to determine several successive values of $g(u, \alpha)$ at points where these curves cross the ellipse. The answers were

$$\begin{aligned} &\text{Max. } g(u, \alpha) = 507.4 \text{ at } u = 192, \alpha = .01459, \\ &\text{Min. } g(u, \alpha) = 360.0 \text{ at } u = 172, \alpha = .02447, \\ &\text{and } g(\hat{u}, \hat{\alpha}) = 409.9. \end{aligned} \tag{6.3}$$

Method 2. (Based on maximum likelihood statistical estimation functions (5.4)). For purposes of comparison the writer carried through the solution using the maximum likelihood estimation functions X_2 and Y_2 defined by (5.4).

[1] Since with non-vanishing Jacobian of (X_1, Y_1) relative to (u, α), no singular point of the (u, α) coordinate system can lie within the ellipse, it is clear from the form of the function $g(u, \alpha)$ that its maximum and minimum values will lie on the boundary of the ellipse. A similar remark applies to Methods 2–4.

In this case the equation of the bounding ellipse was

$$X_2^2 + (.62614)X_2Y_2 + Y_2^2 = 5.4042. \tag{6.4}$$

The determination of the network of α = const., u = const. curves was much more complicated in this case. The results were

Solution of $X = 0$, $Y = 0$, gave $\hat{u} = 180.6$, $\hat{\alpha} = .01924$; $g(\hat{u}, \hat{\alpha}) = 419.7$

$$\begin{aligned} &\text{Max. } g(u, \alpha) = 509.5 \text{ at } u = 187, \alpha = .01426 \\ &\text{Min. } g(u, \alpha) = 364.4 \text{ at } u = 172, \alpha = .02391. \end{aligned} \tag{6.5}$$

The slightly smaller range of estimate of $g(u, \alpha)$ resulting from the use of the second method was forecast from the general theory which predicts a narrowing of range of variation of u and α for same confidence level. Both bivariate distributions involve exact moments of the first and second degree for finite n, and both approach normality rapidly with increasing n. Hence comparable results were to be expected. Of course the form of the function $g(u, \alpha)$ in relation to the different types of estimation functions used in the two cases might modify the comparability of the results.

Method 3. (Based on limiting distribution of maximum likelihood statistics $\hat{u}$ and $\hat{\alpha}$, with variances unknown.) The use of the limiting distribution of the estimation functions $\sqrt{n}\,(\hat{u} - u)$, $\sqrt{n}\,(\hat{\alpha} - \alpha)$ led to results which were not entirely expected by the author. Taking

$$\begin{aligned} &X_3 = A\alpha(\hat{u} - u)/B, \qquad Y_3 = A(\hat{\alpha}/\alpha - 1) \\ &A = \pi\sqrt{n}/\sqrt{6}, \qquad B = \sqrt{\pi^2/6 + (1 - C)^2}, \end{aligned} \tag{6.6}$$

with

$$r = -(1 - C)/B,$$

the equation of the bounding ellipse is the same as (6.4), (no reversal of sign of r occurs because sign of r in (6.4) was reversed by reversing sign of L_u in (5.4)).

Using the inverse method where the range in u and α, with $\hat{u} = 180.6$, $\hat{\alpha} = .01924$, is determined from the range of (X_3, Y_3) within the ellipse (6.4), the maximum and minimum obtained for $g(u, \alpha)$ was

$$\begin{aligned} &\text{Max. } g(u, \alpha) = 490.2 \text{ at } u = 193.2, \alpha = .01549 \\ &\text{Min. } g(u, \alpha) = 353.8 \text{ at } u = 174.0, \alpha = .02558. \end{aligned} \tag{6.7}$$

This result does not agree closely with the previous results. The reason for this discrepancy may be that since the variances indicated by (5.7) are *not* exact for finite n, a variation of α from the central value predicted by (5.6) tends to exaggerate the departure of the distribution of X and Y from the limiting normal distribution through its effect upon the variances. The plausibility of such an explanation is strengthened by the numerical results of a solution of our problem by Method 4.

Method 4 (Based on limiting distribution of maximum likelihood statistics $\hat{u}$ and $\hat{\alpha}$, with variances estimated by taking $\alpha = \hat{\alpha}$ as observed from the sample.) In this case the unknown variances are estimated by taking $\alpha = \hat{\alpha}$ as observed from the sample studied. In order to avoid confusion let α_0 denote this value of α as used in the variance formulae. Thus the estimating functions X_4 and Y_4 become

$$(6.8) \qquad X_4 = A\alpha_0(\hat{u} - u)/B, \qquad Y_4 = A(\hat{\alpha} - \alpha)/\alpha_0$$

and the approximating distribution of (X_4, Y_4) is taken as the same limiting normal distribution used in Method 3. With

$$u_0 = \hat{u} = 180.6, \qquad \alpha_0 = \hat{\alpha} = .01924$$

the extreme values of $g(u, \alpha)$ on the ellipse were

$$(6.9) \qquad \begin{aligned} &\text{Max. } g(u, \alpha) = 507.4 \text{ at } u = 188.6, \alpha = .01443 \\ &\text{Min. } g(u, \alpha) = 362.8 \text{ at } u = 169.7, \alpha = .02382. \end{aligned}$$

These results agree closely with the results obtained by Methods 1 and 2.

The confidence intervals in $g(u, \alpha)$ obtained were, in summary.

Method 1	360.0 to 507.4
Method 2	364.4 to 509.5
Method 3	353.8 to 490.2
Method 4	362.8 to 507.4.

From the analysis of the four methods presented above, one might recommend the following two procedures for finding the confidence interval for x in a problem of the above description, as practicable:

Procedure 1. Use Method 1.

Procedure 2. Determine the maximum likelihood estimates $\hat{u}$ and $\hat{\alpha}$ from (5.6) by trial and error. Then use Method 4. Presumably the second procedure would be more open to question, especially for small values of n.

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ON FUNCTIONS OF SEQUENCES OF INDEPENDENT CHANCE VECTORS WITH APPLICATIONS TO THE PROBLEM OF THE "RANDOM WALK" IN k DIMENSIONS

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1. Summary. Consider a sequence $\{x_i\}$ of independent chance vectors in k dimensions with identical distributions, and a sequence of mutually exclusive events $S_1, S_2, \cdots$, such that S_i depends only on the first i vectors and $\Sigma P(S_i) = 1$. Let φ_i be a real or complex function of the first i vectors in the sequence satisfying conditions: (1) $E(\varphi_i) = 0$ and (2) $E(\varphi_j \mid X_1, \cdots, X_i) = \varphi_i$ for $j \geq i$. Let $\varphi = \varphi_i$ and $n = i$ when S_i occurs. A general theorem is proved which gives the conditions φ_i must satisfy such that $E\varphi = 0$. This theorem generalizes some of the important results obtained by Wald for $k = 1$. A method is also given for obtaining the distribution of φ and n in the problem of the "random walk" in k dimensions for the case in which the components of the vector take on a finite number of integral values.

2. A basic theorem.

2.1 Let $\{X_i\} = \{(X_{1i}, X_{2i}, \cdots, X_{ki})\}$ be a sequence of independent k-dimensional chance variables with identical distributions. Let $S_1, S_2, S_3, \cdots$, be mutually exclusive events such that (1) S_i depends only on $X_1, X_2, \cdots, X_i$, and (2) $\Sigma P(S_i) = 1$. Let $\varphi_i(X_1, X_2, \cdots, X_i)$ be a sequence of real or complex variables satisfying the following two conditions:

Condition 1: $E(\varphi_i) = 0$ for all i.

Condition 2: $E(\varphi_j \mid X_1, X_2, \cdots, X_i) = \varphi_i$ for all $j \geq i$, where $E(\varphi_j \mid X_1, X_2, \cdots, X_i)$ stands for the expected value of φ_j under the condition that $X_1, X_2, \cdots, X_i$ are held constant.[1] Define $\varphi_i = \varphi$ and $n = i$ when the event S_i occurs. We shall assume that $E(n)$ is finite.

A problem of central importance in sequential theory may be formulated as follows: What conditions must φ_i satisfy so that $E(\varphi)$ exists and equals zero? We shall prove the following:

Theorem 2.1. *If there exists a function $f(x_1, x_2, \cdots, x_k) \geq 0$ such that* (a) *$E[f(X_i)]$ is finite and* (b) *$|\varphi_i| \leq \sum_{d=1}^{i} f(X_d)$ when $n \geq i$, then $E(\varphi)$ exists and equals zero.*

Before proceeding to the proof, we consider two consequences of this theorem.

I. Assume that $E(X_{ri}) = a_r$. Let $\varphi_i = \sum_{j=1}^{i} (X_{rj} - a_r)$. It is easily verified that φ_i satisfies conditions 1 and 2. We set $f(x_1, x_2, \cdots, x_k) = | x_r$

[1] Chance variables φ_i satisfying condition 2 have been extensively studied by P. Levy [1] and J. L. Doob [2].

$- a_r|$. Then Theorem 2.1 is applicable and we get $E\varphi = 0$. Now $\varphi = W_r - na_r$ where $W_r = \sum_{i=1}^{n} X_{ri}$. Hence we have

$$E(W_r) = a_r E(n). \tag{2.11}$$

The relationship (2.11) has been proved for $k = 1$ by Wald [3] and subsequently under somewhat more generalized conditions, by one of the authors [4].

II. Let $t_1, t_2, \cdots, t_k$ be any real or complex numbers for which $Ee^{\sum_{r=1}^{k} t_r x_{ri}} = a$ is finite and $|a| \geq 1$. We assume that there exists a positive constant M such that

$$\left|\sum_{i=1}^{m} X_{ri}\right| \leq M, r = 1, 2, \cdots, k, \tag{2.12}$$

when $n > m$. Let

$$\varphi_i = a^{-i} e^{\sum_{j=1}^{i} \sum_{r=1}^{k} t_r x_{rj}} - 1 \tag{2.13}$$

so that

$$\varphi = a^{-n} e^{\sum_{r=1}^{k} t_r W_r} - 1 \tag{2.14}$$

where W_r is defined as above. It is easy to show that φ_i satisfies conditions 1 and 2. Now, in view of (2.12), when $n \geq i$

$$|\varphi_i| \leq |a|^{-i} e^{M\sum_{r=1}^{k} |\tau_r|} e^{\sum_{r=1}^{k} \tau_r x_{ri}} + 1 \leq 1 + Re^{\sum_{r=1}^{k} \tau_r x_{ri}} \tag{2.15}$$

where τ_j is the real part of t_j and $R = e^{M\sum_{r=1}^{k} |\tau_r|}$ is a fixed positive constant. Then, letting

$$f(x_1, x_2, \cdots, x_k) = 1 + Re^{\sum_{r=1}^{k} \tau_r x_{ri}} \tag{2.16}$$

we may apply Theorem 2.1 and obtain

$$E\left(a^{-n} e^{\sum_{r=1}^{n} t_r W_r}\right) = 1 \tag{2.17}$$

which is a generalization of the Fundamental Identity proved by Wald [5] for the case $k = 1$.

PROOF OF THEOREM 2.1. Assume φ_i is real. Define chance variables N_m inductively as follows: $N_0 = 0$. Assuming $N_0, \cdots, N_m$ defined, define $N_{m+1} = N_m + n(X_{N_m+1}, X_{N_m+2}, \cdots)$. Also let $n_m = N_m - N_{m-1}$ and $y_m = f(X_{N_{m-1}+1}) + \cdots + f(X_{N_m})$. It can be shown by induction that N_m is defined for all m with probability one, and that $\{n_m\}$, $\{y_m\}$ are sequences of independent chance variables with identical distributions. Clearly $n_1 = n$.

The Strong Law of Large Numbers asserts that if $z_1, z_2, \cdots$ are independent chance variables with identical distribution, then $\lim_{m\to\infty} \frac{z_1 + z_2 + \cdots + z_m}{m} = c$ with probability one if and only if Ez_1 exists and equals c.

It follows that, with probability one

$$\lim_{m\to\infty} \frac{f(X_1) + \cdots + f(X_m)}{m} = E[f(X_1)] \tag{2.18}$$

and

$$\lim_{m\to\infty} \frac{n_1 + \cdots + n_m}{m} = \lim_{m\to\infty} \frac{N_m}{m} = E(n). \tag{2.19}$$

Since $\dfrac{y_1 + \cdots + y_m}{n_1 + \cdots + n_m} = \dfrac{y_1 + \cdots + y_m}{N_m}$ is a subsequence of $\dfrac{f(X_1) + \cdots + f(X_m)}{m}$, we have with probability one,

$$\lim_{m\to\infty} \frac{y_1 + \cdots + y_m}{N_m} = E[f(X_1)] \tag{2.20}$$

so that

$$\lim_{m\to\infty} \frac{y_1 + \cdots + y_m}{m} = E[f(X_1)]E(n). \tag{2.21}$$

Consequently, $E(y_1)$ exists and equals $Ef(X_1)E(n)$. Since $|\varphi| \leq y_1$, $E(\varphi)$ exists. Also using conditions (2) and (b) which were imposed on φ_i we have

$$\begin{aligned} \left|\int_{S_1+\cdots+S_i} \varphi\, dp\right| &= \left|\sum_{j=1}^{i} \int_{S_j} \varphi_j\, dp\right| = \left|\sum_{j=1}^{i} \int_{S_j} \varphi_i\, dp\right| \\ &= \left|-\int_{n>i} \varphi_i\, dp\right| = \left|\sum_{j>i} \int_{S_j} \varphi_i\, dp\right| \\ &\leq \sum_{j>i} \int_{S_j} |\varphi_i|\, dp \leq \sum_{j>i} \int_{S_j} y_1\, dp \end{aligned} \tag{2.22}$$

which approaches zero as $i \to \infty$. This completes the proof.

If φ_j is a complex valued function, Theorem 2.1 still holds. For writing $\varphi_j = g_j + ih_j$ then Condition 2 becomes $E(g_p + ih_p \mid X_1, \cdots, X_j) = g_j + ih_j$ when $p \geq j$. Hence

$$E(g_p \mid X_1, \cdots, X_j) = g_j \tag{2.23}$$

and

$$E(h_p \mid X_1, \cdots, X_j) = h_j \tag{2.24}$$

when $p \geq j$. Since $|g_j| \leq |\varphi_j|$ and $|h_j| \leq |\varphi_j|$ and φ_j satisfies condition (b) we may apply Theorem 2.1 and get

$$Eg = E(h) = 0. \tag{2.25}$$

Hence $E\varphi = 0$.

3. Applications to the problem of the random walk in k dimensions[2]

3.1. *A theorem concerning decision points.* Let $\{X_j\} = \{(X_{1i}, \cdots, X_{ki})\}$ be a sequence of k-dimensional chance vectors with identical distributions. We assume that X_{ji} $(j = 1, 2, \cdots, k)$, take on a finite number of integral values ranging from $-r_j$ to m_j inclusive, where r_j and m_j are positive integers. We remark that any distribution can be approximated to any degree of accuracy by the distribution of a variate whose values are integral multiples of a constant d, which can be taken as the unit of measurement.

Let $P_{u_1u_2\cdots u_k}$ be the probability that $X_i = (u_1, u_2, \cdots, u_k)$. We define $W_{pi} = \sum_{j=1}^{i} X_{pj}$ and set $U_i = (W_{1i}, W_{2i}, \cdots, W_{ki})$. Then $\{U_i\}$ represents a sequence of points with integral coordinates in a k-dimensional space $S_k = \{(y_1, y_2, \cdots, y_k\}$. Let R be an arbitrary bounded region in S_k. We shall assume, without loss of generality, that the origin is an interior point of R. We now define a random variable n as the smallest subscript i of the sequence $\{U_j\}$ for which W_i is either a boundary point or an exterior point of R. We set $U_n = W = (W_1, W_2, \cdots, W_k)$ and designate W as a decision point of R. Clearly, the number of decision points is finite.

The random variables n and W can be interpreted as follows: Consider a point Q which at the time $t = 0$ is at the origin. At successive intervals of time $t = 1, 2, \cdots$, the point Q moves with integral components in S_k the direction and distance of the motion being determined by chance. The point comes to rest as soon as, but not before it either reaches the boundary of R or falls outside of R. Let U_t be the co-ordinates of the point Q at time t. Then n represents the length of time it takes Q to come to rest, and W represents a possible resting point.[3]

We shall be concerned with the problem of finding the probability distribution of n and W. These will obviously depend on the shape of the region R. In what follows we shall restrict ourselves to the class of regions R which have the property that the intersection of any line parallel to the axes with R is an open interval. In view of the fact that W has integral coordinates, we can without any loss of generality, replace this class of regions by an equivalent class which are bounded by simple polygonal closed surfaces whose vertices have integral coordinates and whose sides are parallel to the planes $y_j = 0$. In the subsequent discussion we assume that the regions R are of this type.

Let

$$\text{(3.10)} \qquad \underset{(y_1, y_2 \cdots y_k) \in R}{\text{l.u.b.}} [y_i]$$

[2] What follows is a generalization of a method previously employed by one of the authors [6] for the case $k = 1$.

[3] That Q will reach a resting point eventually can be asserted with probability one. See A. Wald [5], Lemma 1.

and

$$-b_i = \underset{y_i}{\text{g.l.b}} \; [(y_1, y_2, \cdots, y_k) \in R] \tag{3.11}$$

then a_i and b_i are positive integers.

We now prove the following:

LEMMA 3.1. *For the given sequence of chance vectors $\{X_i\}$ and the given region R, the number of possible decision points N_R is given by*

$$N_R = \prod_{j=1}^{k} (a_j + b_j + r_j + m_j - 1) - \prod_{j=1}^{k} (a_j + b_j - 1). \tag{3.12}$$

PROOF: We shall first prove this theorem for a rectangular region $R = R_1$ where R_1 is defined by $-b_i < y_i < a_i$, $(i = 1, 2, \cdots, k)$ and then generalize the proof to any region of the class specified.

Let R_2 be a closed rectangular region defined by $-(b_i + r_i - 1) \leq y_i \leq (a_i + m_i - 1)$. Then $R_2 \geq R_1$. Let $S = R_2 - R_1$. It is clear that every integral point of S is a possible decision point. Moreover, no point exterior to R_2 is a possible decision point. For assume, for example, that there exists a point $W = (W_1, W_2, \cdots, W_k)$ which is an exterior point of R_2. Then at least one of its coordinates, say W_j, has the property that $W_j > a_j + m_j - 1$ or $W_j < -(b_j + r_j - 1)$. But since $-(b_j - 1) \leq W_{j,n-1} \leq a_j - 1$, it must follow that X_{jn} took on a value greater than m_j or less than $-r_j$ which is contrary to assumption. Now the total number of integral points contained in R_1 is $\prod_{j=1}^{k} (a_j + b_j + r_j + m_j - 1)$ and the total number of integral points in R_1 which by assumption are not decision points, is $\prod_{j=1}^{k} (a_j + b_j - 1)$. Hence the Lemma is proved if R is a rectangular region.

Now, let R be any polygonal region of the type specified and let R_1 be the corresponding rectangular region. Consider two randomly moving points Q and Q_1, each having coordinates W_t at time t. Let the decision points for Q be defined in terms of R and the decision points of Q_1 in terms of R_1. We shall prove that the number of decision points for Q and Q_1 are the same.

By assumption, every line parallel to the axes intersects R in an open interval. Moreover $R_1 \supseteq R$. Hence the sum of the areas of the segments which compose the boundary of R must equal the area of the boundary of R_1. The same must be true for the total number of integral points on the boundaries of the two regions. Thus, the theorem is true for $r_j = m_j = 1$, $(j = 1, 2, \cdots, k)$. We assume that the theorem is true for $r_j = r_j'$ and $m_j = m_j'$ and prove that it must hold for $m_u = m_u' + 1$ for a fixed but arbitrary u. Now it is obvious that if the range of X_{ui} is increased by unity in the positive direction, the point Q can move an extra unit in the positive direction parallel to the y_u axes. Thus, the total number of additional decision points that Q gains by the unit increase in the range of X_{ui} is identical with the total number that Q_1 gains. This proves the theorem.

It is clear that the smallest rectangular region which includes all the decision points of W is R_2. We now prove the following:

THEOREM 3.1. *For any polygonal region R of the class previously specified, and for any random sequence $\{X_i\}$ in which X_i takes on a finite number of integral values, the number of points in the rectangular region R_2 which are not decision points is always equal to* $\prod_{j=1}^{k} (a_j + b_j - 1)$ *where $a_j + b_j$ are the dimensions of the rectangular region R_1.*

PROOF: This Theorem follows from Lemma 3.1 and the fact that the total number of integral points in R_2 is $\prod_{j=1}^{k} (a_j + b_j + r_j + m_j - 1)$.

3.2. *The distribution of W.* Let $\psi(t_1, \cdots, t_k)$ be the joint generating function of X_{ui}, $(u = 1, 2, \cdots, k)$, and $\varphi(t_1, \cdots, t_k)$ the joint generating function of W_j $(j = 1, 2, \cdots, k)$. Then

$$\psi(t_1, \cdots, t_k) = \sum_{u_1=-r_1}^{m_1} \cdots \sum_{u_k=-r_k}^{m_k} P_{u_1 \cdots u_k} t_1^{u_1} \cdots t_k^{u_k} \tag{3.21}$$

$$\phi(t_1, \cdots, t_k) = \sum_{v_1=-(b_1+r_1-1)}^{a_1+m_1-1} \cdots \sum_{v_k=-(b_k+r_k-1)}^{a_k+m_k-1} \xi_{v_1 \cdots v_k} t_1^{v_1} \cdots t_k^{v_k} \tag{3.22}$$

where $\xi_{v_1 \cdots v_k}$ is the probability that $W = (v_1, \cdots, v_k)$. In terms of the generating function ψ the Fundamental Identity (3.17) states that

$$E t_1^{W_1} \cdots t_k^{W_k} [\psi(t_1, \cdots, t_k)]^{-n} = 1 \tag{3.23}$$

for all $t_1, \cdots, t_k$ for which $|\psi(t_1, \cdots, t_k)| \geq 1$. Hence, it follows that for $t_1, \cdots, t_k$ for which $\psi(t_1, \cdots, t_k) = 1$, $\varphi(t_1, \cdots, t_k) = 1$. Let

$$f(t_1, \cdots, t_k) = t_1^{r_1} \cdots t_k^{r_k} [\psi(t_1, \cdots, t_k) - 1] \tag{3.24}$$

and

$$g(t_1, \cdots, t_k) = t_1^{b_1+r_1-1} \cdots t_k^{b_k+r_k-1} [\varphi(t_1, \cdots, t_k) - 1]. \tag{3.25}$$

Then $f(t_1, \cdots, t_k)$ is a polynomial of degree $r_j + m_j$ in t_j and $g(t_1, \cdots, t_k)$ is a polynomial of degree $(a_j + b_j + r_j + m_j - 2)$ in t_j.

We shall assume that $f(t_1, \cdots, t_k)$ is an irreducible polynomial. Then, since $g(t_1, \cdots, t_k)$ vanishes for all values of $t_1, \cdots, t_k$ for which $f(t_1, \cdots, t_k)$ vanishes, it follows[4] that f is a factor of g. That is

$$g(t_1, \cdots, t_k) = f(t_1, \cdots, t_k) \sum_{s_1=0}^{a_1+b_1-1} \cdots \sum_{s_k=0}^{a_k+b_k-1} C_{s_1 \cdots s_k} t_1^{s_1} \cdots t_k^{s_k} \tag{3.26}$$

where the $C_{s_1, \cdots, s_k}$ are unknown. Equating coefficients on both sides of (3.26) we get

[4] See, for example, Bôcher [7], Theorem 7, Chapter 16.

$$\xi_{v_1-b_1-r_1+1\cdots v_k-b_k-r_k+1} = \sum_{u_1=0}^{v_1}\cdots\sum_{u_k=0}^{v_k}(P_{u_1-r_1\cdots u_k-r_k} - \delta_{u_jr_j})\,C_{v_1-r_1\cdots v_k-r_k} + \prod_{j=1}^{k}\delta_{v_j,b_j+r_j-1} \tag{3.27}$$

where δ_{ij} is the Kronecker delta. But by Theorem 3.1, $\prod_{j=1}^{k}(a_j + b_j - 1)$ of the $\xi_{v_1\cdots v_k}$ in $\varphi(t_1, \cdots, t_k)$ are zero since they correspond to values of W which are non-decision points. Hence $\prod_{j=1}^{k}(a_j + b_j - 1)$ terms in (3.27) are zero with the exception of the term $\xi_{b_1+r_1-1\cdots b_k+r_k-1}$ (corresponding to the non-decision point (0, 0)) which is -1. Hence, we have the required number of equations to solve for the unknown C's and consequently for the ξ's provided the determinant of the coefficients is different from zero.

As an illustration, let $R = R_1$, then the C's are obtained by solving the set of linear equations

$$\sum_{u_1=0}^{v_1}\cdots\sum_{u_k=0}^{v_k}\left(\prod_{j=1}^{k}\delta_{u_jr_j} - P_{u_1-r_1\cdots u_k-r_k}\right)C_{v_1-r_1\cdots v_k-r_k} = \prod_{j=1}^{k}\delta_{v_j,b_j+r_j-1} \tag{3.28}$$

where v_j takes on all integral values from r_j to $a_j + b_j + r_j - 2$ inclusive.

3.3. *The distribution of* n. For any random variable U, let $E_{v_1\cdots v_k}U$ stand for the expected value of U under the restriction that $W = (v_1, v_2, \cdots, v_k)$. Let $\varphi_1(t_1, \cdots, t_k; \tau)$ be the joint generating function of $W_1, W_2, \cdots, W_k$, and n. Then

$$\varphi_1(t_1, \cdots, t_k; \tau) = \sum_{u_1}\cdots\sum_{u_k}\xi_{u_1\cdots u_k}t_1^{u_1}\cdots t_k^{u_k}E_{u_1\cdots u_k}\tau^n. \tag{3.31}$$

Let

$$\psi_1(t_1, \cdots, t_k; \tau) = \tau\psi(t_1, t_2, \cdots, t_k) - 1 \tag{3.32}$$

where $\psi(t_1, \cdots, t_k)$ is the joint generating function of $X_{1i}, \cdots, X_{ki}$ and is given by (3.21) and let

$$\psi_2(t_1, \cdots, t_k; \tau) = \varphi_1(t_1, \cdots, t_k; \tau) - 1. \tag{3.33}$$

Then, if we fix τ so that $|\tau| \leq 1$, we see by (3.23) that for all values of $t_1, \cdots, t_k$ for which ψ_1 vanishes, ψ_2 also vanishes. Let

$$f_1(t_1, \cdots, t_k; \tau) = t_1^{r_1}\cdots t_k^{r_k}\psi(t_1, \cdots, t_k; \tau) \tag{3.34}$$

and

$$f_2(t_1, \cdots, t_k; \tau) = t_1^{b_1+r_1-1}\cdots t_k^{b_k+r_k-1}\psi_2(t_1, \cdots, t_k; \tau). \tag{3.35}$$

Then for τ fixed, f_1 is a polynomial of degree $r_j + m_j$ in t_j and f_2 is a polynomial of degree $a_j + b_j + r_j + m_j - 2$ in t_j. Since f_2 vanishes for all values of $t_1, \cdots, t_k$ for which f_1 vanishes then if f_1 is irreducible, f_1 will be a factor of f_2. That is f_2 can then be written as

$$(3.36)\quad f_2(t_1, \cdots, t_k; \tau) = f_1(t_1, \cdots, t_k; \tau) \sum_{v_1=1}^{a_1+b_1-2} \cdots \sum_{v_k=1}^{a_k+b_k-2} d_{v_1} \cdots {}_{v_k} t_1^{v_1} \cdots t_k^{v_k}.$$

The rest of the argument is identical with that employed in section 3.3. The unknowns in the present case, however, are $\xi_{v_1 \cdots v_k} E_{v_1 \cdots v_k} \tau^n$. When $\xi_{v_1 \cdots v_k} E_{v_1 \cdots v_k} \tau_n$ is expanded in a power series in τ, the coefficient of τ^m is the probability that $W = (v_1, \cdots, v_k)$ in exactly m steps. We shall, therefore, examine the validity of the expansion of the above function in the neighborhood of $\tau = 0$.

Let us first consider the rectangular region $R = R_1$. In this case the d's are obtained from the equations

$$(3.37)\quad \sum_{u_1=1}^{v_1} \cdots \sum_{u_k=1}^{v_k} \left(\prod_{j=1}^{k} \delta_{u_j r_j} - \tau P_{u_1 - r_1 \cdots u_k - r_k} \right) d_{v_1 - r_1 \cdots v_k - r_k} = \prod_{j=1}^{k} \delta_{v_j, b_j + r_j - 1},$$

$$(v_j = r_j, \quad r_j + 1, \cdots, \quad a_j + b_j + r_j - 2),$$

so that $\xi_{v_1 \cdots v_k} E_{v_1 \cdots v_k} \tau^n$ will be given as a ratio of two polynomials in τ the denominator of which will be the determinant of the coefficients of (3.37). But this determinant equals unity when $\tau = 0$. Hence the validity of the expansion is established for a rectangular region.

If R is not a rectangle, the value of the determinant of the equations in d will still be unity. This follows from the fact that the number of non-decision points in R_2 is precisely the same as the number of non-decision points contained in R_1, hence by rearranging of the equations they an be made to assume the form (3.37).

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APPROXIMATION OF THE DISTRIBUTION OF THE PRODUCT OF BETA VARIABLES BY A SINGLE BETA VARIABLE

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1. Introduction. In an article published elsewhere in the present issue of the *Annals of Mathematical Statistics* [1] the g-th moments of two statistical test criteria L_{mvc} and L_{vc} were found to have the following expressions, respectively:

$$(1) \qquad (k-1)^{g(k-1)} \prod_{i=1}^{k-1} \left[\frac{\Gamma(\frac{1}{2}(n-1-i)+g)}{\Gamma(\frac{1}{2}(n-1-i))} \right] \cdot \frac{\Gamma(\frac{1}{2}n(k-1))}{\Gamma(\frac{1}{2}n(k-1)+g(k-1))}$$

and

$$(2) \qquad (k-1)^{g(k-1)} \prod_{i=1}^{k-1} \left[\frac{\Gamma(\frac{1}{2}(n-1-i)+g)}{\Gamma(\frac{1}{2}(n-1-i))} \right] \cdot \frac{\Gamma(\frac{1}{2}(n-1)(k-1))}{\Gamma(\frac{1}{2}(n-1)(k-1)+g(k-1))}.$$

If we denote by $(a)_g$ the expression $a(a+1)(a+2)\cdots(a+g-1)$ and make use of the fact that

$$(3) \qquad \Gamma(a+g) = \Gamma(a)\cdot(a)_g$$

and

$$(4) \qquad \Gamma(a+rg) = \Gamma(a)\cdot(a)_{rg} = \Gamma(a)\cdot r^{rg} \prod_{i=1}^{r} \left(\frac{a+i-1}{r}\right)_g$$

where r is a positive integer, the two moments (1) and (2) reduce to

$$(5) \qquad \frac{\prod_{i=1}^{k-1} \left(\frac{n}{2}+\frac{i-k-1}{2}\right)_g}{\prod_{i=1}^{k-1} \left(\frac{n}{2}+\frac{i-1}{k-1}\right)_g} \quad \text{and} \quad \frac{\prod_{i=1}^{k-1} \left(\frac{n-1}{2}+\frac{i-k}{2}\right)_g}{\prod_{i=1}^{k-1} \left(\frac{n-1}{2}+\frac{i-1}{k-1}\right)_g}$$

respectively.

For any given value of i ($i = 1, 2, \cdots, k-1$) the ratio

$$\frac{\left(\frac{n}{2}+\frac{i-k-1}{2}\right)_g}{\left(\frac{n}{2}+\frac{i-1}{k-1}\right)_g} \quad \text{or} \quad \frac{\left(\frac{n-1}{2}+\frac{i-k}{2}\right)_g}{\left(\frac{n-1}{2}+\frac{i-k}{k-1}\right)_g}$$

may be expressed in the form

$$\frac{\Gamma(p_i+g)}{\Gamma(p_i+q_i+g)}$$

which is the g-th moment of a beta variable u_i distributed according to

$$\frac{\Gamma(p_i+q_i)}{\Gamma(p_i)\Gamma(q_i)}\, u_i^{p_i-1}(1-u_i)^{q_i-1}\,du_i\,.$$

Each of the moments in (5) is therefore of the form

$$\prod_{i=1}^{k-1} \frac{\Gamma(p_i + g)}{\Gamma(p_i + q_i + g)}.$$

Thus, L_{mvc} and L_{vc} are each distributed like the product of $k - 1$ independent beta variables.

Each of the moments in (5) can be expressed in the general form

$$M_g = \frac{\prod_{i=1}^{r'} \left(\frac{1}{x} - A_i + 1\right)_g}{\prod_{i=1}^{r'} \left(\frac{1}{x} - B_i + 1\right)_g} \tag{6}$$

where $x = \frac{2}{n}\left(\text{or } \frac{2}{n-1}\right)$, A_i and B_i are real numbers.

Other likelihood ratio statistical test criteria which have been discussed in the literature have moments which can be expressed in the general form (6). For example, the likelihood ratio criterion L_1 for testing the homogeneity of sample variances [2, Neyman and Pearson 1931] has moments of this type. The generalized L_1 criterion for samples from a normal multivariate population [3, Wilks 1933] has such moments. The criterion for testing sphericity [4, Mauchly 1940] of a normal multivariate distribution has moments of this kind. All test criteria having this type of moment lie on the interval (0, 1). The exact distribution functions of the criteria, except possibly for $r = 1$ or 2 in some cases, are very complicated.

The purpose of this note is to consider a method of finding a fractional power of the test criterion which is approximately distributed (in a sense to be described later) according to an incomplete beta (Pearson Type I) distribution function,

$$dF(u) = \frac{\Gamma(p + q)}{\Gamma(p)\Gamma(q)} u^{p-1} (1 - u)^{q-1} du \tag{7}$$

and to find the appropriate values of p, q, and the exponent of the criterion.

2. Generalized hypergeometric series as moment generating functions. Suppose L is a statistical test criterion, or more generally a random variable having as its g-th moment the expression (6). The moment generating function $\varphi(t)$ of L can be expressed as

$$\varphi(t) = \sum_{g=0}^{\infty} M_g t^g = \frac{\prod_{i=1}^{r'} \left(\frac{1}{x} - A_i + 1\right)_g}{\prod_{i=1}^{r'} \left(\frac{1}{x} - B_i + 1\right)_g} \cdot t^g. \tag{8}$$

This can be written as

$$\varphi(t) = {}_{r'+1}F_{r'} \left[\begin{matrix} 1, \frac{1}{x} - A_1, \cdots, \frac{1}{x} - A_{r'}; t \\ \frac{1}{x} - B_1, \cdots, \frac{1}{x} - B_{r'} \end{matrix}\right] \tag{9}$$

where ${}_{r'+1}F_{r'}[\ ]$ is a generalized hypergeometric series [5, Bailey 1935]. We shall not make explicit use of this fact; instead, we shall work with the coefficient of t^g in the series, i.e., M_g.

Let us consider

$$\ln M_g = \sum_{i=1}^{r'} \ln\left(\frac{1}{x} - A_i + 1\right)_g - \sum_{i=1}^{r'} \ln\left(\frac{1}{x} - B_i + 1\right)_g. \tag{10}$$

To expand this in a power series in x consider a single term

$$\begin{aligned} \ln\left(\frac{1}{x} - A + 1\right)_g &= \sum_{j=1}^{g} \ln\left(\frac{1}{x} - A + j\right) \\ &= -g \ln x + g \ln (1 - Ax) + \sum_{j=1}^{g} \ln\left(1 + \frac{jx}{1 - Ax}\right). \end{aligned} \tag{11}$$

Now

$$1 + \frac{jx}{1 - Ax} = 1 + jx + jAx^2 + jA^2x^3 + \cdots .$$

Writing

$$S_m(g) = \sum_{j=1}^{g} j^m,$$

and using the usual expansion for $\ln(1 + x)$, we find

$$\begin{aligned} \ln\left(\frac{1}{x} - A + 1\right) = -g \ln x &+ [S_1(g) - Ag]x + [\tfrac{1}{2}A^2 + AS_1(g) - \tfrac{1}{2}S_2(g)]x^2 \\ &+ [-\tfrac{1}{3}A^3 + A^2S_1(g) - AS_2(g) + \tfrac{1}{3}S_3(g)]x^3 + \cdots. \end{aligned}$$

Applying this expansion to the separate terms in (10) and writing

$$C_m = \sum_{i=1}^{r'} A_i^m - \sum_{i=1}^{r'} B_i^m \tag{12}$$

the terms not involving A_i or B_i cancel out leaving

$$\begin{aligned} \ln M_g = (-C_1g)x &+ [\tfrac{1}{2}C_2 + C_1S_1(g)]x^2 \\ &+ [-\tfrac{1}{3}C_3 + C_2S_1(g) - C_1S_2(g)]x^3 + \cdots . \end{aligned} \tag{13}$$

We shall return to this expression later.

3. Powers of a beta variable. If u has (7) as its distribution function, then

$$E(u^h) = \frac{(p)_h}{(p + q)_h}. \tag{14}$$

If $v = u^r$, r integral, then its g-th moment is given by setting $h = rg$ in (14).

We have

$$E(v^g) = \frac{(p)_{rg}}{(p+q)_{rg}}.$$

But

$$(p)_{rg} = r^{rg}\left(\frac{p}{r}\right)_g \cdot \left(\frac{p+1}{r}\right)_g \cdots \left(\frac{p+r-1}{r}\right)_g$$

so that

$$E(v^g) = \frac{\prod_{i=1}^{r}\left(\frac{p+i-1}{r}\right)_g}{\prod_{i=1}^{r}\left(\frac{p+q+i-1}{r}\right)_g} \tag{15}$$

which is a special case of (6) when p is of order n.

Putting $\frac{1}{x} = \frac{p+q}{r}$, $A_i = 1 + (q - i + 1)/r$, and $B_i = 1 - (i-1)/r$ we have

$$C_1 = q,$$

$$C_2 = \frac{q^2}{r} + q\left(1 + \frac{1}{r}\right).$$

For any given moment of the form (6), from which x, C_1, and C_2 can be computed, we determine p, q, and r so as to satisfy

$$\frac{p+q}{r} = \frac{1}{x} \tag{16}$$

$$q = C_1$$

and to satisfy, as nearly as possible, (with r integral)

$$\frac{q^2}{r} + q\left(1 + \frac{1}{r}\right) = C_2, \tag{17}$$

i.e.,

$$r = \frac{q(q+1)}{C_2 - q}.$$

The use of fractional r is obviously suggested, but its value and validity are not discussed here. Using the values of p, q and r thus obtained, the distribution of the criterion L (having moments (7)), is given approximately by

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\,(\sqrt[r]{L})^{p-1}(1 - \sqrt[r]{L})^{q-1}\,d(\sqrt[r]{L}) \tag{18}$$

where the approximation is such that all moments are correct through terms of order $\left(\frac{r}{p+q}\right)$ (when moments are expanded in series of $\frac{r}{p+q}$) and nearly (exactly if there is an integral value of r satisfying (17)) correct through terms of order $\left(\frac{r}{p+q}\right)^2$.

4. Examples. Returning to the g-th moment of L_{mvc} given by the first expression in (5) we have

$$x = \frac{2}{n}, \qquad r' = k - 1$$

$$A_i = \frac{k+3-i}{2}, \qquad B_i = \frac{k-i}{k-1},$$

$$C_1 = \sum_{i=1}^{k-1} A_i - \sum_{i=1}^{k-1} B_i = \tfrac{1}{4}(k^2 + 3k - 6)$$

$$C_2 = \sum_{i=1}^{k-1} A_i^2 - \sum_{i=1}^{k-1} B_i^2 = \frac{1}{24}[(k+2)(k+3)(2k+5) - 84] - \frac{k(2k-1)}{6(k-1)}.$$

To determine p, q and r for the fitted distribution of L_{mvc} we set

$$\frac{p+q}{r} = \frac{n}{2}$$

$$q = \tfrac{1}{4}(k^2 + 3k - 6)$$

$$r = \frac{q(q+1)}{C_2 - q}$$

and solve for p, q and r. We have the following table of values, p, q and r for various values of k (p being calculated by using the rounded values of r):

k	r	r (rounded)	p	q
3	2	2	$n - 3$	3
4	2.93	3	$1.5n - 5.5$	5.5
5	3.82	4	$2n - 8.5$	8.5
6	4.68	5	$2.5n - 12$	12
7	5 53	6	$3n - 16$	16
8	6.35	6	$3n - 20.5$	20.5
9	7.17	7	$3.5n - 25.5$	25.5
10	7.96	8	$4n - 31$	31
20	15.76	16	$8n - 113.5$	113.5

Thus, by rounding r off to the nearest integer and using this rounded value of r in determining p, we have values of p, q and r for each value of k, which, when substituted in (18) give us the desired fitted beta distribution for L_{mvc}. For $k = 3$, the fitted distribution is the exact distribution.

For the g-th moment of L_{vc} which is given by the second expression in (5), it is convenient to expand in powers of $\frac{2}{n-1}$. Hence we have

$$x = \frac{2}{n-1}, \qquad r' = k - 1$$

$$A_i = \frac{k+2-i}{2}, \qquad B_i = \frac{k-i}{k-1}$$

$$C_1 = \tfrac{1}{4}(k^2 + k - 4)$$

$$C_2 = \tfrac{1}{24}[(k+1)(k+2)(2k+3) - 30] - \frac{k(2k-1)}{6(k-1)}.$$

To determine p, q and r for fitting the distribution function of L_{vc} we put

$$\frac{p+q}{r} = \frac{n-1}{2}$$

$$q = \tfrac{1}{4}(k^2 + k - 4)$$

$$r = \frac{q(q+1)}{C_2 - q}.$$

We have the following table of values of p, q and r for several values of k:

k	r	r (rounded)	p	q
3	2	2	$n - 3$	2
4	2 88	3	$1.5n - 5.5$	4
5	3.71	4	$2n - 8.5$	6.5
6	4.52	5	$2.5n - 12$	9.5
7	5.32	5	$2.5n - 15.5$	13
8	6.14	6	$3n - 20$	17
9	6.88	7	$3.5n - 25$	21.5
10	7.82	8	$4n - 30.5$	26.5
20	15.26	15	$7.5n - 111.5$	104

By rounding r off to the nearest integer, and using the rounded value of r in determining p, we have values of p, q and r for each value of k which, when substituted in (18), give us the desired fitted beta distribution for L_{vc}. For $k = 3$, the fitted distribution is the exact distribution

For a given value of k, approximate 5% and 1% points of $\sqrt[r]{L_{mvc}}$ and $\sqrt[r]{L_{vc}}$ can therefore be obtained from Thompson's [6] tables of the Incomplete Beta Function by entering the tables with $\nu_1 = 2q$, and $\nu_2 = 2p$. For example, for $k = 6$ the 5% and 1% points of $\sqrt[5]{L_{mvc}}$ are obtained by entering Thompson's tables with $\nu_1 = 24$, and $\nu_2 = 5n - 24$.

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SOME FUNDAMENTAL CURVES FOR THE SOLUTION OF SAMPLING PROBLEMS

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1. Summary. In using collateral information in an inverse probability situation to estimate a population fraction from a sample fraction it is necessary to use some particular form for the *a priori* probability function. This paper points out the advantages of using $Kx^r(1 - x)^s$ for this purpose. The application then involves only the Incomplete Beta Function.

Graphs of the 10, 25, 50, 75 and 90 per cent points of the Incomplete Beta Function are given. They cover a range which includes and extends previous tabulations.

2. Introduction. The engineer, scientist or industrialist is often confronted with the following "sampling" problem:

> "The probability, p, of an event happening in a single trial is constant from trial to trial, but the numerical value of this constant is unknown. A series of n trials is made and the event happens c times, $c \leq n$. What light does this statistical data shed on the unknown value of p?"

As a concrete example, suppose that a new type of brakes is proposed for a given class of steam locomotives making the run from Buffalo to Detroit[1] Let each of 30 locomotives be equipped with a set of the new brakes and given a trial run Of these, 26 make satisfactory runs, so far as the behavior of the brakes is concerned; the remaining four encounter difficulties. Here, the event of interest is a satisfactory run, $n = 30$ and $c = 26$. What "weight" (confidence[2]) may the design engineer assign to the assumption that, say, $25/30 \leq p \leq 27/30$?

Practical decisions involving such statistical data are usually based on a combination of the data with "collateral" information. In fact, the applied statistician is all too familiar with the extreme case where the statistical data are so meagre as to provide no information and where a decision must be made *now*—in these cases the decision is made solely on the basis of the collateral information, and rightly so

The methods of statistical analysis and presentation developed up to the present have concentrated on the other extreme case, where the statistical data are so good that collateral information can be neglected.

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[1] This fictitious example convicts the writer of total ignorance of railroad engineering Nevertheless, the illustration brings out, in concrete terms, the class of sampling problems under consideration

[2] The purely intuitive meaning to be attached to "weight" and "confidence" is the same However, the curves presented with this paper are not based on the theory which underlies what are known, in statistical literature, as "confidence intervals".

There is a real need for methods of analysis and presentation to be used where both the statistical data and the collateral information should be used. However, when the significance of the collateral information is adequately expressed by a function $w(x)$, x being a permissible value of the unknown p, the classic Bayes-Laplace theory (see [1]) of inverse probability gives the solution to a sampling problem.

The purpose of this paper is to present a set of sampling curves based on a $w(x)$ function whose form embodies some important properties.[3]

3. Hardy's collateral frequency function. Consider again the locomotive brakes problem. The new design may have been carefully engineered, in accordance with well-known principles, to reduce costs at the expense of a slight reduction in reliability of operation. In such a situation, the collateral information would be somewhat as follows: There is a high "probability" that the unknown value of p is a little below the known value for the old type of brakes. Moreover, it may be assumed that the "probability" drops rapidly for values of p departing materially from this old value. Suppose the latter is $p = .95$; then the collateral information would be presented by some such curve as number 5 in Figure 1, the mode (peak) of this curve being at .90, which is slightly below the old .95 value.

Number 5, of Figure 1, belongs to the family of curves corresponding to the frequency function

$$w(x) = Kx^r(1 - x)^s$$

This form for $w(x)$ was suggested, in 1889, by the British actuary Sir George F. Hardy (see [2]) for the construction of mortality tables. Its mode, mean and variance are given by the equations

$$\begin{aligned}
\text{Mode} &= r/(r + s) \\
\text{Mean} &= (r + 1)/(r + s + 2) \\
\text{Variance} &= (r + 1)(s + 1)/(r + s + 2)^2(r + s + 3)
\end{aligned}$$

G. J. Lidstone (see [3]) has pointed out that the Hardy form for $w(x)$ has two important advantages: First—"By suitable choice of r and s any required values of the mode or mean and the variance of z_x can be reproduced, and thus a great variety of distributions may be approximately represented." Lidstone's z_x is our $w(x)$. Second—"The factors x^r and $(1 - x)^s$ unite in the simplest and most elegant way with similar factors in the Laplacian integrand . . . ".

[3] Many statisticians, including a referee of this paper, feel that it is a common situation to have the collateral information so vague and elusive that it is virtually impossible to take it into account via inverse probability. (The author doubts this.) Such statisticians may wish to use the Clopper-Pearson confidence intervals, using no collateral information, in which case these curves can be used as indicated by Scheffé ("Note on the use of the tables of percentage points of the incomplete beta function to calculate small sample confidence intervals for a binomial p", *Biometrika*, August, 1944)

From this second advantage there follows a third which will be presented in section 6 below.

4. Theory. The Bayes-Laplacian formula gives us

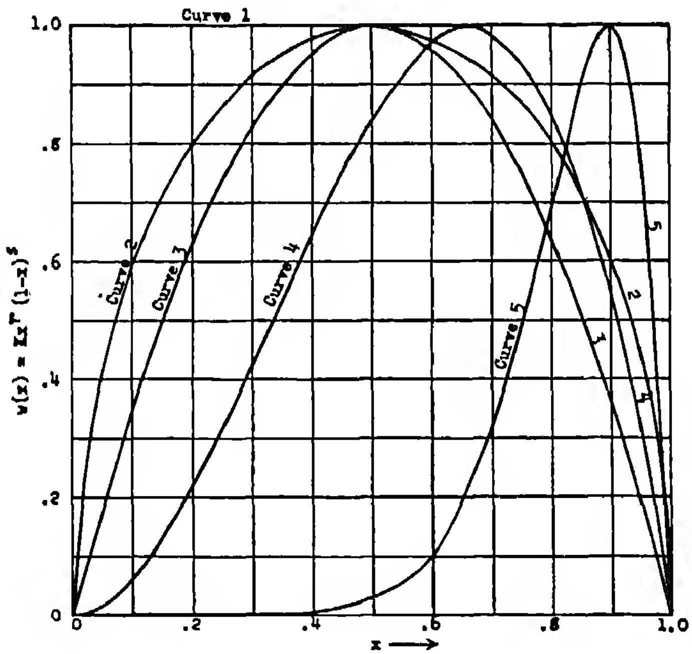
$$(1) \qquad P(p \leqq X) = \int_0^X w(x)x^c(1-x)^{n-c}\,dx \Big/ \int_0^1 w(x)x^c(1-x)^{n-c}\,dx$$

for the "a posteriori probability" that $p \leq X$. In this formula, the product

FIG. 1

Particular forms of the a priori (collateral information) function:

$$w(x) = K\,x^r\,(1-x)^s$$



Curve	r	s	Form
1	0	0	K
2	$\frac{1}{2}$	$\frac{1}{2}$	$Kx^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$
3	1	1	$Kx(1-x)$
4	2	1	$Kx^2(1-x)$
5	9	1	$Kx^9(1-x)$

$x^c(1-x)^{n-c}$ takes care of the fact that the event happened c times in the n trials; the factor $w(x)$ represents, quantitatively, the collateral information.

Adopting, now, Hardy's frequency function, we assume that

$$(2) \qquad w(x) = Kx^r(1-x)^s,$$

r and s being assigned values in accordance with the collateral information pertaining to the particular problem under consideration. Theoretically, the constant K should be such that

$$\int_0^1 w(x)\,dx = 1,$$

but, since $w(x)$ enters in both numerator and denominator of (1), any desirable value may be given to K. Advantage has been taken of this in constructing Figure 1; to facilitate comparison of the five curves shown therein, for each curve K is such that the maximum ordinate is equal to 1.

The second advantage, pointed out by Lidstone, of the form adopted in this paper for the function $w(x)$ becomes apparent immediately on substitution of (2) in (1). We obtain

$$(3) \qquad P(p \leq X) = \int_0^X x^C(1-x)^{N-C}\,dx \Big/ \int_0^1 x^C(1-x)^{N-C}\,dx$$

with $C = c + r$ and $N = n + r + s$. Therefore, a *single family of fundamental curves*, plotted with reference to C and N, will give the solutions for a multitude of different practical problems. To solve a particular problem, for which the values of n, c, r and s are specified, we merely enter the curves with $C = c + r$ and $N = n + r + s$. These linear relations transform all a posteriori curves, published on the assumption that $w(x)$ is a constant, into fundamental curves; namely, that they are applicable with the more general form (2). For example: The information given on the sheets of inverse curves (inserted in the back cover pocket) of Col. Leslie E. Simon's *Engineer's Manual of Statistical Methods* includes the restriction "that prior to sampling, one lot fraction defective is as likely as another". It is now obvious that the use of Col. Simon's curves is not so limited; his curves may be used in any situation wherein the available collateral information is covered by the assumption that $w(x)$ has the Hardy form. Likewise, the "Weight = .98" and "Weight = .8" curves ("confidence", in the intuitive sense), presented by R. P Crowell and the writer in their paper now have a much wider range of applicability.

5. Curves. The ratio of definite integrals in equation (3) is tabulated, in a different notation, in "Tables of the Incomplete Beta Functions", edited by Karl Pearson.

This paper	Pearson Tables	Thompson Tables (see [5])
C	$p - 1$	$(v_2 - 2)/2$
$N - C$	$q - 1$	$(v_1 - 2)/2$
X	x	tabulated value
$P(p \leq X)$	tabulated value	caption to Table

The range of values of C and $(N - C)$ covered by the Pearson Tables is indicated by the shaded area in Figure 7. For curve points falling outside this

range (except for $C = 1$ and 2, found from the binomial summation by trial and error) recourse was had to a series developed by the writer for the solution of some problems confronting him, as Switching Theory Engineer, in the Bell Telephone Laboratories Many points of the $C = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12$

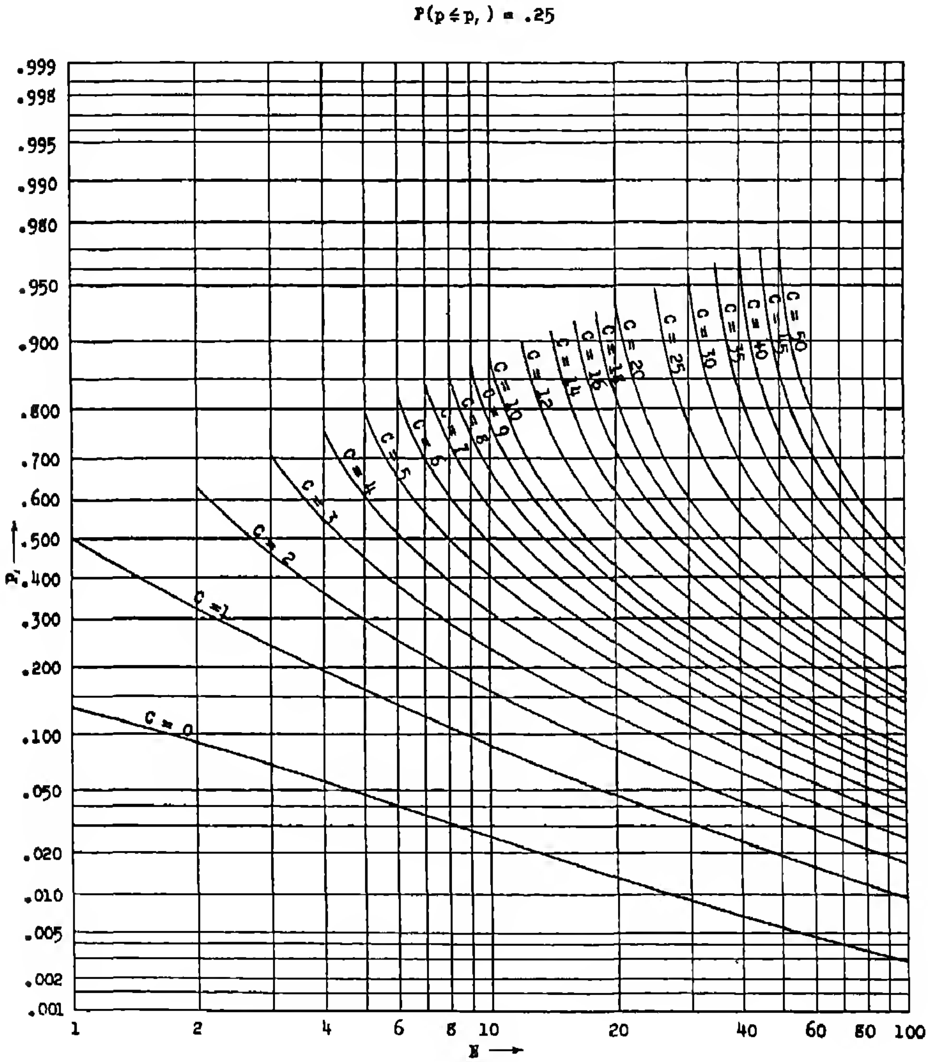


FIG. 2

and 14 curves can be obtained directly from the Thompson Tables. They do not, however, give any points for the $C = 16, 18, 20, 25, 30, 40, 45$ and 50 curves. It may be added that, except for certain marginal values, the Thompson Tables were also derived from the Pearson Tables

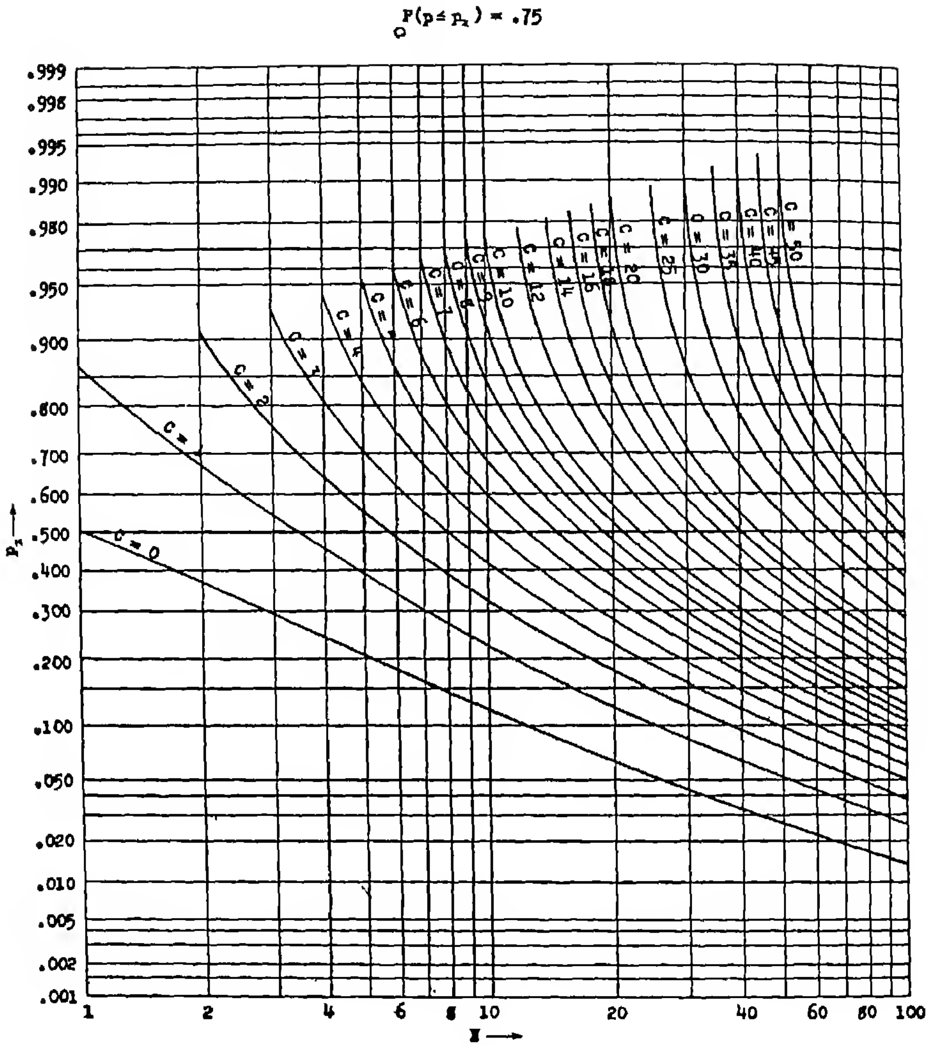


Fig. 3

Five sets of fundamental curves are submitted, namely,

Figure 2,	$P(p \leq X)$	$= .25,$	$X = p_1$
" 3,	"	$= .75,$	$X = p_2$
" 4,	"	$= .10,$	$X = p_1$
" 5,	"	$= .90,$	$X = p_2$
" 6,	"	$= .50,$	$X = p_0$

It will be noted that p_1 has been written instead of X for the curves such that

$P\ (p \leq X)$ is *less* than .50; likewise, p_2 for X for those corresponding to $P\ (p \leq X)$ *greater* than 50; p_0 for X for the $P\ (p \leq X) =$ 50 curves.

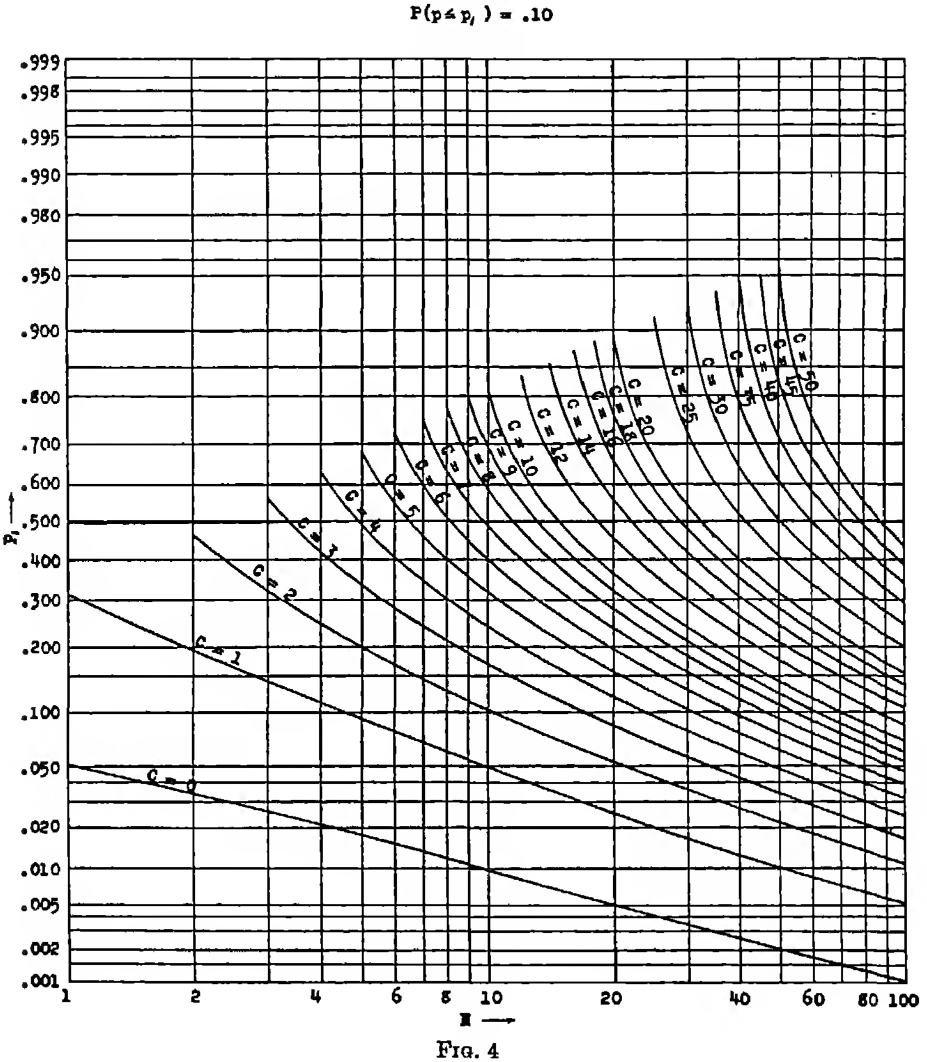


FIG. 4

For each pair of values of C and N, the curves of Figures 2 and 3 give the range

$$P(p_1 \leq p \leq p_2) = .50$$

whereas, the curves of Figures 4 and 5 give the range

$$P(p_1 \leq p \leq p_2) = .80$$

As an example of the applicability of the fundamental curves, let us reconsider the locomotive problem for which $n = 30$ and $c = 26$ It was suggested that

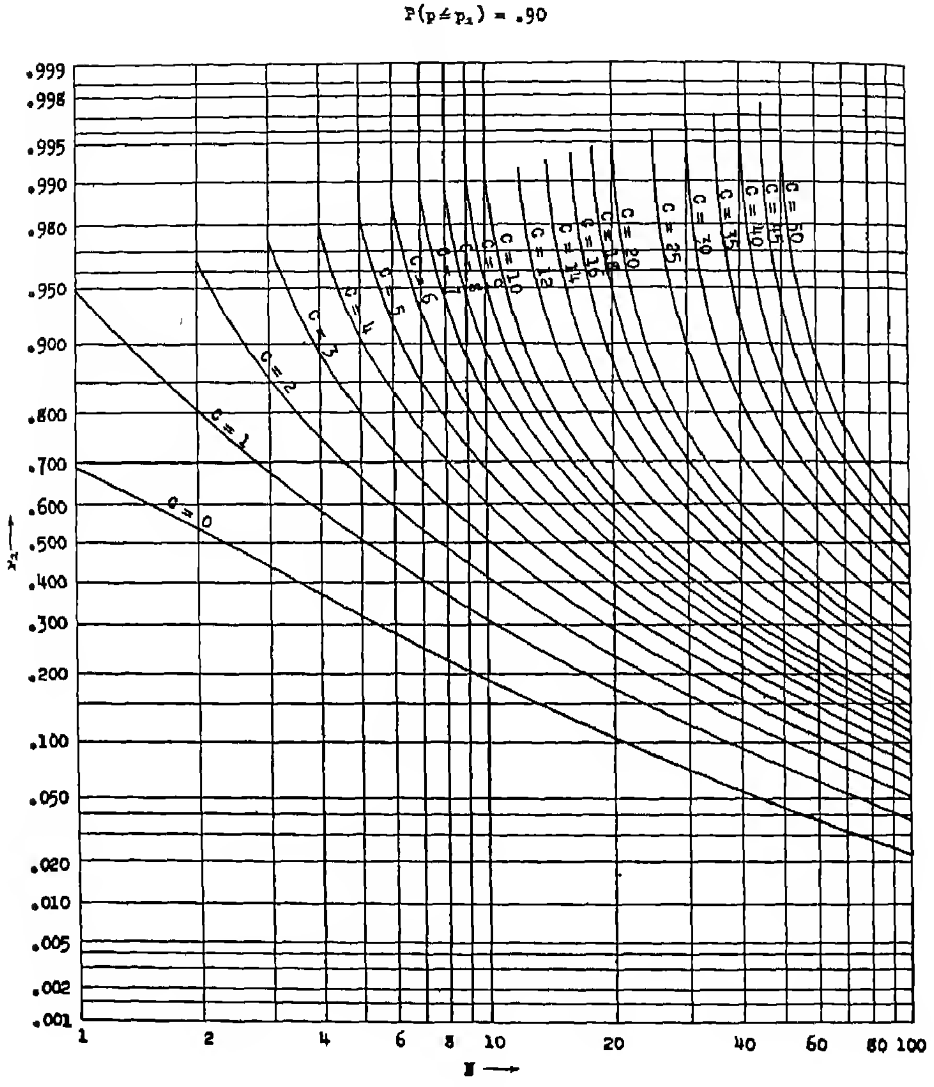


FIG. 5

the $r = 9$, $s = 1$ curve of Figure 1 might well represent the collateral information available. Therefore we take $N = 30 + 9 + 1 = 40$ and $C = 26 + 9 = 35$. Entering Figures 2, 3, 4 and 5 with this data we find

Fig.	$P(p \leq p_1)$	p_1		Fig.	$P(p \leq p_2)$	p_2
2	.25	.83		3	.75	.89
4	.10	.79		5	.90	.92

$P(p \leq p_0) = .50$

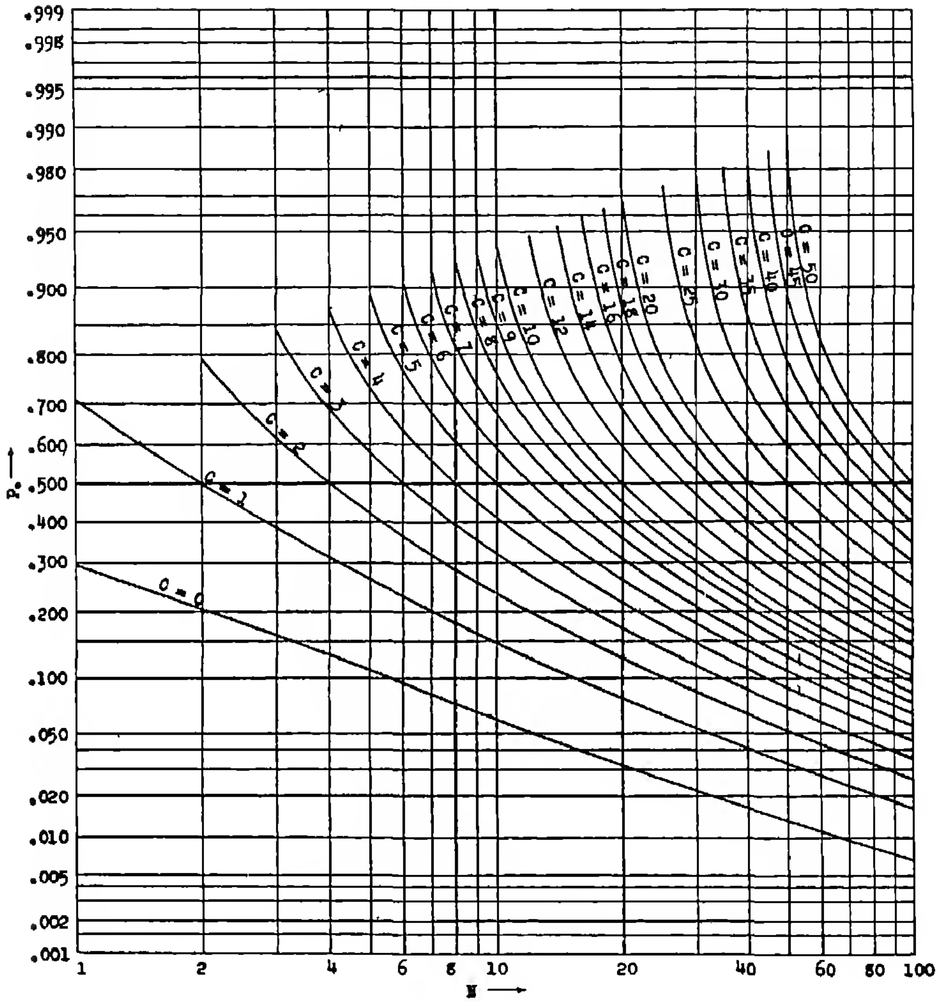


FIG. 6

Thus we have, for the unknown probability of a successful run with a new set of brakes,

$$.83 < p \leq .89, \quad \text{with weight } .50$$

and

$$.79 < p \leq .92, \quad \text{with weight } .80$$

6. Sequential property of the curves. The original draft of this paper was submitted to Dr. W. V. Houston[4] in connection with the solution of a problem

[4] Of the California Institute of Technology and now President of Rice Institute, Houston, Texas. It was Dr. Houston who gave the impetus to the publication of this paper.

"Tables of The Incomplete Beta-Function," edited by Karl Pearson, can be used for evaluation of

$$P = \frac{\int_0^P x^C(1-x)^{N-C}\,dx}{\int_0^1 x^C(1-x)^{N-C}\,dx}$$

only when values of $(N - C)$ and C are in .

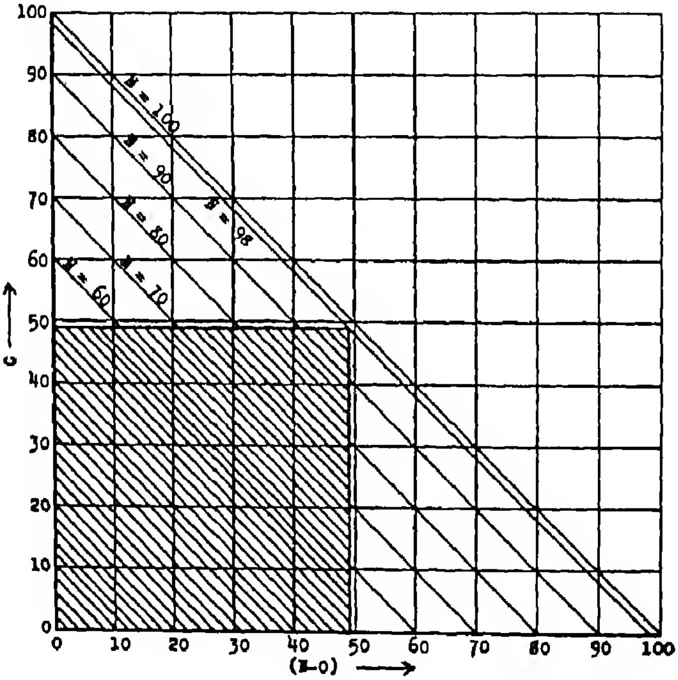


Fig. 7

in which he was interested. Regarding equation (3), Dr. Houston made a very significant comment, the burden of which may be stated as follows: Suppose that before the series of n trials had been made, it was known that, at some *earlier time*, a series of $r + s$ trials had resulted in r successful outcomes. Suppose, moreover, that the collateral information called for the assumption that, a priori, all values of p were equally likely. Under these circumstances equation (3), derived by substitution of (2) in (1), gives $P(p \leq X)$ for *two consecutive series of trials*, one of $r + s$ with r successes followed by another of n with c successes. An immediate generalization of Dr. Houston's thought shows that the fundamental curves may be entered with

$$N = n_1 + n_2 + \cdots + n_i + \cdots + n_m + r + s,$$

$$C = c_1 + c_2 + \cdots + c_i + \cdots + c_m + r,$$

for the solution of a problem involving *m consecutive series of trials*, n_i and c_i being the number of trials and successes, respectively, in the ith series; the introduction of r and s removing the restriction that all values of p were a priori equally likely.

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ENLARGEMENT METHODS FOR COMPUTING THE INVERSE MATRIX

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1. Summary. The enlargement principle provides techniques for inverting any nonsingular matrix by building the inverse upon the inverses of successively larger submatrices. The computing routines are relatively easily learned since they are repetitive. Three different enlargement routines are outlined: first-order, second-order, and geometric. None of the procedures requires much more work than is involved in squaring the matrix.

2. Introduction. A set of methods is presented here for computing the inverse matrix, based on what we shall call an *enlargement principle*. The principle is to build the inverse upon the inverses of successively larger submatrices. This leads to simple repetitive routines that are not unlike iterative steps, but afford a direct solution.

The basis for such routines has also been noticed before,[1] but does not seem to have attracted the attention it merits. A possible reason for this lack of attention may be the belief that the methods apply only to a restricted class of matrices. We establish a simple lemma in this paper which shows that the enlargement methods apply to *all* nonsingular matrices, so that their use is perfectly general.

The enlargement principle may be considered an opposite of the "condensation" principle that governs Gauss' method of elimination and its variants such as the Doolittle procedure and Aitken's "pivotal condensation."[2] It is interesting that the same formula upon which the enlargement methods are based can also serve as a foundation for the condensation methods, as is shown in section 7 below.

The enlargement methods have the following characteristics:

(1) The first-order procedure outlined in the next section has been learned by statistical clerks in about ten minutes. People who calculate inverses only occasionally and forget the process between times should find the method as economical as those who must constantly compute inverses.

(2) They are direct methods, and yield an exact answer with not much more work than is involved in squaring the matrix.

(3) They can be adapted to electric punch-card systems, which will be efficient when very large matrices are to be inverted.

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[1] It has appeared earlier in [2]. Waugh's recent note [10] also rediscovers the basic formula although only a specialized use is suggested there. Professor Harold Hotelling has called my attention to reference [1], which overlaps substantially with the present paper, and to a use of an enlargement approach to computing latent roots and vectors [9]. I am also indebted to Professor Hotelling for other helpful comments on the present paper.

[2] For an excellent summary and bibliography of direct and iterative methods for computing the inverse matrix see ([5], [6]).

(4) A sequence of inverses is yielded. Exact inverses of successively larger submatrices are computed in the routines, and these inverses are often themselves of interest. For correlation problems, this means that a sequence of sets of successively higher order multiple correlation constants is produced routinely.

(5) The general formula upon which the methods are based allows many variations in procedure, so that special adaptations can be easily made for special matrices

A "first-order" enlargement procedure for computing the inverse matrix will be outlined in the next section. The proof for the method follows from the general formula in section 4. This procedure and formula are also described in [2]. Other enlargement routines are described in subsequent sections. Some additional formulas of relevance are discussed in section 8.

3. First-order enlargement. Let the matrix whose inverse is desired be

$$A_n = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{Vmatrix}.$$

The following sequence of successively larger principal submatrices will be assumed to be nonsingular:

$$A_2 = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}, \quad A_3 = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}, \cdots, A_n.$$

If necessary, the rows and columns of A_n can always be shifted to obtain such a sequence. The following additional notation will be used:

$$B_i = (a_{1,i+1} a_{2,i+1} \cdots a_{i,i+1})$$

$$C_i = (a_{i+1,1}\, a_{i+1,2} \cdots a_{i+1,i})$$

$$d_i = a_{i+1,i+1}.$$

Thus, we can write

$$A_{i+1} = \begin{Vmatrix} A_i & B_i' \\ C_i & d_i \end{Vmatrix}, \quad (i = 2, 3, \cdots, n-1).$$

The first-order enlargement procedure is to compute in turn $A_2^{-1}, A_3^{-1}, \cdots, A_n^{-1}$. The inverse of A_2 is computed by the traditional steps:

{1} Compute $\Delta = a_{11}a_{22} - a_{21}a_{12}$, and compute $1/\Delta$

{2} Then

$$A_2^{-1} = \begin{Vmatrix} \Delta^{-1} a_{22} & -\Delta^{-1} a_{12} \\ -\Delta^{-1} a_{21} & \Delta^{-1} a_{11} \end{Vmatrix}.$$

Remember that $B_2 = (a_{13}\,a_{23})$, $C_2 = (a_{31}\,a_{32})$, and that $d_2 = a_{33}$ The steps for computing A_3^{-1} are as follows:

{3} Compute $E_2' = A_2^{-1}B_2'$.

{4} Compute $f_2 = d_2 - C_2E_2'$.

{5} Compute $1/f_2$.

{6} Compute $G_2' = f_2^{-1}E_2'$, and compute $H_2 = f_2^{-1}C_2A_2^{-1}$.

{7} To each element in A_2^{-1} add the product of the corresponding elements in E_2 and H_2 to form $K_2 = A_2^{-1} + E_2'H_2$.

Then the third order inverse is

$$A_3^{-1} = \begin{Vmatrix} K_2 & -G_2' \\ -H_2 & 1/f_2 \end{Vmatrix}.$$

In general, to obtain A_{i+1}^{-1} from A_i^{-1}, $(i = 2, 3, \cdots, n - 1)$, imitate[3] steps {3} through {7}:

{3′} Compute $E_i' = A_i^{-1}B_i'$.

{4′} Compute $f_i = d_i - C_iE_i'$,

{5′} Compute f_i^{-1}

{6′} Compute $G_i' = f_i^{-1}E_i'$, and compute $H_i = f_i^{-1}C_iA_i^{-1}$.

{7′} Compute $K_i = A_i^{-1} + E_i'H_i$. Then

$$A_{i+1}^{-1} = \begin{Vmatrix} K_i & -G_i' \\ -H_i & 1/f_i \end{Vmatrix}.$$

By repeated applications of steps {3′} through {7′} to the successively larger A_i^{-1}, A_n^{-1} is attained.

If A_n is symmetric, then almost half the work is saved, for then $B_i = C_i$, $G_i = H_i$, and K_i is symmetric, $(i = 2, 3, \cdots, n - 1)$.

To help gauge the amount of work needed to arrive at A_n^{-1}, let us compare it with the work that would be needed to square A_n. For the general asymmetric case, n^2 product sums of n terms each are required for A_n^2, a total of n^3 multiplications. With calculating machines, the sums of the products are accumulated, so that no separate process of addition is involved. To reach A_n^{-1} by the above enlargement method, $n^3 - n$ multiplications are required. Most of the addition is accomplished in the process by accumulative multiplication, but an additional $\dfrac{n(n-1)(2n-1)}{6} + n - 3$ terms have to be added otherwise. Furthermore, $n - 1$ reciprocal numbers are needed. Thus, A_n^{-1} involves somewhat less multiplications than does A_n^2, but needs more additions, as well as some reciprocal numbers.

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[3] Actually, these steps could be used immediately in place of steps {1} and {2} to compute A_2^{-1}, by letting $i = 1$, and letting $A_1 = a_{11}$ (which may be assumed different from zero). The traditional method, however, is quicker for the 2x2 matrix.

In linear multiple correlation problems, if A_{i+1} is the correlation matrix of the first $i + 1$ variates, then E_i consists of the regression coefficients of the first i variates for predicting the $(i + 1)$th variate, and f_i is the square of the multiple correlation coefficient for this regression.

4. A lemma and the general formula. The enlargement procedure just outlined is one of many possible routines which can be developed from a general formula for the inverse matrix in partitioned form. This formula seems to have appeared first in [2], where it is stated that the method applies only to the cases where $f_i \neq 0$ in step {4} We shall establish here a lemma that shows that this is no restriction, for the submatrix in step {4} is always nonsingular Our lemma proves that the enlargement methods will invert *any* nonsingular matrix.

Let A_n be a nonsingular matrix of order n, partitioned in the form

$$A_n = \left\| \begin{matrix} A & B' \\ C & D \end{matrix} \right\|. \tag{1}$$

where A is of order m, $(1 \leq m < n)$, and will be assumed nonsingular B and C are of $n - m$ rows and m columns, and D is of order $n - m$.

The following lemma is needed to show that enlargement methods will invert any nonsingular matrix:

LEMMA *If in (1), both A_n and A are nonsingular, then the matrix*

$$F = D - CA^{-1}B' \tag{2}$$

is nonsingular.

For the proof, postmultiply the first submatric column of A_n by $A^{-1}B'$ and subtract from the second, leaving

$$M = \left\| \begin{matrix} A & 0 \\ C & F \end{matrix} \right\|.$$

M differs from A_n only by an elementary transformation; hence its rank is that of A_n But clearly the rank of M is the sum of the ranks of A and F. Therefore, the rank of F is $n - m$, and F is nonsingular.

The inversion formula itself is the following identity:

$$\left\| \begin{matrix} A & B' \\ C & D \end{matrix} \right\|^{-1} = \left\| \begin{matrix} A^{-1} + A^{-1}B'F^{-1}CA^{-1} & -A^{-1}B'F^{-1} \\ -F^{-1}CA^{-1} & F^{-1} \end{matrix} \right\|. \tag{3}$$

A direct verification that the identity holds can be obtained by multiplying the right member in either direction by the right member of (1), yielding the unit matrix.

In section 3, the formula exhibited for A_{i+1}^{-1} at step {7′} is easily identified as a special case of formula (3) where $n = i + 1$, $m = i$. F corresponds to f_i, which is a scalar number; hence F^{-1} is easily computed in this case.

5. Second order enlargement. In formula (3), once A^{-1} is given, the rest of the work is essentially straightforward matric multiplication, except for computing F^{-1}. In section 3, F was easily inverted since it was of order unity. F can also be easily inverted if it is of order two, so that a *second order* enlargement procedure is feasible, computing A_{i+2}^{-1} from A_i^{-1}. The steps are similar to those in section 3 but involve larger matrices.

Letting A_i have the same meaning as in section 3, define now B_i, C_i, and D_i according to the partitioning

$$A_{i+2} = \begin{Vmatrix} A_i & B_i' \\ C_i & D_i \end{Vmatrix}.$$

Then B_i and C_i are of two rows and i columns, and D_i is of order two. Compute A_2^{-1} as in section 3. From then on, to compute A_{i+2}^{-1} from A_i, the steps are:

{3″} Compute $E_i' = A_i^{-1}B_i$.

{4″} Compute $F_i = D_i - C_i E_i'$.

{5″} Compute F_i^{-1} by steps [1] and [2] of section 3.

{6″} Compute $G_i' = F_i^{-1}E_i'$, and compute $H_i = F_i^{-1}C_iA_i^{-1}$.

{7″} Compute $K_i = A_i^{-1} + E_i'H_i$.

Then

$$A_{i+2}^{-1} = \begin{Vmatrix} K_i & -G_i \\ -H_i & F_i^{-1} \end{Vmatrix}.$$

If n is even, successive enlargements will lead A_n^{-1}. If n is odd, then A_{n-1}^{-1} is attained, from which A_n^{-1} can be computed according to section 3.

The number of multiplications and additions for this procedure is the same as for section 2 However, less writing is involved since only about half as many A_i are inverted. A disadvantage is that it is more complicated at each stage than is the procedure of section 3.

6. Geometric enlargement. Another routine is that which may be called *geometric* enlargement. Here, A_{2i}^{-1} is computed from A_i^{-1}. The steps may be described as follows. Letting A_i have the same meaning as previously, redefine B_i, C_i, and D_i according to the partitioning

$$A_{2i} = \begin{Vmatrix} A_i & B_i' \\ C_i & D_i \end{Vmatrix}.$$

Then B_i, C_i, and D_i are all, like A_i, square matrices of order i. Compute A_2^{-1} according to steps {1} and {2}, and compute A_4^{-1} according to steps {3″} through {7″}. From then on, to compute A_{2i}^{-1} from A_i^{-1}, the steps are formally the same as before, with a complication in step {5‴}:

{3'''} Compute $E'_\iota = A_\iota^{-1}B'_\iota$.
{4'''} Compute $F_\iota = D_\iota - C_\iota E'_\iota$.
{5'''} Compute F_ι^{-1} by *geometric enlargement* in the same way as A_ι^{-1}.
{6'''} Compute $G'_\iota = F_\iota^{-1}E'_\iota$, and compute $H_\iota = F_\iota^{-1}C_\iota A_\iota^{-1}$.
{7'''} Compute $K_\iota = A_\iota^{-1} + E'_\iota H_\iota$.

Then,

$$A_{2\iota}^{-1} = \begin{Vmatrix} K_\iota & -G_\iota \\ -H_\iota & F_\iota^{-1} \end{Vmatrix}.$$

This method involves less writing than the others, but is more complicated.

7. Condensation methods; special cases. Formula (3) also affords a basis for condensation methods by "back solution." For example, let A be of order m, where m is one or two so that A is easily inverted. Then F is of order $n - m$, and we will denote it by F_{n-m}. Partition F_{n-m} into the form

$$F_{n-m} = \begin{Vmatrix} A^{(2)} & B'^{(2)} \\ C^{(2)} & D^{(2)} \end{Vmatrix}$$

where $A^{(2)}$ is again of order m, defining F_{n-2m} of order $n - 2m$. Continue the process until an F_ι is reached which is easily inverted, and solve backwards to reach F_{n-m}^{-1}, and then A_n^{-1}, by repeated use of (3).

Formula (3) is of great help in those special cases where A is large but easily inverted, such as a diagonal matrix, orthogonal matrix, etc. The labor can then be focussed on inverting an F which is much smaller than A_n.

8. Further identities. It is of some interest to exhibit some matric identities relevant to formula (3). Using the notation of section 4, let us seek the inverse of A_n partitioned in the form

(4) $$A_n^{-1} = \begin{Vmatrix} W & X' \\ Y & Z \end{Vmatrix}.$$

An equation to be satisfied is

$$\begin{Vmatrix} W & X' \\ Y & Z \end{Vmatrix} \cdot \begin{Vmatrix} A & B' \\ C & D \end{Vmatrix} = \begin{Vmatrix} I & 0 \\ 0 & I \end{Vmatrix},$$

which yields the equations

(5) $$WA + X'C = I$$

(6) $$WB' + X'D = 0$$

(7) $$YA + ZC = 0$$

(8) $$YB' + ZD = I.$$

If A and D are nonsingular, then from (6) and (7),

$$X' = -WB'D^{-1}, \qquad Y = -ZCA^{-1}. \tag{9}$$

Using (9) in (5) and (8), and remembering the lemma of section 4, we obtain

$$W = (A - B'D^{-1}C)^{-1}, \qquad Z = (D - CA^{-1}B')^{-1}. \tag{10}$$

Using (10) in (9) yields

$$X' = -(A - B'D^{-1}C)^{-1}B'D^{-1}, \qquad Y = -(D - CA^{-1}B')^{-1}CA^{-1}. \tag{11}$$

Putting (10) and (11) into (4) completes the formula

$$\begin{Vmatrix} A & B' \\ C & D \end{Vmatrix}^{-1} = \begin{Vmatrix} (A - B'D^{-1}C)^{-1} & -(A - B'D^{-1}C)^{-1}B'D^{-1} \\ -(D - CA^{-1}B')^{-1}CA^{-1} & (D - CA^{-1}B')^{-1} \end{Vmatrix}. \tag{12}$$

Comparing (3) with (12), we have the identities

$$(A - B'D^{-1}C)^{-1} = A^{-1} + A^{-1}B'(D - CA^{-1}B')^{-1}CA^{-1} \tag{13}$$

$$(A - B'D^{-1}C)^{-1}B'D^{-1} = A^{-1}B'(D - CA^{-1}B')^{-1}, \tag{14}$$

which may of course be verified by direct simplification

An important feature of each of these identities is that the matrix in parentheses on the left is of order m, while that in parentheses on the right is of order $n - m$.

A special case of (13) was noticed by the writer [3], [4] and of (14) by Ledermann ([7], [8]) and the writer ([3], [4]), in connection with regression problems of factor analysis In this special case, A is a diagonal matrix and hence easily inverted, $n - m$ is the number of common factors, which is usually small compared with m, the correlation matrix of m observed variates is given factored into the form $A - B'D^{-1}C$; and the work of inverting the correlation matrix of order m is simplified essentially into inverting a much smaller matrix.

It should be noticed that (12), (13), and (14) assume that both A and D are nonsingular, where (3) assumes only that A is nonsingular (since then F must be nonsingular from the lemma of section 4)

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THE FREQUENCY DISTRIBUTION OF DEVIATES FROM MEANS AND REGRESSION LINES IN SAMPLES FROM A MULTIVARIATE NORMAL POPULATION

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1. Summary. The joint frequency distribution has been found for any set of the $(n - k)$ deviates from their sample mean of each of the t variates in a sample from a multivariate normal population. Expressions for the variance of any single deviate in this distribution, the correlation coefficient between any pair of deviates, and certain partial correlation coefficients between any pair have also been obtained.

These results have been generalized so as to include the corresponding properties of deviates from a set of t multiple linear regression equations estimated from the sample, the m independent variates being the same for each of the t dependent.

2. Introduction. Some years ago, Irwin published results relating to the frequency distribution of the deviations of individual observations from the mean of a sample drawn from a normal population (see [1]). He derived an expression for the joint distribution of any number of these deviates, which distribution is always of the normal multivariate form, and thence obtained the total and partial correlation coefficients between any pair of the deviates.

The purpose of this paper is to discuss the generalization of Irwin's problem, firstly to the properties of the deviates of individual observations from the mean in a sample from a multivariate normal population and secondly to the properties of deviates from a regression equation instead of from a mean. So far as is known to the writer Irwin's results are of little practical importance, and these generalizations are probably of no practical value whatsoever. Nevertheless, they have some interest as additions to the knowledge of the mathematical properties of the normal frequency function, and for that reason alone they are put on record here.

3. Deviations from the sample mean. Irwin based his discussion on a normal population with mean m and variance σ^2, but the algebra is simplified a little, without any real loss of generality in the final results, if, by means of a preliminary transformation, these parameters of position and scale are made zero and unity respectively. The multivariate normal distribution in the t variates ${}_iy$, $(i = 1, 2, \cdots t)$, each with mean zero and variance unity, has the frequency function

$$(1) \qquad \frac{1}{(2\pi)^{\frac{1}{2}t} R^{\frac{1}{2}}} \exp\left\{-\frac{1}{2R} \rho^{ij}\, {}_iy\, {}_jy\right\},$$

where $i, j = 1, 2, \cdots t$; ρ^{ij} is the cofactor of the element ρ_{ij} in the determinant of population correlation coefficients

$$R = \begin{vmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1t} \\ \rho_{12} & 1 & \rho_{23} & \cdots & \rho_{2t} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \rho_{1t} & \rho_{2t} & \rho_{3t} & \cdots & 1 \end{vmatrix} \tag{2}$$

A summation convention for the affixes i, j is understood throughout this paper, except when the contrary is explicitly stated.

Let $({}_iy_p)$ represent a sample of n independent sets of values of the t variates randomly selected from the population, $(p = 1, 2, \cdots, n)$. Then the element of probability for the sample is

$$\frac{1}{(2\pi)^{\frac{1}{2}nt} R^{\frac{1}{2}n}} \exp\left\{-\frac{1}{2R}\rho^{ij}\sum_p {}_iy_p\,{}_jy_p\right\} \prod_{\substack{i=1,\cdots,t\\ p=1,\cdots,n}} \{d({}_iy_p)\}. \tag{3}$$

If ${}_i\bar{y}$ is the mean of the n sample values of ${}_iy$, the deviates from the mean are $({}_iY_p)$, where

$${}_iY_p = {}_iy_p - {}_i\bar{y} = \sum_q \left(\delta_{pq} - \frac{1}{n}\right) {}_iy_q, \tag{4}$$

the summation being taken over $q = 1, 2, \cdots n$ with

$$\delta_{pq} = \begin{cases} 1 & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}$$

Now the ${}_iY$ are linear combinations of normally distributed variates, and are therefore themselves normally distributed. Clearly

$$E({}_iY_p) = 0 \tag{5}$$

and, from an expansion by means of equation (4) using

$$\begin{aligned} E({}_iy_p\,{}_jy_q) &= \delta_{pq}\rho_{ij}, \\ E({}_iY_p\,{}_jY_q) &= \left(\delta_{pq} - \frac{1}{n}\right)\rho_{ij}, \end{aligned} \tag{6}$$

where $\rho_{ii} = 1$ (not summed). Consequently the variance of any one deviate is

$$\sigma^2({}_iY_p) = \frac{n-1}{n}, \tag{7}$$

and the correlation coefficient between any pair is

$$\rho({}_iY_p, {}_jY_q) = \frac{n\delta_{pq} - 1}{n - 1}\rho_{ij}. \tag{8}$$

Equation (7) and equation (8) for the particular case of $i = j$ agree with the well-known results that Irwin has already given as equations (10) of his paper.

For any i, only $(n - 1)$ of the deviates ${}_iY_p$ are functionally independent. The joint distribution of these for $p = 1, 2, \cdots, (n - k)$ may be obtained from an inversion of the matrix of correlation coefficients. If Δ is the determinant of this matrix and $\Delta({}_iY_p, {}_jY_q)$ the cofactor corresponding to the two elements specified, this inversion shows that

$$\frac{\Delta({}_iY_p, {}_jY_q)}{\Delta} = \left(\delta_{pq} + \frac{1}{k}\right)\frac{\rho_{ij}}{R} \cdot \frac{n-1}{n}. \tag{9}$$

The joint distribution is therefore

$$\text{const.} \times \exp\left\{-\frac{1}{2R}\rho^{ij} \sum_{p \leq q \leq n-k} \left(\delta_{pq} + \frac{1}{k}\right) {}_iY_p\, {}_jY_q\right\} \prod(dY). \tag{10}$$

Now Δ may be evaluated as

$$\Delta = \left(\frac{n}{n-1}\right)^{t(n-k-1)} \left(\frac{k}{n-1}\right)^{t} R^{n-k},$$

and the constant multiplier in equation (10) is therefore

$$\frac{\left(\dfrac{n}{k}\right)^{\frac{1}{2}t}}{\{(2\pi)^t R\}^{\frac{1}{2}(n-k)}}. \tag{11}$$

From equation (9), the partial correlation coefficient between any two of the variates in the distribution (10), the remaining $t(n - k) - 2$ being held constant, is written down as

$$\text{partial correlation coefficient between } {}_iY_p, {}_jY_q = -\frac{k\delta_{pq} + 1}{k + 1} \cdot \frac{\rho^{ij}}{(\rho^{ii}\rho^{jj})^{\frac{1}{2}}}; \tag{12}$$

the summation convention is suspended for this equation.

4. Deviations from regression equations. The results obtained in section three may be generalized so as to relate to the frequency distribution of deviates from linear or polynomial regression equations instead of to deviates from means. Suppose that there are m independent variates x^α, $(\alpha = 1, 2, \cdots, m)$, which take values x_p^α corresponding to the sample observations ${}_iy_p$; polynomial regressions may be included by taking powers of an x as separate variates. If a conventional variate x^0, whose value is always unity, be introduced, the regression equation of ${}_iy$ on x^α, $(\alpha = 0, 1, 2 \cdots, m)$, may be written

$${}_i\eta = {}_ib^\alpha x^\alpha, \tag{13}$$

where a summation convention is understood for $\alpha = 0, 1, \cdots, m$ and the regression coefficients are the solutions of the normal equations.

$${}_ib^\alpha \sum_p x_p^\alpha x_p^\beta = \sum_p {}_iy_p x_p^\beta. \tag{14}$$

Write

$$B^{\alpha\beta} = \sum_p x_p^\alpha x_p^\beta \tag{15}$$

and let $(B_{\alpha\beta})$ be the inverse matrix of $(B^{\alpha\beta})$.

Then the solutions of equations (14) are

$${}_ib^\alpha = B_{\alpha\beta} \sum_p {}_iy_p x_p^\beta. \tag{16}$$

If the deviation of ${}_iy_p$ from the regression equation (13) is ${}_iZ_p$, then

$$\begin{aligned} {}_iZ_p &= {}_iy_p - {}_i\eta_p \\ &= \sum_q (\delta_{pq} - B_{\alpha\beta} x_p^\alpha x_q^\beta)\, {}_iy_q, \end{aligned} \tag{17}$$

the summation for q being over $q = 1, 2, \cdots, n$. As for equation (5),

$$E({}_iZ_p) = 0. \tag{18}$$

Also

$$E({}_iZ_p {}_jZ_q) = (\delta_{pq} - B_{\alpha\beta} x_p^\alpha x_q^\beta)\rho_{ij}. \tag{19}$$

since by definition

$$B^{\alpha\beta} B_{\alpha\gamma} = \delta_{\beta\gamma}.$$

Write now θ for the square matrix of $(m + 1)$ rows and columns whose elements are the $B^{\alpha\beta}$, and X_p for the single column matrix of values x corresponding to the pth observation; i.e.

$$\theta = (B^{\alpha\beta}) \tag{20}$$

and

$$X_p = \begin{pmatrix} x_p^0 \\ x_p^1 \\ x_p^2 \\ \cdot \\ \cdot \\ \cdot \\ x_p^m \end{pmatrix} \tag{21}$$

Write also

$$\theta_{p,q,r,\cdot} = \theta - X_p X'_p - X_q X'_q - X_r X'_r - \cdots. \tag{22}$$

Then

$$|\theta_p| = |\theta| \cdot (1 - B_{\alpha\beta} x_p^\alpha x_p^\beta),$$

and

$$|\theta_{pq}| - |\theta_{pq} + X_p X'_q| = -|\theta| \cdot B_{\alpha\beta} x_p^\alpha x_q^\beta.$$

Hence, from equation (19), the variance of a deviate may be written

$$\sigma^2({}_iZ_p) = \frac{|\theta_p|}{|\theta|}, \tag{23}$$

and the correlation coefficient between any pair of deviates is

$$\rho({}_iZ_p, {}_jZ_q) = \begin{cases} \rho_{ij} & (p = q) \\ \rho_{ij} \dfrac{|\theta_{pq}| - |\theta_{pq} + X_p X_q'|}{\{|\theta_p| \cdot |\theta_q|\}^{\frac{1}{2}}} & (p \neq q) \end{cases} \tag{24}$$

For any i, only $(n - m - 1)$ of the deviates ${}_iZ_p$ are functionally independent. The joint distribution of these for $p = 1, 2, \cdots, (n - k)$ and any $k \geq m + 1$ may be found by inversion of the matrix of correlation coefficients obtained from equation (24). The multiplier of the exponential in this distribution of $t(n - k)$ variables is

$$\frac{|\theta|^{\frac{1}{2}t(n-k)}}{(2\pi)^{\frac{1}{2}t(n-k)} R^{\frac{1}{2}(n-k)} D^{\frac{1}{2}t}},$$

where

$$D = \begin{vmatrix} |\theta_1| & |\theta_{12}| - |\theta_{12} + X_1X_2'| & \cdots & |\theta_{1,n-k}| - |\theta_{1,n-k} + X_1X_{n-k}'| \\ |\theta_{12}| - |\theta_{12} + X_1X_2'| & |\theta_2| & \cdots & |\theta_{2,n-k}| - |\theta_{2,n-k} + X_2X_{n-k}'| \\ \cdots & \cdots & \cdots & \cdots \\ |\theta_{1,n-k}| - |\theta_{1,n-k} + X_1X_{n-k}'| & |\theta_{2,n-k}| - |\theta_{2,n-k} + X_2X_{n-k}'| & \cdots & |\theta_{n-k}| \end{vmatrix}$$

Since θ is positive definite, there exists a non-singular matrix K such that

$$K\theta K' = I.$$

Then the X_p may be transformed to new column matrices W_p by

$$KX_p = W_p = \begin{pmatrix} w_p^0 \\ w_p^1 \\ w_p^2 \\ \cdot \\ \cdot \\ \cdot \\ w_p^m \end{pmatrix}$$

and consequently

$$X_p = K^{-1}W_p.$$

It follows that

$$|\theta_p| = |\theta| \cdot |I - W_pW_p'|,$$

which may be reduced to the form

$$|\theta_p| = |\theta| \cdot (1 - w_p^\alpha w_p^\alpha).$$

Similarly

$$| \theta_{pq} | - | \theta_{pq} + X_p X_q' | = -| \theta | w_p^\alpha w_q^\alpha .$$

Hence

$$D = | \theta |^{n-k} \begin{vmatrix} 1 - w_1^\alpha w_1^\alpha & -w_1^\alpha w_2^\alpha & \cdots & -w_1^\alpha w_{n-k}^\alpha \\ -w_1^\alpha w_2^\alpha & 1 - w_2^\alpha w_2^\alpha & \cdots & -w_2^\alpha w_{n-k}^\alpha \\ \cdots & \cdots & \cdots & \cdots \\ -w_1^\alpha w_{n-k}^\alpha & -w_2^\alpha w_{n-k}^\alpha & \cdots & 1 - w_{n-k}^\alpha w_{n-k}^\alpha \end{vmatrix}$$

This may be transformed into

$$D = | \theta |^{n-k} \begin{vmatrix} & & & & W_1' \\ & & & & W_2' \\ & & & & \cdot \\ & & I_{n-k} & & \cdot \\ & & & & \cdot \\ & & & & W_{n-k}' \\ W_1 & W_2 & \cdots & W_{n-k} & I_{m+1} \end{vmatrix}$$

$$= | \theta |^{n-k} \cdot | I - W_1 W_1' - W_2 W_2' - \cdots - W_{n-k} W_{n-k}' |$$

$$= | \theta |^{n-k-1} \cdot | \theta_{1,2,\cdots,(n-k)} |.$$

Thus, finally, the constant in the distribution is found to be

$$\frac{1}{\{(2\pi)^t R\}^{\frac{1}{2}(n-k)}} \left\{ \frac{|\theta|}{|\Omega_k|} \right\}^{\frac{1}{2}t}, \tag{25}$$

in which Ω_k has been written for $\theta_{1,2,\ \ ,(n-k)}$, a matrix of the same form as θ but calculated from the last k sets of observations only.

The cofactors of the matrix of correlation coefficients, required for the coefficients of the quadratic form in the distribution, can be derived in a similar manner. The distribution may be written

$$\begin{aligned} &\frac{1}{\{(2\pi)^t R\}^{\frac{1}{2}(n-k)}} \left\{ \frac{|\theta|}{|\Omega_k|} \right\}^{\frac{1}{2}t} \\ &\quad \cdot \exp. \left\{ -\frac{1}{2R} \rho^{ij} \sum_{p \leq q \leq n-k} \left(\delta_{pq} - 1 + \frac{|\Omega_k + X_p X_q'|}{|\Omega_k|} \right) {}_i Z_p \, {}_j Z_q \right\} \Pi \, (dZ), \end{aligned} \tag{26}$$

of which the distribution (10) is easily seen to be the particular case for $m = 0$.

From (26), the partial correlation coefficient between any pair of deviates, ${}_i Z_p$ and ${}_j Z_q$, may be written down as

$$- \frac{|\Omega_k + X_p X_q'| + (\delta_{pq} - 1) |\Omega_k|}{\{|\Omega_k + X_p X_p'| \cdot |\Omega_k + X_q X_q'|\}^{\frac{1}{2}}} \frac{\rho^{ij}}{(\rho^{ii} \rho^{jj})^{\frac{1}{2}}}; \tag{27}$$

in this expression the summation convention is again suspended.

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ON THE ASYMPTOTIC DISTRIBUTIONS OF CERTAIN STATISTICS USED IN TESTING THE INDEPENDENCE BETWEEN SUCCESSIVE OBSERVATIONS FROM A NORMAL POPULATION

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1. The statistics to be considered here have the general expression

$$T = \frac{Q}{S}, \qquad Q = \sum_{i=1}^{N} a_{ij}(x_i - \bar{x})(x_j - \bar{x}), \qquad S = \sum_{i=1}^{N} (x_i - \bar{x})^2,$$

where $(x_1, \cdots, x_N)$ is a sample from a normal population whose mean and variance can evidently be assumed to be 0 and 1 respectively.[1] The purpose of this note is to study the asymptotic distribution of T assuming that the x_i are independent. The whole work may be regarded as a straightforward application of Cramér's theory of asymptotic expansion (see [1], pp. 69–88).

If $A = [a_{ij}]$ and γ is the row vector $N^{-\frac{1}{2}}[1, 1, \cdots, 1, 1]$ the quadratic form Q has the matrix $(I - \gamma'\gamma)A(I - \gamma'\gamma)$. The latent roots of this matrix, which are also the latent roots of $A(I - \gamma'\gamma)^2 = A(I - \gamma'\gamma)$, will be denoted by $0, \lambda_1, \cdots, \lambda_n$, with $n = N - 1$. Then Q and S can be simultaneously diagonalized (by a rotation of the N-dimensional space), so that

$$Q = \sum_{r=1}^{N} \lambda_r y_r^2, \qquad S = \sum_{r=1}^{n} y_r^2,$$

where the y_r are again independently and normally distributed with zero mean and unit variance.

We shall make the following assumptions

(a) $|\lambda_r| \leq 1$ for all r.

(b) There is a positive number c independent of n such that

$$\sum_{r=1}^{n} (\lambda_r - \bar{\lambda})^2 > cn, \quad \text{where} \quad \bar{\lambda} = \frac{1}{n}\sum_{r=1}^{n} \lambda_r.$$

Write

$$z = \frac{\sqrt{2\sum_{r=1}^{n}(\lambda_r - \bar{\lambda})^2 x}}{\sqrt{n^2 - 2nx^2}}, \qquad s_m(x) = \sum_{r=1}^{n} (\lambda_r - \bar{\lambda} - z)^m,$$

$$X_r = (\lambda_r - \bar{\lambda} - z)(y_r^2 - 1), \qquad G(x) = Pr\{T \leq \bar{\lambda} + z\}.$$

[1] The exact and the approximate distribution of such statistics were a recent subject of study by a number of statisticians See W. J. Dixon, "Further contributions to the problem of serial correlation," *Annals of Math Stat.*, Vol 15 (1944), pp. 119–144. Further references are listed in Dixon's paper

Then it can easily be verified that

$$G(x) = Pr\left\{\frac{\sum_{i=1}^{r} X_r}{\sqrt{2s_2(x)}} \leq x\right\}.$$

This expression of $G(x)$ shows that the application of Cramér's expansion is at hand, since $E(X_r) = 0$ and $2s_2(x)$ is the variance of ΣX_r. Let ρ_{kn} and T_{kn} stand for the same quantities as defined in Cramér's work (see [1], pp. 70–71). Since moments of all order of X_r exist, we may use $2k + 2$ in place of k. We have

$$\rho_{2k+2,n} = \frac{\frac{1}{n} m_k s_{2k+2}(x)}{\left(\frac{2}{n} s_2(x)\right)^{k+1}}, \qquad T_{2k+2,n} = \frac{\sqrt{n}}{4\rho_{2k+2,n}^{3/2k+2}},$$

where $m_k = E(y^2 - 1)^{2k+2}$ and y is a normal variate with mean 0 and variance 1.

By virtue of assumption (a) $| T | \leq 1$ Therefore we may confine ourselves to the range of values for which $| \bar{\lambda} + z | \leq 1$. Then $| \lambda_r - \bar{\lambda} - z | \leq 2$. Also, by assumption (b), $s_2(x) \geq \Sigma(\lambda_r - \bar{\lambda})^2 > cn$. Hence $\rho_{2k+2,n}$, and in consequence $\sqrt{n}T_{2k+2,n}^{-1}$, are less than some constant independent of n and x. The remainder of Cramer s expansion, if it is justifiable, will therefore be less than Mn^{-k}, where M is independent of n and x. The justification consists in verifying that the following condition is satisfied: if $f_r(t)$ is the characteristic function of X_r and A is any positive number, then

$$\text{l.u.b.} \prod_{r=1}^{n} | f_r(t) | \quad \text{for} \quad | t | > \frac{T_{2k+2,n}}{\sqrt{2s_2(x)}}$$

is less than $M_1 T_{2k+2,n}^{-A}$, where M_1 is independent of n and x (see [1], p. 85). Since $T_{2k+2,n} \leq \frac{1}{4}\sqrt{n}$ [2] and $s_2(x) > c\sqrt{n}$, it is sufficient to show that, if a and A are any positive numbers and if

$$U = \text{l.u.b.} \prod_{r=1}^{n} | f_r(t) | \quad \text{for} \quad | t | > a,$$

then $U \leq M_2 n^{-A}$, where M_2 is independent of n and x. Now

$$| f_r(t) | = \{1 + 4t^2(\lambda_r - \bar{\lambda} - z)^2\}^{-\frac{1}{4}}$$

whence

$$U = \prod_{r=1}^{n} \{1 + 4a^2(\lambda_r - \bar{\lambda} - z)^2\}^{-\frac{1}{4}}.$$

Let μ be the number of λ_r for which $(\lambda_r - \bar{\lambda} - z)^2 < \frac{1}{2}c$ Then $cn < s_2(x) \leq \frac{1}{2}c(n - \mu) + 4\mu$; hence $cn < (8 - c)\mu$ and

$$U \leq (1 + 2a^2c)^{-\frac{1}{4}\mu} < (1 + 2a^2c)^{-(cn/4(8-c))}$$

This shows that the desired condition on U is satisfied, and that therefore Cramér's procedure can be adopted.

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[2] This follows from the fact that $P_{2k+2,n} > 1$. Cf. Cramér, [1], p. 70.

Wherever Cramér's asymptotic expansion is valid, the terms in the expansion are most conveniently obtained with the help of Cornish and Fisher's symbolic expression (see [2]):

$$e^{-(1/3!)\gamma_3(d^3/dx^3)+(1/4!)\gamma_4(d^4/dx^4)-\cdots}\Phi(x),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}\,dy$$

and γ_j is the jth semi-invariant of the random variable whose distribution is under asymptotic expansion. In the present case we have

$$\frac{\gamma_j}{j!} = \frac{\beta_j(x)}{n^{\frac{1}{2}(j-2)}},$$

where

$$\beta_j(x) = \frac{2^{\frac{1}{2}(j-2)}}{j}\,\frac{\frac{1}{n}\,s_j(x)}{\left(\frac{1}{n}\,s_2(x)\right)^{\frac{1}{2}j}}.$$

Hence we may express our result as follows:

$$G(x) = \exp\left[\sum_{j=3}^{2k+1}\frac{(-1)^j\beta_j(x)}{n^{\frac{1}{2}(j-2)}}\left(\frac{d}{dx}\right)^j\right]\Phi(x) + R_k(x), \tag{1}$$

where $|R_k(x)| \leq Mn^{-k}$, and M is independent of n and x. The symbolic exponential in (1) is to be expanded as far as and including the term in $n^{-\frac{1}{2}(2k-1)}$.

2. Let us apply the result (1) to the following three statistics: $T_\alpha = Q_\alpha/S$, $(\alpha = 1, 2, 3)$, where

$$Q_1 = \sum_{i=1}^{N}(x_i - \bar{x})(x_{i+1} - \bar{x}) \quad \text{with} \quad x_{N+1} = x_1,$$

$$Q_2 = \tfrac{1}{2}(x_1 - \bar{x})^2 + \tfrac{1}{2}(x_N - \bar{x})^2 + \sum_{i=1}^{N-1}(x_i - \bar{x})(x_{i+1} - \bar{x}),$$

$$Q_3 = \sum_{i=1}^{N-1}(x_i - \bar{x})(x_{i+1} - \bar{x}).$$

T_2 is simply related with $T^* = Q^*/S$, where

$$Q^* = \sum_{i=1}^{N-1}(x_i - x_{i+1})^2;$$

for we have $Q_2 = S - \frac{1}{2}Q^*$, whence $T_2 = 1 - \frac{1}{2}T.^*$ We shall write $\lambda_r^{(\alpha)}$ for the λ's corresponding to Q_α, and

$$b_{m\alpha} = \sum_{r=1}^{n}(\lambda_r^{(\alpha)})^m, \qquad (\alpha = 1, 2, 3).$$

(i) For Q_1 we have $\lambda_r^{(1)} = \cos \frac{2\pi r}{N}$ (see [3]). Since

$$\cos^m \theta = \frac{1}{2^m}(e^{i\theta} + e^{-i\theta})^m = \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j} e^{i(2j-m)\theta},$$

we have

$$b_{m1} = \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}\sum_{r=1}^{n}\xi^r, \quad \text{where} \quad \xi = e^{2\pi(2j-m)i/N}.$$

If $m < n$, then

$$\sum_{r=1}^{n}\xi^r = -1 \quad \text{if} \quad j \neq \tfrac{1}{2}m, \qquad = n \quad \text{if} \quad j = \tfrac{1}{2}m.$$

Hence, for $m < n$, $b_{m1} = -1$ if m is odd, $b_{m1} = \frac{N}{2^m}\binom{m}{\frac{1}{2}m} - 1$ if m is even.

In particular

$$\bar{\lambda}^{(1)} = -\frac{1}{n}, \qquad \sum_{r=1}^{n}(\lambda_r^{(1)} - \bar{\lambda}^{(1)})^2 = \frac{n^2 - n - 2}{2n} > 0.4n \quad \text{if} \quad n \geq 7.$$

Hence assumptions (a) and (b) are true (for $n \geq 7$). The $s_j(x)$ are conveniently computed with the help of b_{m1}. The $\beta_j(x)$ are then computed to yield the terms in (1).

(ii) The λ's corresponding to Q^* are $4\sin^2 \frac{r\pi}{2N}$ (see [4]). Hence

$$\lambda_r^{(2)} = \cos\frac{r\pi}{N}.$$

By a computation similar to that in (i) we easily obtain $b_{m2} = \frac{N}{2^m}\binom{m}{\frac{1}{2}m} - 1$ for even m and $b_{m2} = 0$ for odd m, provided $m < 2n$. In particular, $\bar{\lambda}^{(2)} = 0$, $\Sigma(\lambda_r^{(2)} - \bar{\lambda}^{(2)})^2 = \frac{n-1}{2} \geq \cdot 4n$ for $n \geq 5$. Hence assumptions (a) and (b) are true (for $n \geq 5$).

(iii) In the case of Q_3 the matrix A is

$$A = \begin{Vmatrix} 0 & \frac{1}{2} & & & & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & & & \\ & \frac{1}{2} & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & 0 & \frac{1}{2} \\ 0 & & & & \frac{1}{2} & 0 \end{Vmatrix}$$

whose latent roots are $\cos \pi t/(N+1)$, $(t = 1, \cdots, N)$ (see [5]), all less than or equal to unity in absolute value. It follows that the same is true for the $\lambda_r^{(3)}$.

Hence assumption (a) is true. Unlike the two previous cases, there is no simple expression for b_{m3} With the help of the formula

$$b_{m3} = \operatorname{tr} \{A(I - \gamma'\gamma)\}^m$$

we may compute b_{m3} for small values of m. Thus

$$b_{13} = -\frac{n}{n+1}$$

$$b_{23} = \frac{n}{2} - \frac{2n-1}{n+1} + \frac{n^2}{(n+1)^2}$$

$$b_{33} = -\frac{3(n-1)}{n+1} + \frac{3n(2n-1)}{2(n+1)^2} - \frac{n^3}{(n+1)^3}$$

$$b_{43} = \frac{3n-2}{8} - \frac{8n-11}{2(n+1)} + \frac{4n(n-1)}{(n+1)^2} + \frac{(2n-1)^2}{2(n+1)^2} - \frac{2n^2(2n-1)}{(n+1)^3} + \frac{n^4}{(n+1)^4}$$

$$b_{53} = -\frac{5(4n-7)}{4(n+1)} + \frac{5n(8n-11)}{8(n+1)^2} + \frac{5(2n-1)(n-1)}{2(n+1)^2} - \frac{5n^2(n-1)}{(n+1)^3} - \frac{5n(2n-1)^2}{4(n+1)^3} + \frac{5n^3(2n-1)}{2(n+1)^5} - \frac{n^5}{(n+1)^5}$$

$$\overline{\lambda^{(3)}} = -\frac{1}{n+1}, \sum_{r=1}^{n} (\lambda_r^{(3)} - \overline{\lambda^{(3)}})^2 = \frac{n}{2} - \frac{2n-1}{n+1} + \frac{n^2-n}{(n+1)^2} > 0.4n \text{ for } n \geq 10.$$

Hence assumption (b) is true (for $n \geq 10$). Using these values of b_{m3} we may compute $\beta_3(x)$, $\beta_4(x)$ and $\beta_5(x)$. By (1) we have

$$G(x) = \Phi(x) - \frac{1}{n^{\frac12}}\beta_3(x)\Phi^{(3)}(x) + \frac{1}{n}\left(\beta_4(x)\Phi^{(4)}(x) + \tfrac{1}{2}\beta_3^2(x)\Phi^{(6)}(x)\right) - \frac{1}{n^{\frac32}}\left(\beta_5(x)\Phi^{(5)}(x) - \beta_3(x)\beta_4(x)\Phi^{(7)}(x) + \tfrac{1}{6}\beta_3^3(x)\Phi^{(9)}(x)\right) + R(x),$$

where $|R(x)| \leq Mn^{-2}$ and M is independent of n and x.

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NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

ESTIMATING THE PARAMETERS OF A RECTANGULAR DISTRIBUTION

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1. Introduction. In this note, the range and midrange of the sample are shown to be a pair of sufficient statistics, and maximum likelihood estimates, for the true range and true mean of a rectangular distribution, exact and limiting distribution of midrange, range, and their ratio are derived; the "efficiencies" of the sample mean and median as estimates of the true mean are calculated; and the limiting distribution of the difference between two sample midranges is derived. All the limiting distributions are non-normal, and the error of estimate is of order n^{-1} rather than the customary order $n^{-\frac{1}{2}}$. The limiting distribution of midrange, and the limiting ratio of variances of the midrange and sample mean were given by Fisher [1].

$f(x)$ and $F(x)$ are used throughout to designate the probability density function of x and the distribution function (cumulative probability function) of x; the argument will also indicate the random variable being considered

2. Exact distribution of midrange, range, and their ratio. Let $x_1, \cdots, x_n$ be a set of n independent observations on a random variable having the rectangular distribution $f(x) = 1/L$, $(\theta - L/2 \leq x \leq \theta + L/2)$, where θ is the true mean, and L the true range. The minimum observation u and the maximum observation v are a pair of sufficient statistics for θ and L, as the conditional distribution of the remaining observations for given u and v is independent of θ and L:

$$f(x_1, \cdots, x_n \mid u, v) = (v - u)^{-(n-2)}$$

The midrange $\bar{\theta} = \frac{1}{2}(u + v)$ and the range $\bar{L} = v - u$ are maximum likelihood estimates of θ and L, respectively, as they are the parameter values which uniquely maximize $f(x_1, \cdots, x_n)$ for the given set of observations. We shall assume that the random variable is normalized by change of origin and change of scale so that $\theta = 0$ and $L = 1$. The joint probability density function of u and v is

$$f(u, v) = \frac{d^2 F(u, v)}{dv\, d(-u)} = \frac{d^2(v - u)^n}{dv\, d(-u)} = n(n - 1)(v - u)^{n-2}, \qquad (-\tfrac{1}{2} \leq u \leq v \leq \tfrac{1}{2}). \tag{1}$$

Making the transformation $\bar{\theta} = \frac{1}{2}(u + v)$, $\bar{L} = v - u$ in (1),

$$f(\bar{\theta}, \bar{L}) = n(n - 1)\bar{L}^{n-2} \qquad (0 \leq 2\,|\,\bar{\theta}\,| \leq 1 - \bar{L} \leq 1). \tag{2}$$

Integrating out $\bar{L}$ from 0 to $(1 - 2\,|\bar{\theta}\,|)$,

$$\begin{aligned} f(\bar{\theta}) &= n(1 - 2\,|\,\bar{\theta}\,|)^{n-1}, && (|\,\bar{\theta}\,| \leq \tfrac{1}{2}). \\ |\,F(\bar{\theta}) - F(0)\,| &= \tfrac{1}{2} - \tfrac{1}{2}(1 - 2\,|\,\bar{\theta}\,|)^{n}, && (|\,\bar{\theta}\,| \leq \tfrac{1}{2}). \end{aligned} \tag{3}$$

Odd moments vanish by symmetry; even order moments are

$$\mu_{2k}(\bar{\theta}) = \int_{-\frac{1}{2}}^{\frac{1}{2}} n\bar{\theta}^{2k}(1 - 2\,|\,\bar{\theta}\,|)^{n-1}\,d\bar{\theta} = 2^{-2k} \Big/ \binom{2k + n}{2k}. \tag{4}$$

In (2), integrating out $\bar{\theta}$ from $\frac{1}{2}(\bar{L} - 1)$ to $\frac{1}{2}(1 - \bar{L})$,

$$\begin{aligned} f(\bar{L}) &= n(n - 1)\bar{L}^{n-2}(1 - \bar{L}), && (0 \leq \bar{L} \leq 1). \\ F(\bar{L}) &= n(n - 1)\int_0^{\bar{L}} \bar{L}^{n-2}(1 - \bar{L})\,d\bar{L} = n(n - 1)B_{\bar{L}}(n - 1, 2), \\ & && (0 \leq \bar{L} \leq 1). \\ \mu_k(\bar{L}) &= n(n - 1)\int_0^{1} \bar{L}^{n-2+k}(1 - \bar{L})\,d\bar{L} = \frac{n(n - 1)}{(n + k)(n + k - 1)}. \end{aligned} \tag{5}$$

Thus $\mu_1(\bar{L}) = (n - 1)/(n + 1)$; hence the bias of $\bar{L}$ can be removed by multiplying $\bar{L}$ by $(n + 1)/(n - 1)$.

The statistic $t = \bar{\theta}/\bar{L}$ can be used to test the hypothesis that the mean of a rectangular distribution of unknown range is 0. To obtain the distribution of t when the hypothesis is true, set $t = \bar{\theta}/\bar{L}$ and $\bar{L} = \bar{L}$ in (2):

$$\begin{aligned} f(t, \bar{L}) &= n(n - 1)\bar{L}^{n-1}, && (\bar{L} \leq (1 + 2\,|\,t\,|)^{-1}). \\ f(t) &= (n - 1)(1 + 2\,|\,t\,|)^{-n} \\ |\,F(t) - F(0)\,| &= \tfrac{1}{2} - \tfrac{1}{2}(1 + 2\,|\,t\,|)^{1-n}. \end{aligned} \tag{6}$$

Moments of t do not exist for order greater than $(n - 2)$; for $k \leq n - 2$, odd moments vanish by symmetry and

$$\mu_{2k}(t) = 2(n - 1)\int_0^{\infty} t^{2k}(1 + 2t)^{-n}\,dt = 2^{2k} \Big/ \binom{n - 2}{2k}.$$

3. Limiting distributions. $\bar{\theta}$, $\bar{L}$, and t have non-normal limiting distributions, although $\bar{\theta}$ and $\bar{L}$ are maximum likelihood estimates; this is explained by the discontinuity of $f(x, \bar{\theta})$ at $x = \bar{\theta} \pm \frac{1}{2}$. We obtain the limiting distributions of $q = n\bar{\theta}$ and $r = n(1 - \bar{L})$. Substituting q and r in (2), and proceeding to the limit for increasing n,

$$\lim f(q, r) = \lim \frac{n - 1}{n}\left(1 - \frac{r}{n}\right)^{n-2} = e^{-r}, \qquad (0 \leq 2\,|\,q\,| \leq r < \infty).$$

The necessary simple integrations yield the following limiting distributions:

$$(7)\qquad \begin{aligned} f(q) &= e^{-2|q|}. \\ |F(q) - F(0)| &= \tfrac{1}{2} - \tfrac{1}{2}e^{-2|q|}. \\ \mu_{2k}(q) &= (2k)!/2^{2k};\ \mu_{2k+1} = 0. \\ f(r) &= re^{-r}, && (r \geq 0) \\ F(r) &= 1 - (1 + r)e^{-r}, && (r \geq 0) \\ \mu_k(r) &= (k + 1)! \end{aligned}$$

The limiting distribution of $s = nt$ is the same as that of $n\bar{\theta}$, as is seen by comparing (3) and (6).

4. Comparison of $\bar{\theta}$ with $\bar{x}$ and $\tilde{x}$ as estimates of θ. The sample mean $\bar{x}$ and median $\tilde{x}$ are unbiased estimates of θ.

$$(8)\qquad \begin{aligned} \mu_2(\bar{x}) &= \frac{1}{n}\int_{-\frac{1}{2}}^{\frac{1}{2}} x^2\, dx = 1/(12n). \\ \mu_2(\tilde{x}) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{x}^2 f(\tilde{x})\, d\tilde{x} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{x}^2 \frac{(2m+1)!}{m!m!}(\tfrac{1}{2} - \tilde{x})^m(\tilde{x} + \tfrac{1}{2})^m\, d\tilde{x}, \end{aligned}$$

for $n = 2m + 1$, m an integer. Substituting $z = 1 - 4\tilde{x}^2$, then simplifying the Beta function obtained on integration,

$$(9)\qquad \mu_2(\tilde{x}) = \frac{(2m+1)!}{m!m!2^{2m+3}}\int_0^1 z^n(1 - z)^{\frac{1}{2}}\, dz = \frac{1}{4(2m+3)} = \frac{1}{4(n+2)}$$

(4), with $k = 1$, gives $\mu_2(\bar{\theta}) = \dfrac{1}{2(n+1)(n+2)}$. Comparison of this with (8) and (9) shows that $\mu_2(\bar{\theta})/\mu_2(\bar{x}) = \dfrac{6n}{(n+1)(n+2)}$ and $\mu_2(\tilde{x})/\mu_2(\bar{x}) = 3n/(n+2)$. As n increases, $\mu_2(\bar{\theta})/\mu_2(\bar{x}) \to 6/n \to 0$; and $\mu_2(\tilde{x})/\mu_2(\bar{x}) \to 3$ Thus the "efficiency" of the mean is zero, and the median is only one-third as "efficient" as the mean (The concept of efficiency is not strictly applicable as $\bar{\theta}$ does not have a normal limiting distribution.)

5. Limiting distribution of difference between two midranges. Let $\bar{\theta}_1$ and $\bar{\theta}_2$ be the midranges of samples of n_1 and n_2 observations, respectively, from two normalized rectangular populations, and let $z = q_1 - q_2 = n_1\bar{\theta}_1 - n_2\bar{\theta}_2$. Applying the formula for composition of random variables, one obtains from (7),

$$(10)\qquad \begin{aligned} f(z) &= \int_{-\infty}^{\infty} f(z - q)f(q)\, dq = \int_{-\infty}^{\infty} e^{-2|z-q|}e^{-2|q|}\, dq \\ &= \int_{-\infty}^{0} e^{-2|z|}e^{-4q}\, dq + \int_0^{|z|} e^{-2|z|}\, dq + \int_{|z|}^{\infty} e^{2|z|}e^{-4q}\, dq \\ &= \tfrac{1}{4}e^{-2|z|} + |z|e^{-2|z|} + \tfrac{1}{4}e^{-2|z|} = (|z| + \tfrac{1}{2})e^{-2|z|} \\ |F(z) - F(0)| &= \tfrac{1}{2} - \frac{|z| + 1}{2}e^{-2|z|}. \\ \mu_{2k}(z) &= (k + 1)(2k)!/2^{2k}. \end{aligned}$$

$z = \frac{n_1}{2}\frac{v_1 + u_1}{v_1 - u_1} - \frac{n_2(v_2 + u_2)}{2(v_2 - u_2)}$ can be used to test the hypothesis of equality of means of any two rectangular populations, and has in the limit the distribution (10), if the means of the populations are equal.

6. The one-parameter rectangular distribution. If $f(x) = 1/\lambda$, $(0 \leq x \leq \lambda)$, then $f(x_1, \cdots, x_n \mid v) = v^{1-n}$. Thus v is a sufficient statistic and is evidently the maximum likelihood estimate of λ Here $F(v) = (v/\lambda)^n$, $f(v) = nv^{n-1}\lambda^{-n}$, and $\mu_k(v) = \lambda^k n/(n + k)$. The normalized error $y = n(\lambda - v)/\lambda$ has the probability density function $f(y) = (1 - y/n)^{n-1}$, which tends to e^{-y} as n increases.

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ON THE POWER FUNCTION OF THE SIGN TEST FOR SLIPPAGE OF MEANS

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1. Summary. This note compares the power functions of the sign test for slippage with the power functions of the most powerful test for the case of normal populations. The sign test is found to be approximately 95% efficient for small samples.

2. Introduction. Let us consider a univariate population whose mean equals its median and whose cumulative distribution function is continuous at the mean. A sampling method of testing the supposition that the mean of this population exceeds a given constant value μ_0 (slippage to the right) is furnished by considering how many values of the sample are less than μ_0. An analogous method applies for testing whether the mean is less than μ_0 (slippage to the left). A particular class of populations for which the sign test is valid are the normal populations. This note compares the power functions of the sign test with the power functions of the most powerful test for slippage for the case in which the population is normal (Table I). It is shown that the sign test is approximately 95% as efficient as the most powerful test (the Student t-test) for samples of size 4, 5 and 6, and that although the relative efficiency of the sign test decreases as the sample size increases, its efficiency is approximately 75% for samples of size 13. This supports the idea that for normal populations little efficiency is lost by using attributes instead of continuous variables if the sample size is small

In choosing between the sign and Student t-tests for slippage the following considerations may be of interest:

(a) The sign test is valid for a more general class of populations than the t-test.

(b) The sign test is almost as efficient as the t-test for small samples from normal populations.

(c) The sign test is much more easily computed than the t-test.

(d) The sign test has a very limited choice of significance levels for small samples while the t-test can have any desired significance level for any size sample.

The considerations (a) to (d) also apply in choosing between the sign test and the Daly test based on $(\bar{x} - \mu_0)/R$, where $\bar{x}$ is the mean and R the range of the sample used for the test (see [1]).

In section 5, Table II shows that for small size samples the significance levels of the sign test do not change greatly if the mean is only approximately equal to the median.

3. Statement of sign test. Let $x_1, \cdots, x_n$ be a sample of size n from a univariate population whose mean equals its median and whose cumulative distribution function is continuous at the mean, that is, which has the property that

$$Pr(x < \mu) = Pr(x > \mu) = \tfrac{1}{2}, \tag{1}$$

where μ is the population mean.

The significance test to decide whether μ exceeds a given constant value μ_0 is defined by

(2) *If m or less of the sample values $x_1, \cdots, x_n$ are less than μ_0, accept $\mu > \mu_0$.*

The significance test to decide whether $\mu < \mu_0$ is given by

(3) *If m or less of $x_1, \cdots, x_n$ are greater than μ_0, accept $\mu < \mu_0$.*

It is to be observed that in both (2) and (3) the null hypothesis tested is that $\mu = \mu_0$. In (2) the alternative is $\mu > \mu_0$ and in (3) the alternative is $\mu < \mu_0$.

From (1) it follows immediately that (2) and (3) both have the same significance level $\alpha(m, n)$, where

$$\alpha(m, n) = (\tfrac{1}{2})^n \sum_{j=0}^{m} \frac{n!}{j!(n-j)!}.$$

Appropriate choices of m and n will result in values of $\alpha(m, n)$ suitable for significance tests For example

$$\begin{aligned} \alpha(0, 4) &= .0624, & \alpha(1, 8) &= 0352 \\ \alpha(0, 5) &= .0312, & \alpha(1, 9) &= .0195 \\ \alpha(0, 6) &= .0156, & \alpha(1, 10) &= .0107 \\ \alpha(1, 7) &= .0625, & \alpha(2, 13) &= .0112. \end{aligned}$$

If the population has a continuous distribution function, $Pr(x_i = x_j ; i \neq j) = 0$ In this case let $x_{(i)}$ be the ith largest of $x_1, \cdots, x_n$. Then (2) can be restated as

$$\textit{If } x_{(m+1)} > \mu_0, \textit{ accept } \mu > \mu_0. \tag{4}$$

Test (3) is seen to be equivalent to

$$\text{(5)} \qquad \textit{If } x_{(n-m)} < \mu_0 \text{, } \textit{accept } \mu < \mu_0 .$$

Thus for the case of populations with continuous distribution functions it is only necessary to determine one order statistic and compare it with μ_0 in order to apply a test.

It is to be observed that a particular class of populations which satisfy (1) are those which have distribution functions which are symmetrical and continuous. Thus the normal populations represent a particular class for which (4) and (5) are valid.

4. Comparison with Student *t*-test. Consider the case in which the population is normal with mean μ and variance σ^2. Then the power function for (4) is given by

$$\begin{aligned} \text{Power Function} &= Pr(x_{(m+1)} > \mu_0) \\ &= Pr\left(\frac{x_{(m+1)} - \mu}{\sigma} > \frac{\mu_0 - \mu}{\sigma}\right) \\ &= \frac{n!}{m!(n-m-1)!}\int_\delta^\infty \left(\int_{-\infty}^x f(y)\,dy\right)^m \left(\int_x^\infty f(y)\,dy\right)^{n-m-1} f(x)\,dx, \end{aligned}$$

where

$$f(y) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}y^2} \qquad \text{and} \qquad \delta = \frac{\mu_0 - \mu}{\sigma}.$$

For a normal population, however, it is well known that the most powerful Studentized test of the one-sided alternative $\mu > \mu_0$ is the appropriate Student t-test. Values of the power function for the t-test are found for given values of δ by using the normal approximation given in [2].

The method of measuring the relative efficiencies of the two types of tests will be different from the common method of measuring the relative efficiencies of estimates, which consists in taking the ratio of the variances of the two estimates as the measure of their relative efficiency. The principle followed here will be to consider a sign test based on a given sample size and vary the degrees of freedom of the t-test having the same significance level until the power functions of the sign test and t-test agree in the sense that in the half-plane $\delta \leqq 0$ the area between the two power curves for which the sign test power function exceeds the t-test power function is equal to the analogous area for which the sign test power function is less than the t-test power function. The considerations are limited to the half-plane $\delta \leqq 0$ because the test is one-sided. The size of the t-test sample having this property divided by the size of the sign test sample is called the relative efficiency of that sign test. Intuitively this relative efficiency measures how much more data must be added if the sign test is to

furnish an amount of information equivalent to that supplied by the t-test. In obtaining the relative efficiencies in the manner described above, the degrees of freedom of the t-test are allowed to assume fractional values and the values of the power function are computed using the normal approximation as if it were valid for fractional degrees of freedom. The number of degrees of freedom, of course, can only be integral This method, however, gives an interpolated

TABLE I

A comparison of the power functions of the sign and t tests

Test	m	n	Approximate Relative Efficiency	Significance Level	Values of Power Function			
					$\delta=-\frac{1}{2}$	$\delta=-1$	$\delta=-1\frac{1}{2}$	$\delta=-2$
t		3.8		.0624	.219	.484	.755	.920
sign	0	4	95%	.0624	.229	.500	.755	.908
t		4.8		.0312	.150	.402	.709	.909
sign	0	5	96%	.0312	.159	.420	.703	.888
t		5.7		.0156	.098	.330	.660	.899
sign	0	6	95%	.0156	.110	.355	.655	.863
t		5.6		.0625	.306	.695	.932	.995
sign	1	7	80%	.0625	.311	.711	.920	.988
t		6.4		.0352	.225	.619	.908	.989
sign	1	8	80%	.0352	.239	.630	.869	.978
t		7.4		.0195	.171	.565	.893	.988
sign	1	9	82%	.0195	182	.573	.879	.974
t		8		.0107	.117	.468	.848	.983
sign	1	10	80%	.0107	.137	.515	853	.964
t		9.75		.0112	.162	631	.950	.998
sign	2	13	75%	.0112	.165	.661	.949	.998

measure of the size sample of the t-test having the properties outlined above. Table I supplies a comparison of the relative efficiencies and the powers of the sign test and the t-test obtained in the manner just described. Thus for samples of size 4, 5 and 6 the sign test is approximately 95% as efficient as the Student t-test. The relative efficiency decreases as the size of the sample increases but even for samples as large as 13 is approximately 75%.

For normal populations it is also well known that the most powerful Studentized test of the alternative $\mu < \mu_0$ is given by the appropriate Student t-test. It is clear that Table I can also be considered as a comparison of test (5) with the corresponding Student t-test if δ is replaced by $-\delta$ and m by $n - m$.

5. Approximate cases. Suppose that (1) is only approximately satisfied by the population in question.

Let $Pr(x < \mu) = \frac{1}{2} + r$. Then the significance level of (2) is

$$\sum_{j=0}^{m} \frac{n!}{j!(n-j)!} \left(\tfrac{1}{2} + r\right)^j \left(\tfrac{1}{2} - r\right)^{n-j}. \tag{6}$$

Significance levels of (2) for small size samples are given in Table II as a function of r.

TABLE II

A comparison of the significance levels of the sign test when the mean differs from the median

m	n	Significance Level				
		$r=0$	$r=-.02$	$r=-.05$	$r=.02$	$r=.05$
0	4	.0624	.073	.091	.053	.041
0	5	.0312	.038	.050	.026	.019
0	6	.0156	.020	.028	.012	.008

Table II shows that for small samples the significance level of (2) does not change greatly from $\alpha(m, n)$ if (1) is only approximately satisfied. Expression (6) shows, however, that for large size samples even a small value of r can cause a large change in the significance level of (2).

For $Pr(x < \mu) = \frac{1}{2} + r$ it is apparent that the significance level of (3) is (6) with r replaced by $-r$ so that Table II applies to tests (3) if this replacement is made.

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AN APPROXIMATION TO THE PROBABILITY INTEGRAL

BY J. D. WILLIAMS

United States Naval Ordnance Test Station, Inyokern, California

1. Summary. It is shown that

$$\frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{1}{2}t^2}\,dt \le [1 - e^{-(2/\pi)x^2}]^{\frac{1}{2}}$$

and that the equality is never in error by as much as three-fourths of one percent. Other approximations are discussed.

2. For use on those occasions when an approximate analytic expression for the integral

$$p(x) = \frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{1}{2}t^2}\,dt \tag{1}$$

is desired, the approximation

$$p'(x) = [1 - e^{-(2/\pi)x^2}]^{\frac{1}{2}} \tag{2}$$

is simple and reasonably accurate. An approximation equivalent to this is quite commonly used in problems involving a bivariate normal distribution, but its use in the one-dimensional case seems to be less well known.

We shall first show that $p(x) \le p'(x)$ and then estimate, by calculation, the relative error made when the equality is accepted.

$$\begin{aligned} p(x) &= \frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{1}{2}t^2}\,dt \\ &= \left[\frac{1}{2\pi}\int_{-x}^{x}\int_{-x}^{x} e^{-\frac{1}{2}(t_1^2+t_2^2)}\,dt_1\,dt_2\right]^{\frac{1}{2}} \\ &\le \left[\frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{(2x/\sqrt{\pi})} re^{-\frac{1}{2}r^2}\,dr\,d\theta\right]^{\frac{1}{2}} \\ &= [1 - e^{-(2/\pi)x^2}]^{\frac{1}{2}} = p'(x), \qquad \text{q.e.d.} \end{aligned} \tag{3}$$

The approximation, introduced at the stage of passage to polar coordinates, comprises replacement of the square region of integration $-x \le x_i \le x$ by a circular region, $0 \le r \le \frac{2}{\sqrt{\pi}}x$, having the same area. Since we are dealing with a circular normal distribution with zero means, the region of fixed area which covers the greatest density is a circle whose center is at the origin. Therefore our square region of area $4x^2$ must contain less density than the circular region of area $4x^2$ by which we have replaced it.

The maximum value of the relative error,

$$\epsilon_p = \frac{p'(x)}{p(x)} - 1, \tag{4}$$

is found by calculation to be about seven-tenths of one percent, as may be judged from Table 1, column 3.

The question may be asked: Can the relative error be reduced by suitable choice of the parameter c in

$$p'(x) = [1 - e^{-cx^2}]^{\frac{1}{2}}? \tag{5}$$

Calculation indicates that by taking $c = 0.6302$ the relative error is reduced to about one-half of one percent; but this gain is offset, for many purposes, by the loss of the inequality (3).

The density function implied by (2), namely

$$\rho'(x) = \frac{|x|}{\pi} e^{-(2/\pi)x^2}[1 - e^{-(2/\pi)x^2}]^{-\frac{1}{2}}, \tag{6}$$

has the variance

$$\sigma^2 = \pi (1 - \log 2) = 0.964. \tag{7}$$

If c is determined so that the density function will have unit variance, then (5) becomes

$$p'(x) = \left[1 - \left(\frac{e}{2}\right)^{-2x^2}\right]^{\frac{1}{2}}; \tag{8}$$

this approximation to (1) leads to relative errors of almost two percent, which occur when x is small.

The density function (6) may be used to judge the quality of (2) in approximating to an integral of the form

$$p(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-\frac{1}{2}t^2}\, dt, \tag{9}$$

the approximation being

$$p'(x_1, x_2) = \tfrac{1}{2}[p'(x_2) - p'(x_1)] \tag{10}$$

when x_1 and x_2 are positive (which is the severe case). It is evident that the relative error in accepting (10) for (9) cannot exceed the greatest relative discrepancy ϵ_ρ, in the interval $x_1 \leq x \leq x_2$, between density function (6) and the normal density

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \tag{11}$$

The quantity

$$\epsilon_\rho = \frac{\rho'(x)}{\rho(x)} - 1 \tag{12}$$

is tabulated in Table 1, column 6, from which it appears that the relative error committed in using (10) for (9) will surely be less than one-and-a-half percent

provided $0 \leq x_t \leq 1.8$; but the relative error may be very great when the interval of integration lies beyond $x = 1.8$.

The approximations described herein were suggested by the following situation, encountered in work done by the Applied Mathematics Panel, NDRC: The probability P of at least one success, defined by $-x \leq x_t \leq x$, in a sample

TABLE 1

x	$p'(x)$	$p(x)$	ϵ_p	$\rho'(x)$	$\rho(x)$	ϵ_ρ
.0	0	0	0	.3989	.3989	0
.1	.0797	.0797	.0002	.3969	.3970	.0005
.2	.1586	.1585	.0005	.3914	.3910	.0010
.3	.2360	.2358	.0008	.3821	.3814	.0018
.4	.3112	.3108	.0013	.3695	.3683	.0033
.5	.3836	.3829	.0018	.3539	.3521	.0051
.6	.4526	.4515	.0024	.3356	.3332	.0072
.7	.5177	.5161	.0031	.3151	.3123	.0089
.8	.5785	.5763	.0038	.2929	.2897	.0111
.9	.6347	.6319	.0044	.2695	.2661	.0128
1.0	.6862	.6827	.0051	.2454	.2420	.0141
1.1	.7329	.7287	.0058	.2211	.2179	.0147
1.2	.7747	.7699	.0063	.1971	.1942	.0149
1.3	.8118	.8064	.0067	.1738	.1714	.0140
1.4	.8443	.8385	.0069	.1516	.1497	.0127
1.5	.8725	.8664	.0070	.1306	.1295	0085
1.6	.8967	.8904	.0070	.1113	.1109	.0036
1.7	.9171	.9109	.0068	.0937	.0940	−.0032
1.8	.9341	.9281	.0065	.0781	.0790	−.0114
1.9	9485	9426	0063	.0640	.0656	−.0244
2.0	.9600	.9545	.0058	.0520	.0540	−.0370

of n pairs (x_1, x_2) from a population in which the independent component probabilities are $p(x)$, is

$$P = 1 - [1 - p^2(x)]^n. \tag{13}$$

A little numerical exploration, supplemented by examination of the limiting values as $x \to 0$ and $x \to \infty$, revealed that when P is fixed the quantity $\log n$ is very nearly a linear function, of slope minus two, of $\log x$; so nearly, in fact, that one was encouraged to posit the linearity and observe the consequences This yielded (5), which became (2) by requiring that it go to zero with x in the same manner as (1).

DISTRIBUTION OF THE RATIO OF SAMPLE RANGE TO SAMPLE STANDARD DEVIATION FOR NORMAL AND COMBINATIONS OF NORMAL DISTRIBUTIONS

By G. A. Baker

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1. Introduction. The distribution of sample ranges in terms of the standard deviation of the sampled population for homogeneous populations has been dealt with in some detail by mathematical methods for the normal parent and by empirical sampling methods for non-normal parents. These results are presented in summary in Tables XXII, XXIII, and XXIV of [1]. Bliss [2] suggests that the range in different sized samples from a normal parent at various levels of significance, in terms of the standard deviation computed with varying degrees of freedom, would be a valuable table. It is not clear whether he means that the standard deviation is to be estimated from the same sample as the range or from a second independent sample, as is done by Newman [3], Pearson and Hartley [4], and Hartley [5].

In natural hybridization of distinct types of plants and subsequent back crossing with parental types distinctly bimodal populations may develop. Heiser [6] has described such a situation for sunflowers. Similar situations may occur in natural and artificial crossing of peaches and apricots as shown by the work of Hesse [7] of this station. In studying such genetical material it often would be helpful to know the expected distributions of the sample ranges in terms of the sample standard deviations estimated from the same sample for certain typical nonhomogenous populations. Applications to such data will be published elsewhere.

Since the mathematical situation for the distributions of the sample range (R) in terms of the sample standard deviation (s) appears somewhat complex, empirical sampling methods were resorted to for obtaining the distributions for a normal parent (N), a symmetrical distinctly bimodal nonhomogeneous parent (A), and a weakly bimodal but strongly skewed parent (B). Populations A and B are pictured in charts A (p. 341) and B (p. 348) of [8].

Population N is approximately represented by

$$(N) \qquad \frac{1296}{5\sqrt{2\pi}} \exp. -\tfrac{1}{2}\frac{(X-15.5)^2}{25},$$

population A by

$$(A) \qquad \frac{648}{5\sqrt{2\pi}}\left(\exp. -\tfrac{1}{2}\frac{(X-15.5)^2}{25} + \exp. -\tfrac{1}{2}\frac{(X-32.5)^2}{25}\right),$$

and population B by

$$(B) \qquad \frac{972}{5\sqrt{2\pi}}\left(\exp. -\tfrac{1}{2}\frac{(X-15.5)^2}{25} + \tfrac{1}{2}\exp. -\tfrac{1}{2}\frac{(X-31.5)^2}{25}\right).$$

The method of drawing samples is the same as that originally described in [9]. N, A, and B each have a total area of 1296. Thus, 1296 integers distributed over a proper range and with the frequencies indicated by the corresponding areas under the curves N, A, and B were entered on charts with 6 big rows and 6 big columns of squares which were subdivided into 6 little rows and 6 little columns. In each case the 1296 integers were distributed in a non-systematic way among the 1296 little squares. By throwing 4 differentiated dice (one die assigned to a big row, one to a big column, one to a little row, and one to a little column) it was possible to draw random individuals from populations that are approximately N, A, and B.

Fisher [10] has defined g_1 which measures the skewness of a distribution and g_2 which measures the flatness. These g's are equivalent to the square root of β_1 and $\beta_2 - 3$, respectively in Karl Pearson's older notation. For population A, $g_1 = 0$ and $g_2 = -1.10$. For population B, $g_1 = 0.62$ and $g_2 = -0.29$.

TABLE 1

Distribution of range in terms of sample standard deviation for samples of specified sizes from a normal parent population (N), $g_1 = 0$, $g_2 = 0$

Sample Size	Number of Samples	Mean	Standard Devia-tion	g_1	Standard Error of g_1 (Normal)	g_2	Standard Error of g_2 (Normal)
2		1.4142	0.0	0.0		0 0	. .
4	1220	2.2238	0.1564	−0.660	0.0700	0.434	0.1400
16	305	3.5112	0.3879	0.115	0.1396	0.135	0.2783
36	135	4.4014	0.6076	0.607	0.2085	0.332	0.4142
64	76	4.8272	0.6409	0.492	0.2756	−0.751	0.5448
100	48	5.1215	0.6616	−0 077	0.3432	1.038	0.6744

2. Empirical random sampling results. The sample sizes considered are 2, 4, 16, 36, 64, 100. The distribution functions for various sample sizes are characterized by giving means, standard deviations, g_1's, and g_2's. The results are given in Tables 1, 2, and 3. The standard deviations of the samples were computed by dividing the sum of squares by one less than the number in the sample. When the size of the sample is two then the range divided by the standard deviation of the sample is always a constant, square root of 2.

The constants for the distributions for all sample sizes except four were computed without grouping. The constants for the distributions for samples of four were computed from grouped data with a small class interval.

3. Discussion. The mean values of the range divided by the standard deviation of the sample for population A run lower than for populations N and B. The standard deviations of the distributions for all parents increase from zero and continue to increase throughout the range considered for population N.

The standard deviations cut down much more quickly for population A than for population B. The values of g_1 and g_2 show that the distributions are significantly non-normal for certain sample sizes but perhaps not seriously so for other sample sizes.

The distributions of range divided by the sample standard deviation are quite different from the corresponding distributions of range in terms of the standard deviations of the population as can be seen by reference to the tables in [1].

TABLE 2

Distribution of range in terms of sample standard deviation for samples of specified sizes from a bimodal symmetrical population (A), $g_1 = 0$, $g_2 = -1.10$

Sample Size	Number of Samples	Mean	Standard Deviation	g_1	Standard Error of g_1 (Normal)	g_2	Standard Error of g_2 (Normal)
2		1.4142	0.0	0.0	.	0.0	. .
4	1040	2.2050	0.1551	−0.468	0.0758	−0.356	0.1515
16	259	3.5742	0.5283	1.025	0.1514	1.182	0.3015
36	115	4.0690	0.4604	0.561	0.2255	−0.279	0.4474
64	64	4.3194	0.3377	0.106	0.2993	−1.829	0.5905
100	41	4.4846	0.3194	0.426	0.3695	−0.890	0.7245

TABLE 3

Distribution of range in terms of sample standard deviation for samples of specified sizes from a skewed bimodal population (B), $g_1 = 0.62$, $g_2 = -0.29$

Sample Size	Number of Samples	Mean	Standard Deviation	g_1	Standard Error of g_1 (Normal)	g_2	Standard Error of g_2 (Normal)
2	.	1.4142	.	0.0		0.0	. .
4	1061	2.2258	0.1459	−0.470	0.0751	−0.142	0.1500
16	265	3.9277	0.5938	0.540	0.1496	0.405	0.2982
36	117	4.4792	0.5476	0.400	0.2236	0.018	0.4437
64	66	4.8485	0.5249	0.534	0.2950	1.028	0.5906
100	42	5.0481	0.3626	−0.092	0.3655	−0.632	0.7166

At the suggestion of the referee it is noted that the empirical results for the means in Table 1 are rather well approximated by $E(R)/E(s)$. It is necessary to remember that $E(s) \neq \sigma$ for small samples. For a discussion of $E(s)$ see Kenney [11] equation 28, page 135.

It is also noted that if

$$X = \log(\log \text{sample size} - \log 2)$$

$$Y = \log\left(\text{mean}\left(\frac{R}{s}\right) - \sqrt{2}\right)$$

then the plots of the (X, Y) values in each case are approximately straight lines for the present range in sample sizes.

The standard deviation and range when determined from the same sample are correlated. For the normal population this correlation decreases and practically disappears for samples of 100 or greater. This is not true for populations A and B. For these populations the correlation between sample range and sample standard deviation decreases much more slowly and seems to be of the order of 0.5 for samples of 100.

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NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute new items of interest

Personal Items

Dr. Theodore W. Anderson, Jr. of the Cowles Commission for Economic Research has been awarded a Guggenheim Memorial Foundation Fellowship.

Assistant Professor Theodore A. Bancroft of Iowa State College has been appointed to an associate professorship at the University of Georgia

Dr. Z. W. Birnbaum is now an associate professor in the mathematics department at the University of Washington.

Mr. Albert H. Bowker and Mr. Edward Paulson, formerly with the Statistical Research Group, of Columbia University, have been awarded pre-doctoral fellowships in mathematical statistics by the National Research Council. They are now studying at Columbia University.

Mr Oscar K. Buros, of Rutgers University, is Review Editor of the Journal of the American Statistical Association. He is making the review section a very important part of the Journal with such features as replicating reviews and bibliographies of statistical methodology. Members of the Institute who are authors of papers and books (both English and non-English) on statistical methodology are urged to send a reprint, review copy, or bibliographic information to Mr. Buros as soon after publication as possible.

Professor Harold Cramér, Director of the Institute of Mathematical Statistics at the University of Stockholm will be a visiting professor at Princeton University during the fall semester of the 1946–1947 academic year. He will give a course of graduate lectures on the theory of probability.

Dr. J. H. Curtiss has been appointed assistant to the Director of the National Bureau of Standards, where his duties will include the administration of the mathematical and statistical activities of the Bureau. Dr. Curtiss served in the U. S Naval Reserve during the war, and recently received a Commendation Ribbon from the Secretary of the Navy for his work in statistical engineering for the Bureau of Ships and the Office of the Commander-in-Chief. He will continue to be on leave of absence from Cornell University throughout the academic year 1946–1947 Administrative direction of the Mathematical Tables Project of the National Bureau of Standards has been assigned to Dr. Curtiss. Members of the Institute are cordially invited to visit the Project when in New York City, and to confer with the Project Director, Dr. Arnold Lowan, concerning their computational problems. The address of the Project is 150 Nassau Street, New York City. The Project is currently supported by funds transferred to the Bureau from the Office of Research and Inventions of the Navy Department An Advisory Panel of mathematicians interested in the computation of tables is being formed to define the long range program of the Project. An announce-

ment as to the personnl of this panel will appear in a later issue of the Annals.

Assistant Professor W. J. Dixon of the University of Oklahoma has been appointed to an associate professorship at the University of Oregon

Dr Hallett H. Germond has returned from war service to his teaching duties in the Department of Mathematics at the University of Florida

Dr. Earl L. Green, has accepted a position as Associate Professorof Zoology at Ohio State University.

Mr. John C. Hintermaier, formerly supervisory chemist with the Forstmann Woolen Company of Passaic has accepted a position as chief chemist of the Vanity Fair Mills at Reading.

Mr William Hodgkinson, Jr, has returned from war service to his position with the American Telephone and Telegraph Company at New York.

Mr. Robert H. Hoskins, discharged from the Navy in March, is employed in the Actuarial Ordinary General Division of the John Hancock Mutual Life Insurance Company at Boston.

A testimonial dinner was given to Professor Harold Hotelling on May 3, 1946 at the Columbia University Men's Faculty Club as a farewell by the Statistical Techniques Group, New York Chapter, American Statistical Association. Professor Hotelling is leaving Columbia at the end of the academic year to become Professor of Mathematical Statistics at the University of North Carolina. Professor Helen M. Walker, on behalf of the Group, presented gifts to Professor and Mrs. Hotelling. The Chairman, Professor Irving Lorge, introduced the distinguished visitors who came to honor Professor Hotelling Among the speakers were Professor P. C. Mahalanobis of Presidency College, Calcutta, India, Dr. Stuart Rice, Chairman of the Statistical Commission of the Economic and Social Council of the United Nations, and Dean Pegram of the Graduate Faculties of Columbia University. Professor Hotelling reviewed the changes in statistical theory and techniques that were developed during the 15 years of his professorship at Columbia University.

Mr. Calvin J. Kirchen, who has recently accepted a position with the technical department of Remington Arms Company at Bridgeport, Conn., addressed the Rochester Society of Quality Control Engineers on Sept. 17 on "The Applications of Sequential Analysis to Acceptance Inspection".

Dr. Walter Leighton of the Rice Institute has been appointed to a professorship at Washington University.

Miss Dorothy Marrow has been appointed to an assistant professorship at George Washington University

Professor D E. Morton of the National Bureau of Econlmic Research is joining the faculty of Cornell University

Assistant Professor Cecil J. Nesbitt of the University of Michigan has been promoted to an associate professorship.

Dr. A. C. Olshen has accepted a position as Actuary of the West Coast Life Insurance Company at San Francisco.

Mr. William B. Rice has opened an office as Consulting Business Statistician at 1011 South Los Angeles Street, Los Angeles.

Mr. John Salerno, formerly a draftsman (statistical) with the War Department is now Mathematician with the U. S. Coast and Geodetic Survey.

Assistant Professor Henry Scheffé of the Mathematics department of Syracuse University has been appointed associate professor of engineering at the University of California at Los Angeles. Professor Scheffé has been awarded a Guggenheim Memorial Foundation Fellowship.

Mr. William B. Simpson has returned from overseas and is attending the University of Chicago

Professor Geoge W. Tyler has returned to his position in the Mathematics Department at Virginia Polytechnic Institute, having spent two years at the University of California Division of War Research.

Professor W. Allen Wallis, who returned to his position at Stanford University in April after serving for nearly four years as Director of Research with the Statistical Research Group of Columbia University, has accepted a position as Professor of Statistics and Economics in the School of Business of the University of Chicago effective September 1, 1946.

Mr. Frank A. Weck who served during the war as a Captain in the Office of the Surgeon General is now in the Actuarial division of the Metropolitan Life Insurance Company.

The University of Pennsylvania held a conference on "Measurement of Consumers Interest" at Philadelphia on May 17–18, 1946. This conference was sponsored by the Departments of Philosophy, Psychology, Statistics, Marketing, and Foreign Commerce. Among the speakers were the following members of the Institute: Professor L. L. Thrustone of the University of Chicago, Professor Louis Guttman of Cornell University, Dr. W. Edwards Deming of the Bureau of the Budget, Professor C. West Churchman of the University of Pennsylvania, Dr. John H. Curtiss of the National Bureau of Standards, Professor Paul Peach of the University of North Carolina, and Professor S. S. Wilks of Princeton University.

The following four doctorates, with mathematical statistics as a major subject, were conferred during 1945 in the United States. The name, University, month in which the degree was conferred, and the title of the dessertation are given in each case:

T. W. Anderson, Jr., Princeton, June, "The Non-Central Wishart distribution and its application to Problems in Multivariate Statistics."

Frances Campbell, Michigan, June, "A Study of Truncated Bivariate Normal Distributions."

W. M. Chen, California, June, "Power Function of the Analysis of Variance and Convariance of a Normal Bivariate Population."

J. J. Livers, Michigan, February, "Use of Partitions in Multivariate Moment Sampling Theory."

Professor A. R. Crathorne of the University of Illinois, a Fellow of the Institute and one of its founders, died on March 7, 1946 at the age of 72

Announcement of New Preliminary Actuarial Examinations

On June 7, 1947, three new Preliminary Actuarial Examinations will be given to undergraduate students of mathematics and others who may be interested in going into the actuarial profession. These new examinations are sponsored jointly by the Actuarial Society of America and the American Institute of Actuaries.

The new series of examinations will replace Parts 1, 2, and 3 of the actuarial examinations which have been given heretofore, but will carry the same credit toward Associateship in the two actuarial organizations. These examinations have been prepared under the direction of a joint committee of actuaries and mathematicians. They will be administered by the College Entrance Examination Board at centers throughout the United States and Canada.

Descriptions of the three new examinations are as follows:

1. *Language Aptitude Examination.* This is a three-hour aptitude examination testing reading comprehension and precise knowledge of the meaning of words. It is similar to the well-known Scholastic Aptitude Test of the College Entrance Examination Board, except that it is pitched at approximately the college sophomore level. Verbal facility and command of the English language, as well as mathematical ability, are important in the actuarial profession. This is not the type of an examination for which specific preparation can be made; it is an aptitude rather than an achievement examination.

2. *General Mathematics Examination.* This is a three-hour achievement examination on material usually covered in the first two years of mathematics in colleges and universities in the United States and Canada. More specifically, it is based on college algebra, trigonometry, analytical geometry, and differential and integral calculus. It is designed to be taken by the mathematically talented undergraduate at the end of his sophomore year, although it is not restricted to this group.

3. *Special Mathematics Examination.* This is a three-hour achievement examination based on the material usually covered in undergraduate courses in finite differences, probability, and statistics. It is designed to be given at the end of the junior or senior year to college mathematics majors who have either taken courses or done concentrated reading in these fields, but it is not restricted to this group.

The two actuarial bodies will jointly award one $200 and eight $100 prizes to the nine highest-ranking contestants on the basis of performance on the first two of the examinations described above. In determining these awards the General Mathematics Examination will be weighted twice as much as the Language Aptitude Examination.

Information regarding these new examinations, and applications for taking them, may be obtained from either of the following organizations:

The Actuarial Society of America
393 Seventh Avenue
New York 1, New York

The American Institute of Actuaries
720 North Michigan Avenue
Chicago, Illinois

Announcement of Cowles Fellowships for Women

Two Sarah Frances Hutchinson Cowles Fellowships for women will be awarded by the University of Chicago for the academic year 1947–48 upon nomination by the Cowles Commission for Research in Economics. Applicants must be students of outstanding promise, preparing for the degree of master or doctor in the field of social sciences and statistics, preferably in quantitative economics or mathematical statistics. The Fellowships amount to $1000 each, but may be supplemented by an additional grant of $500 if the work of the Fellowship holder lies within the Cowles Commission's field of interest. Holders will be expected to be in residence at the University of Chicago. Application and supporting documents must be filed before March 1, 1947. Application blanks and further particulars may be secured from the Cowles Commission for Research in Economics, The University of Chicago, Chicago 37, Illinois, U. S. A.

New Members

The following persons have been elected to membership in the Institute:

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ON SOME USEFUL "INEFFICIENT" STATISTICS

By Frederick Mosteller

Princeton University

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Summary. Several statistical techniques are proposed for economically analyzing large masses of data by means of punched-card equipment; most of these techniques require only a counting sorter. The methods proposed are designed especially for situations where data are inexpensive compared to the cost of analysis by means of statistically "efficient" or "most powerful" procedures. The principal technique is the use of functions of order statistics, which we call *systematic statistics.*

It is demonstrated that certain order statistics are asymptotically jointly distributed according to the normal multivariate law.

For large samples drawn from normally distributed variables we describe and give the efficiencies of rapid methods:

i) for estimating the mean by using 1, 2, $\cdots$, 10 suitably chosen order statistics; (cf p. 386)

ii) for estimating the standard deviation by using 2, 4, or 8 suitably chosen order statistics; (cf. p. 389)

iii) for estimating the correlation coefficient whether other parameters of the normal bivariate distribution are known or not (three sorting and three counting operations are involved) (cf. p. 394).

The efficiencies of procedures ii) and iii) are compared with the efficiencies of other estimates which do not involve sums of squares or products

1. Introduction. The purpose of this paper is to contribute some results concerning the use of order statistics in the statistical analysis of large masses of data. The present results deal particularly with estimation when normally distributed variables are present. Solutions to all problems considered have

been especially designed for use with punched-card equipment although for most of the results a counting sorter is adequate.

Until recently mathematical statisticians have spent a great deal of effort developing "efficient statistics" and "most powerful tests." This concentration of effort has often led to neglect of questions of economy. Indeed some may have confused the meaning of technical statistical terms "efficient" and "efficiency" with the layman's concept of their meaning. No matter how much energetic activity is put into analysis and computation, it seems reasonable to inquire whether the output of information is comparable in value to the input measured in dollars, man-hours, or otherwise. Alternatively we may inquire whether comparable results could have been obtained by smaller expenditures In some fields where statistics is widely used, the collection of large masses of data is inexpensive compared to the cost of analysis. Often the value of the statistical information gleaned from the sample decreases rapidly as the time between collection of data and action on their interpretation increases. Under these conditions, it is important to have quick, inexpensive methods for analyzing data, because economy demands militate against the use of lengthy, costly (even if more precise) statistical methods. A good example of a practical alternative is given by the control chart method in the field of industrial quality control. The sample range rather than the sample standard deviation is used almost invariably in spite of its larger variance. One reason is that, after brief training, persons with slight arithmetical knowledge can compute the range quickly and accurately, while the more complicated formula for the sample standard deviation would create a permanent stumbling block. Largely as a result of simplifying and routinizing statistical methods, industry now handles large masses of data on production adequately and profitably. Although the sample standard deviation can give a statistically more efficient estimate of the population standard deviation, if collection of data is inexpensive compared to cost of analysis and users can compute a dozen ranges to one standard deviation, it is easy to see that economy lies with the less efficient statistic.

It should not be thought that inefficient statistics are being recommended for all situations There are many cases where observations are very expensive, and obtaining a few more would entail great delay. Examples of this situation arise in agricultural experiments, where it often takes a season to get a set of observations, and where each observation is very expensive. In such cases the experimenters want to squeeze every drop of information out of their data. In these situations inefficient statistics would be uneconomical, and are not recommended.

A situation that often arises is that data are acquired in the natural course of administration of an organization. These data are filed away until the accumulation becomes mountainous. From time to time questions arise which can be answered by reference to the accumulated information. How much of these data will be used in the construction of say, estimates of parameters, depends on the precision desired for the answer. It will however often be less expensive to

get the desired precision by increasing the sample size by dipping deeper into the stock of data in the files, and using crude techniques of analysis, than to attain the required precision by restricting the sample size to the minimum necessary for use with "efficient" statistics.

It will often happen in other fields such as educational testing that it is less expensive to gather enough data to make the analysis by crude methods sufficiently precise, than to use the minimum sample sizes required by more refined methods. In some cases, as a result of the type of operation being carried out sample sizes are more than adequate for the purposes of estimation and testing significance. The experimenters have little interest in milking the last drop of information out of their data. Under these circumstances statistical workers would be glad to forsake the usual methods of analysis for rapid, inexpensive techniques that would offer adequate information, but for many problems such techniques are not available.

In the present paper several such techniques will be developed. For the most part we shall consider statistical methods which are applicable to estimating parameters. In a later paper we intend to consider some useful "inefficient" tests of significance.

2. Order statistics. If a sample $O_n = x'_1, x'_2, \cdots, x'_n$ of size n is drawn from a continuous probability density function $f(x)$, we may rearrange and renumber the observations within the sample so that

$$x_1 < x_2 < \cdots < x_n \tag{1}$$

(the occurrence of equalities is not considered because continuity implies zero probability for such events). The x_i's are sometimes called *order statistics*. On occasion we write $x(i)$ rather than x_i. Throughout this paper the use of primes on subscripted x's indicates that the observations are taken without regard to order, while unprimed subscripted x's indicate that the observations are order statistics satisfying (1). Similarly $x(n_i)$ will represent the n_ith order statistic, while $x'(n_i)$ would represent the n_ith observation, if the observations were numbered in some random order. The notation here is essentially the *opposite* of usual usage, in which attention is called to the order statistics by the device of primes or the introduction of a new letter. The present reversal of usage seems justified by the viewpoint of the article—that in the problems under consideration the use of order statistics is the natural procedure.

An example of a useful order statistic is the median; when $n = 2m + 1$ ($m = 0, 1, \cdots$), x_{m+1} is called the median and may be used to estimate the population median, i.e. u defined by

$$\int_{-\infty}^{u} f(t)\, dt = \tfrac{1}{2}.$$

In the case of symmetric distributions, the population mean coincides with u and x_{m+1} will be an unbiased estimate of it as well. When $n = 2m$ ($m = 1, 2,$

$\cdots$), the median is often defined as $\frac{1}{2}(x_m + x_{m+1})$. The median so defined is an unbiased estimate of the population median in the case of symmetric distributions; however for most asymmetric distributions $\frac{1}{2}(x_m + x_{m+1})$ will only be unbiased asymptotically, that is in the limit as n increases without bound. For another definition of the sample median see Jackson [8, 1921]. When x is distributed according to the normal distribution

$$N(x, a, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(1/2\sigma^2)(x-a)^2},$$

the variance of the median is well known to tend to $\pi\sigma^2/2n$ as n increases.

It is doubtful whether we can accurately credit anyone with the introduction of the median. However for some of the results in the theory of order statistics it is easier to give credit. In this section we will restrict the discussion to the order statistics themselves, as opposed to the general class of statistics, such as the range $(x_n - x_1)$, which are derived from order statistics We shall call the general class of statistics which are derived from order statistics, and use the value ordering (1) in their construction, *systematic statistics*.

The large sample distribution of extreme values (examples x_r, x_{n-s+1} for r, s fixed and $n \to \infty$) has been considered by Tippett [17, 1925] in connection with the range of samples drawn from normal populations; by Fisher and Tippett [3, 1928] in an attempt to close the gap between the limiting form of the distribution and results tabled by Tippett [17], by Gumbel [5, 1934] (and in many other papers, a large bibliography is available in [6, Gumbel 1939]), who dealt with the more general case $r \geq 1$, while the others mentioned considered the special case of $r = 1$, and by Smirnoff who considers the general case of x_r, in [15, 1935] and also [16] the limiting form of the joint distribution of x_r, x_s, for r and s fixed as $n \to \infty$.

In the present paper we shall not usually be concerned with the distribution of extreme values, but shall rather be considering the limiting form of the joint distribution of $x(n_1)$, $x(n_2)$, $\cdots$, $x(n_k)$, satisfying

Condition 1. $\lim_{n\to\infty} \frac{n_i}{n} = \lambda_i; \qquad i = 1, 2, \cdots, k;$

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k .$$

In other words the proportion of observations less than or equal to $x(n_i)$ tends to a fixed proportion which is bounded away from 0 and 1 as n increases. K. Pearson [13, 1920] supplies the information necessary to obtain the limiting distribution of $x(n_1)$, and limiting joint distribution of $x(n_1)$, $x(n_2)$. Smirnoff gives more rigorous derivations of the limiting form of the marginal distribution of the $x(n_i)$ [15, 1935] and the limiting form of the joint distribution of $x(n_i)$ and $x(n_j)$ [16] under rather general conditions. Kendall [10, 1943, pp. 211–14] gives a demonstration leading to the limiting form of the joint distribution.

Since we will be concerned with statements about the asymptotic properties of the distributions of certain statistics, it may be useful to include a short dis-

cussion of their implications both practical and theoretical. If we have a statistic $\hat{\theta}(O_n)$ based on a sample $O_n: x_1', x_2', \cdots, x_n'$ drawn from a population with cumulative distribution function $F(x)$ it often happens that the function $(\hat{\theta} - \theta)/\sigma_n = y_n$, where σ_n is a function of n is such that

$$\text{(A)} \qquad \lim_{n\to\infty} P(y_n < t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\, dx.$$

When this condition (A) is satisfied we often say: *$\hat{\theta}$ is asymptotically normally distributed with mean θ and variance σ_n^2.* We will not be in error if we use the statement in italics provided we interpret it as synonymous with (A). However there are some pitfalls which must be avoided In the first place condition (A) may be true even if the distribution function of y_n, or of $\hat{\theta}$, has no moments even of fractional orders for any n. Consequently we do not imply by the italicized statement that $\lim_{n\to\infty} E[\hat{\theta}(O_n)] = \theta$, nor that $\lim_{n\to\infty} \{[E(\hat{\theta}^2) - [E(\hat{\theta})]^2\} = \sigma_n^2$, for, as mentioned, these expressions need not exist for (A) to be true. Indeed we shall demonstrate that Condition (A) is satisfied for certain statistics even if their distribution functions are as momentless as the startling distributions constructed by Brown and Tukey [1, 1946]. Of course it may be the case that all moments of the distribution of $\hat{\theta}$ exist and converge as $n \to \infty$ to the moments of a normal distribution with mean θ and variance σ_n^2. Since this implies (A), but not conversely, this is a stronger convergence condition than (A). (See for example J H. Curtiss [2, 1942].) However the important implication of (A) is that for sufficiently large n each percentage point of the distribution of $\hat{\theta}$ will be as close as we please to the value which we would compute from a normal distribution with mean θ and variance σ_n^2, independent of whether the distribution of $\hat{\theta}$ has these moments or not.

Similarly if we have several statistics $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k$, each depending upon the sample $O_n: x_1', x_2', \cdots, x_n'$, we shall say *that the $\hat{\theta}_i$ are asymptotically jointly normally distributed with means θ_i, variances $\sigma_i^2(n)$, and covariances $\rho_{ij}\sigma_i\sigma_j$*, when

$$\text{(B)} \qquad \begin{aligned} &\lim_{n\to\infty} P(y_1 < t_1, y_2 < t_2, \cdots, y_k < t_k) \\ &\qquad = K \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_k} e^{-\frac{1}{2}Q^2}\, dx_1\, dx_2 \cdots dx_k, \end{aligned}$$

where $y_i = (\hat{\theta}_i - \theta_i)/\sigma_i$, and Q^2 is the quadratic form associated with a set of k jointly normally distributed variables with variances unity and covariances ρ_{ij}, and K is a normalizing constant. Once again the statistics $\hat{\theta}_i$ may not have moments or product moments, the point that interests us is that the probability that the point with coordinates $(\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k)$ falls in a certain region in a k-dimensional space can be given as accurately as we please for sufficiently large samples by the right side of (B).

Since the practicing statistician is very often really interested in the probability that a point will fall in a particular region, rather than in the variance

or standard deviation of the distribution itself, the concepts of asymptotic normality given in (A) and (B) will usually not have unfortunate consequences. For example, the practicing statistician will usually be grateful that the sample size can be made sufficiently large that the probability of a statistic falling into a certain small interval can be made as near unity as he pleases, and will not usually be concerned with the fact that, say, the variance of the statistic may be unbounded.

Of course, a very real question may arise: how large must n be so that the probability of a statistic falling within a particular interval can be sufficiently closely approximated by the asymptotic formulas? If in any particular case the sample size must be ridiculously large, asymptotic theory loses much of its practical value. However for statistics of the type we shall usually discuss, computation has indicated that in many cases the asymptotic theory holds very well for quite small samples

For the demonstration of the joint asymptotic normality of several order statistics we shall use the following two lemmas.

LEMMA 1. *If a random variable $\hat{\theta}(O_n)$ is asymptotically normally distributed converging stochastically to θ, and has asymptotic variance $\sigma^2(n) \underset{n\to\infty}{\longrightarrow} 0$, where n is the size of the sample $O_n : x_1', x_2', \cdots, x_n'$, drawn from the probability density function $h(x)$, and $g(\hat{\theta})$ is a single-valued function with a nonvanishing continuous derivative $g'(\hat{\theta})$ in the neighborhood of $\hat{\theta} = \theta$, then $g(\hat{\theta})$ is asymptotically normally distributed converging stochastically to $g(\theta)$ with asymptotic variance $\sigma_n^2[g'(\theta)]^2$*

PROOF. By the conditions of the lemma

$$\lim_{n\to\infty} P\left[\frac{\hat{\theta}-\theta}{\sigma_n} < t\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\frac{1}{2}u^2}\,du.$$

Now if $t\sigma_n = \Delta\theta$, $\Delta\theta = \hat{\theta} - \theta$, using the mean value theorem there is a θ_1 in the interval $[\theta, \hat{\theta}]$, such that

$$g(\hat{\theta}) = g(\theta) + (\hat{\theta} - \theta)g'(\theta_1),$$

which implies

$$\lim_{n\to\infty} P\left(\frac{\hat{\theta}-\theta}{\sigma_n} < t\right) = \lim_{n\to\infty} P\left(\frac{g(\hat{\theta})-g(\theta)}{\sigma_n g'(\theta_1)} < t\right), \qquad g'(\theta_1) \neq 0,$$

where θ_1 is a function of n. However $\lim_{\Delta\theta\to 0} g'(\theta_1) = g'(\theta)$ so we may write

$$\lim_{n\to\infty} P\left(\frac{\hat{\theta}-\theta}{\sigma_n} < t\right) = \lim_{n\to\infty} P\left(\frac{g(\hat{\theta})-g(\theta)}{\sigma_n g'(\theta)} < t\right), \qquad g'(\theta) \neq 0.$$

where the form of the expression on the right is the one required to complete the proof of the lemma.

Of course if we have several random variables $\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_k$, we can prove by an almost identical argument that

LEMMA 2. *If the random variables $\hat{\theta}_i(O_n)$ are asymptotically jointly normally*

distributed converging stochastically to θ_i, *and have asymptotic variances* $\sigma_i^2(n) \underset{n\to\infty}{\longrightarrow} 0$, *and covariances* $\rho_{ij}\sigma_i\sigma_j$, *where* n *is the size of the sample* $O_n : x_1', x_2', \cdots, x_n'$ *drawn from the probability density function* $h(x)$, *and* $g_i(\hat{\theta}_i)$, $i = 1, 2, \cdots, k$, *are single-valued functions with nonvanishing continuous derivatives* $g_i'(\hat{\theta}_i)$ *in the neighborhood of* $\hat{\theta}_i = \theta_i$, *then the* $g_i(\hat{\theta}_i)$ *are jointly asymptotically normally distributed with means* $g_i(\theta_i)$, *variances* $\sigma_i^2[g_i'(\theta_i)]^2$ *and covariances* $\rho_{ij}\sigma_i\sigma_j g_i'(\theta_i)g_j'(\theta_j)$

The following condition represents restrictions on the probability density function $f(x)$ sufficient for the derivation of the limiting form of the joint distribution of the $x(n_i)$ satisfying Condition 1

CONDITION 2. *The probability density function* $f(x)$ *is continuous, and does not vanish in the neighborhood of* u_i, *where*

$$\int_{-\infty}^{u_i} f(x)\, dx = \lambda_i, \qquad i = 1, 2, \cdots, k.$$

If we recall the discussion of condition (*B*) above, the theorem of Pearson and Smirnoff may be stated:

THEOREM 1. *If a sample* $O_n : x_1, x_2, \cdots, x_n$ *is drawn from* $f(x)$ *satisfying Condition* 2, *and if* $x(n_1)$, $x(n_2)$ *satisfy Condition* 1 *as* $n \to \infty$, *then* $x(n_1)$, $x(n_2)$ *are asymptotically distributed according to the normal bivariate distribution with means* u_1, u_2,

$$\int_{-\infty}^{u_i} f(x)\, dx = \lambda_i,$$

and variances

$$\sigma_i^2 = \frac{\lambda_i(1 - \lambda_i)}{n[f(u_i)]^2}, \qquad i = 1, 2,$$

and covariance

$$\rho_{12}\sigma_1\sigma_2 = \frac{\lambda_1(1 - \lambda_2)}{nf(u_1)f(u_2)}.$$

Theorem 1 has an obvious generalization which seems not to have been carried out in the literature The generalization may be stated:

THEOREM 2 *If a sample* $O_n : x_1, x_2, \cdots, x_n$ *is drawn from* $f(x)$ *satisfying Condition* 2, *and if* $x(n_1), x(n_2), \cdots, x(n_k)$ *satisfy Condition* 1 *as* $n \to \infty$, *then the* $x(n_i)$, $i = 1, 2, \cdots, k$, *are asymptotically distributed according to the normal multivariate distribution, with means* u_i,

$$\int_{-\infty}^{u_i} f(x)\, dx = \lambda_i,$$

and variances

$$\sigma_i^2 = \frac{\lambda_i(1 - \lambda_i)}{nf(u_i)^2}, \qquad i = 1, 2, \cdots, k,$$

and covariances

$$\rho_{ij}\sigma_i\sigma_j = \frac{\lambda_i(1-\lambda_j)}{nf(u_i)f(u_j)}, \qquad 1 \le i < j \le k.$$

PROOF. We shall carry out the demonstration for the uniform distribution

$$f(x) = \begin{cases} 1, \; 0 \le x \le 1, \\ 0, \text{ elsewhere,} \end{cases}$$

and then utilize the fact that by a suitable transformation of the uniform distribution we may get any $f(x)$ satisfying Condition 2. Of course for the particular case of the uniform distribution all moments of the $x(n_i)$ exist and converge to those of the asymptotic theory.

The joint probability density of the $x(n_i)$, satisfying Condition 1 and drawn from $f(x)$, is given by

$$(2) \quad g[x(n_1), x(n_2), \cdots, x(n_k)] = \frac{n!}{(n_1-1)!(n-n_k)!\prod_{i=2}^{k}(n_i - n_{i-1} - 1)!} \left(\int_0^{x(n_1)} dt_1\right)^{n_1-1} \left(\int_{x(n_k)}^{1} dt_{k+1}\right)^{n-n_k} \prod_{i=2}^{k}\left[\int_{x(n_{i-1})}^{x(n_i)} dt_i\right]^{n_i-n_{i-1}-1}.$$

Performing the indicated integrations we get from the right of (2)

$$(3) \qquad Cx(n_1)^{n_1-1}\prod_{i=2}^{k}[x(n_i) - x(n_{i-1})]^{n_i-n_{i-1}-1}[1-x(n_k)]^{n-n_k},$$

where C is the multinomial coefficient on the right of (2). It is well known that for the uniform distribution $E[x(n_i)] = \dfrac{n_i}{n+1}$, or asymptotically $\dfrac{n_i}{n}$, $i = 1, 2, \cdots, k$. We make the transformation $y_i = \left(x(n_i) - \dfrac{n_i}{n}\right)\sqrt{n}$, leading to

$$(4) \quad C_1\left(\frac{n_1}{n} + \frac{y_1}{\sqrt{n}}\right)^{n_1-1}\prod_{i=2}^{k}\left(\frac{n_i - n_{i-1}}{n} + \frac{[y_i - y_{i-1}]}{\sqrt{n}}\right)^{n_i-n_{i-1}-1} \cdot \left(\frac{n-n_k}{n} - \frac{y_k}{\sqrt{n}}\right)^{n-n_k}.$$

Using the usual technique of factoring out expressions like

$$\left(\frac{n_i - n_{i-1}}{n}\right)^{n_i-n_{i-1}-1},$$

we rewrite (4) with C_2 as a new constant, and setting $\lambda_i = \dfrac{n_i}{n}$

$$(5) \quad C_2\left(1 + \frac{y_1}{\lambda_1\sqrt{n}}\right)^{n_1-1} \cdot \prod_{i=2}^{k}\left(1 + \frac{(y_i - y_{i-1})}{(\lambda_i - \lambda_{i-1})\sqrt{n}}\right)^{n_i-n_{i-1}-1}\left(1 - \frac{y_k}{(1-\lambda_k)\sqrt{n}}\right)^{n-n_k}.$$

Now taking the logarithm of (5), expanding, neglecting terms $O\left(\frac{1}{\sqrt{n}}\right)$ and higher, collecting terms and taking the antilogarithm we get the approximate asymptotic distribution of the order statistics

$$
\begin{aligned}
&g(x(n_1), x(n_2), \cdots, x(n_k)) \\
(6)\qquad &= C_3 \exp\left[-\frac{1}{2}\left\{\sum_{i=1}^{k} y_i^2 \frac{\lambda_{i+1} - \lambda_{i-1}}{(\lambda_{i+1} - \lambda_i)(\lambda_i - \lambda_{i-1})} - 2\sum_{i=2}^{k} \frac{y_i y_{i-1}}{\lambda_i - \lambda_{i-1}}\right\}\right],
\end{aligned}
$$

where $\lambda_0 = 0$, $\lambda_{k+1} = 1$. Now setting up the matrix of the coefficients of the quadratic expression in the exponent

$$A_{ii} = \frac{\lambda_{i+1} - \lambda_{i-1}}{(\lambda_{i+1} - \lambda_i)(\lambda_i - \lambda_{i-1})}; \qquad A_{i,i-1} = A_{i-1,i} = -\frac{1}{\lambda_i - \lambda_{i-1}},$$

$i = 1, 2, \cdots, k$; $A_{ij} = 0$, $|i - j| > 1$. To obtain the variances and covariances we need

$$A^{ij} = \frac{\text{cofactor of } A_{ij} \text{ in } \|A_{ij}\|}{\text{determinant } A_{ij}}$$

(see for example Wilks [18, p. 63 et seq.]). Now

$$
\begin{aligned}
(7)\qquad &|A| = \text{determinant } A_{ij} = \prod_{1}^{k+1} \frac{1}{\lambda_i - \lambda_{i-1}}; \\
&\text{cofactor of } A_{ii} = \lambda_i(1 - \lambda_i)\,|A|,\ i = 1, 2, \cdots, k. \\
&\text{cofactor of } A_{ij} = \begin{cases} \lambda_i(1 - \lambda_j)\,|A|, & i < j \\ \lambda_j(1 - \lambda_i)\,|A|, & j < i. \end{cases}
\end{aligned}
$$

This completes the proof for the uniform distribution.

If the uniform distribution is transformed into a probability density function $f(x)$ satisfying Condition 2, by an order preserving transformation, we appeal to Lemma 2. We notice that the $x(n_i)$ are transformed into $g[x(n_i)]$, and that the probability that $x(n_i)$ falls in the interval $[u_i, u_i + \Delta u_i]$ is transformed into the probability that $g[x(n_i)]$ falls in the interval $[g(u_i), g(u_i + \Delta u_i)]$. Using the mean value theorem we may write

$$g(u_i + \Delta u_i) = g(u_i) + \Delta u_i g'(u_i'),$$

where u_i' lies in the interval $[u_i, u_i + \Delta u_i]$. However

$$\lim_{\Delta u_i \to 0} g'(u_i') = g'(u_i).$$

The density for the uniform distribution in the interval $[u_i, u_i + \Delta u_i]$ is just Δu_i, and this same density will tend to $f(u_i)\Delta u_i g'(u_i)$. Therefore $g'(u_i) = 1/f(u_i)$, which completes the proof of Theorem 2.

It would often be useful to know the small sample distribution of the order statistics, particularly in the case where the sample is drawn from a normal.

Fisher and Yates' tables [4] give the expected values of the order statistics up to samples of size 50 However it would be very useful in the development of certain small sample statistics to have further information. It is perhaps too much to expect tabulated distribution functions, but at least the variances and covariances would be useful. A joint effort has resulted in the calculation for samples $n = 2, 3, \cdot \cdot, 10$ of the expected values to five decimal places, the variances to four decimal places, and the covariances to nearly two decimal places. It is expected that these tables will be published shortly.

3. Estimates of the mean of a normal distribution. It will be important in what follows to define efficiency and to indicate its interpretation. Then we shall construct some estimates of the means of certain distributions and compute their efficiencies. Except for the tables given, the discussion is applicable to the estimation of the mean of any symmetric distribution; and, of course, the concept of efficiency is still more general in its application. A statistic $\hat{\theta}(O_n)$, where O_n is the sample, is said to be an *efficient* estimate of θ if

i) $\sqrt{n}\,(\hat{\theta} - \theta)$ is asymptotically normally distributed with zero mean and finite variance, $\sigma^2(\hat{\theta})$, and
ii) for any other statistic $\hat{\theta}'$ with $\sqrt{n}(\hat{\theta}' - \theta)$ asymptotically normally distributed with zero mean and variance $\sigma^2(\hat{\theta}')$, $\sigma^2(\hat{\theta}) \leq \sigma^2(\hat{\theta}')$

The ratio $\sigma^2(\hat{\theta})/\sigma^2(\hat{\theta}')$ is termed the efficiency of $\hat{\theta}'$ if $\hat{\theta}$ is an efficient estimate of θ. For discussion see Wilks [18, 1943]. The concepts of efficient statistic or estimate and of efficiency were introduced by R. A. Fisher They serve as one measure of the amount of information a statistic draws from a sample. It is also common practice to speak of relative efficiencies, for example, of the statistics $\hat{\theta}'$ and $\hat{\theta}''$ described in ii) above, we say if $\sigma^2(\hat{\theta}') < \sigma^2(\hat{\theta}'')$ that the efficiency of $\hat{\theta}''$ relative to $\hat{\theta}'$ is the ratio of the smaller variance to the larger. This concept of efficiency has sometimes been used when the normality assumption has been violated by one or both statistics, when one or both are biased, and when small samples are considered. When used under these conditions the concept of efficiency becomes more difficult to interpret, although a comparison of the variation of two statistics about the value they are commonly estimating is often of value

In the case of estimates of the mean a of a variable which is normally distributed according to $N(x, a, \sigma^2)$ from a sample of n, we can often express the variance of an asymptotically unbiased estimate as $\sigma^2(\hat{\theta}_i) = k_i\sigma^2/n$. The sample mean $\hat{\theta} = \Sigma x_i'/n$ is an efficient estimate of a with variance σ^2/n. Then in such cases the efficiency of $\hat{\theta}_i$ in estimating a is $1/k_i$. The interpretation is merely that to obtain the same precision using $\hat{\theta}_i$ as is possible with $\hat{\theta}$, one must use a sample k_i times as large

Bearing in mind that we are at present searching for economical methods for analyzing large samples, it is clear that the concept of efficiency offers us a practical way of comparing cost of information with cost of obtaining it.

In the present section and in sections 4 and 5 we shall develop certain systematic estimates of parameters of normally distributed variables. Our procedure then will be to compare the efficiency of the systematic estimates with the efficient statistic for estimating the parameter in question, and also in sections 4 and 5 we compare our estimates with a statistic not involving squares or products. Of course the efficient statistic for estimating the mean of a normal is the sample mean, therefore in this section we will only compare our estimates with the sample mean.

We can construct unbiased estimates of the mean of a normal distribution from linear combinations of suitably chosen order statistics. These systematic statistics will be asymptotically normally distributed if the order statistics from which they are derived satisfy Condition 1. We will restrict ourselves to a useful practical case where equal weights are used. In other words the estimate discussed is just the average of k order statistics $k^{-1}\Sigma x(n_i)$. Suppose $x(n_i)$, $i = 1, 2, \cdots, k$ satisfy Condition 1, that $E[x(n_i)] = E[x(n_{k-i+1})]$, so that $E[\Sigma x(n_i)] = a$. An important unsolved question is to discover what spacing of the $x(n_i)$ will yield minimum variance, and thereafter at what rate does the efficiency of this optimumly spaced estimate increase with k. Computational methods bog down rapidly after $k = 3$. Because so little is known about this problem it seems worthwhile to offer some results for three arbitrary spacings (these results are of course useful in analyzing data).

If the $x(n_i)$ satisfy Theorem 2 we may approximate the variance of the systematic statistic $\hat{\theta}_k = \Sigma x(n_i)/k$ by the usual formula

$$\sigma^2(\hat{\theta}_k) = E[\Sigma x(n_i)/k]^2 - [E(\Sigma x(n_i)/k)]^2. \tag{8}$$

We lose no generality by assuming the mean and variance of the underlying normal to be 0 and 1 respectively Then using the fact that $\Sigma u_i = 0$, and the result of Theorem 1 we rewrite (8) as

$$\sigma^2(\hat{\theta}_k) = E[\Sigma(x(n_i) - u_i)/k]^2 = \frac{1}{k^2 n}\left[\sum_{i=1}^{k} \frac{\lambda_i(1-\lambda_i)}{f_i^2} + 2\sum_{i<j} \frac{\lambda_i(1-\lambda_j)}{f_i f_j}\right], \tag{9}$$

where $f_m = f(u_m)$.

Using the symmetry which makes $\lambda_i = 1 - \lambda_{k-i+1}$, $f_i = f_{k-i+1}$, and the fact that for $k = 2r + 1$, $f_{r+1} = 1/\sqrt{2\pi}$, $\lambda_{r+1} = \frac{1}{2}$, we may simplify the right side of equation (9) with the following results for $k = 1, 2, \cdots, 7$. The factor $1/k^2$ has not been disturbed. We also write the general formulas for the simplified form of (9), but we omit a rather lengthy combinatorial argument which establishes the generalization

$$k = 1:\ \frac{\pi}{2n}$$

$$k = 2:\ \frac{2\lambda_1}{4nf_1^2}$$

$$
(10)\quad
\begin{aligned}
k &= 3:\ \frac{2}{9n}\left[\frac{\lambda_1}{f_1^2} + \frac{\lambda_1\sqrt{2\pi}}{f_1} + \frac{\pi}{4}\right]\\
k &= 4:\ \frac{2}{16n}\left[\frac{\lambda_1}{f_1^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{\lambda_2}{f_2^2}\right]\\
k &= 5:\ \frac{2}{25n}\left[\frac{\lambda_1}{f_1^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{\lambda_2}{f_2^2} + \sqrt{2\pi}\left(\frac{\lambda_1}{f_1} + \frac{\lambda_2}{f_2}\right) + \frac{\pi}{4}\right]\\
k &= 6:\ \frac{2}{36n}\left[\frac{\lambda_1}{f_1^2} + \frac{\lambda_2}{f_2^2} + \frac{\lambda_3}{f_3^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{2\lambda_1}{f_1 f_3} + \frac{2\lambda_2}{f_2 f_3}\right]\\
k &= 7:\ \frac{2}{49n}\left[\frac{\lambda_1}{f_1^2} + \frac{\lambda_2}{f_2^2} + \frac{\lambda_3}{f_3^2} + \frac{2\lambda_1}{f_1 f_2} + \frac{2\lambda_1}{f_1 f_3} + \frac{2\lambda_2}{f_2 f_3}\right.\\
&\qquad\qquad\left. + \sqrt{2\pi}\left(\frac{\lambda_1}{f_1} + \frac{\lambda_2}{f_2} + \frac{\lambda_3}{f_3}\right) + \frac{\pi}{4}\right]\\
k &= 2r:\ \frac{2}{(2r)^2 n}\left[\sum_{i=1}^{r}\frac{\lambda_i}{f_i^2} + 2\sum_{1\le i<j\le r}\frac{\lambda_i}{f_i f_j}\right], \qquad r \ge 1\\
k &= 2r+1:\ \frac{2}{(2r+1)^2 n}\left[\frac{(2r)^2 n}{2}\sigma^2(\hat{\theta}_{2r}) + \sqrt{2\pi}\sum_{i=1}^{r}\frac{\lambda_i}{f_i} + \frac{\pi}{4}\right], \qquad r \ge 1.
\end{aligned}
$$

In addition to the possibility of minimizing the equations of (10) by numerical methods, three other procedures suggest themselves: i) to space the order statistics uniformly in probability; ii) to choose those k order statistics whose expected values are equal to the expected values of the order statistics in a sample of size k drawn from a unit normal; iii) to choose $\lambda_i = (i - \frac{1}{2})/k$. The following table lists for $k = 1, 2$, and 3 the expected values u_i of the order statistics and the probability to the left of the expected values λ_i for each of the procedures. The chosen order statistics are counted from left to right. It will be noticed that the third method gives very good results, and has the value of simplicity of formula. The following table gives a comparison between the efficiencies resulting from spacing by the three methods. The three optimum cases are included for completeness.

Statisticians planning to use the method of expected values suggested above will find Fisher and Yates [4, 1943] table of the expected values of the order statistics in samples of size k drawn from a unit normal helpful for computing the λ_i. Alternatively the following table of λ_i might be used.

As an example of the use of Table III, suppose we are using the expected value method for estimating the mean of a large sample drawn from a normal distribution $N(x, a, \sigma^2)$. If we are willing to use 6 observations out of 1000 for this purpose Table III indicates the selection of x_{103}, x_{261}, x_{421}, x_{580}, x_{740}, x_{898}. Furthermore Table II indicates that the variance of the estimate of a based on the average of these six observations will be approximately $\sigma^2/.948n$, $n = 1000$.

4. Estimates of the standard deviation. The statistic

$$s^2 = \sum_{i=1}^{n} (x'_i - \bar{x})^2/(n - 1),$$

TABLE I

Comparison of the order statistics which would be chosen according to each of the four procedures for subsamples of k = 1, 2, 3

k	Order Statistic	Optimum		Equal Probability		Expected Values		$\lambda_i = (i - \frac{1}{2})/k$	
		u_i	λ_i	u_i	λ_i	u_i	λ_i	u_i	λ_i
1	First	.0000	.5000	.0000	.5000	.0000	.5000	.0000	.5000
2	First	−.6121	.2702	−.4307	.3333	−.5642	.2863	−.6745	.2500
	Second	.6121	.7298	.4307	.6667	.5642	.7137	.6745	.7500
3	First	−.9056	.1826	−.6745	.2500	−.8463	.1967	−.9674	.1667
	Second	.0000	.5000	.0000	.5000	.0000	.5000	.0000	.5000
	Third	.9056	.8174	.6745	.7500	.8463	.8033	.9674	.8333

TABLE II

Comparison of the efficiencies of four methods of spacing k order statistics used in the construction of an estimate of the mean

k	$\lambda_i = i/(k+1)$	Expected Values*	$\lambda_i = (i - \frac{1}{2})/k$	Optimum
1	.637	.637	.637	.637
2	.793	.809	.808	.810
3	.860	.878	.878	.879
4	.896	.914	913	
5	918	.933	934	
6	.933	.948	.948	
7	.944	.956	.957	
8	.952	.963	.963	
9	.957	.968	.969	
10	.962	.972	.973	

* The u_i are chosen equal to the expected values of the order statistics of a sample of size k.

where $\bar{x} = \sum_{i=1}^{n} x'_i/n$ is well known to be an unbiased estimate of the population variance σ^2, for $n > 1$. However s is not in general an unbiased estimate of σ. We are not interested here in the question of when we should estimate σ and when it is more advantageous to estimate σ^2. All we want is to have an

unbiased estimate of σ, based on sums of squares, to compare with another unbiased estimate based on order statistics. In the case of observations drawn from a normal distribution

$$s' = \frac{(\frac{1}{2}n)^{\frac{1}{2}}\Gamma(\frac{1}{2}[n-1])}{\Gamma(\frac{1}{2}n)}\sqrt{\frac{\Sigma(x_i' - \bar{x})^2}{n}}, \tag{11}$$

is an unbiased estimate of σ (see for example Kenney [11], with variance

$$\sigma^2(s') = \left\{\frac{1}{2}\left[\frac{\Gamma(\frac{1}{2}[n-1])}{\Gamma(\frac{1}{2}n)}\right]^2(n-1) - 1\right\}\sigma^2. \tag{12}$$

TABLE III*

$P(x < u_{i|k}) \times 10^4$, $u_{i|k} = E(x_{i|k})$, $x_{i|k}$ is the ith order statistic in a sample of size k drawn from a normal distribution $N(x, 0, 1)$

$k \backslash i$	1	2	3	4	5	6	7	8	9	10
1	5000									
2	2863	7137								
3	1987	5000	8013							
4	1516	3832	6168	8484						
5	1224	3103	5000	6897	8776					
6	1025	2605	4201	5799	7395	8975				
7	0881	2244	3622	5000	6378	7756	9119			
8	0773	1971	3182	4394	5606	6818	8030	9227		
9	0688	1756	2837	3919	5000	6082	7163	8244	9312	
10	0619	1584	2559	3536	4512	5488	6464	7441	7416	9381

* The table is given to more places than necessary for the purpose suggested because it may be of interest in other applications The $E(x_{i|k})$ from which the table was derived were computed to five decimal places

For most practical purposes however, when $n > 10$, the bias in s is negligible. For large samples $\sigma^2(s')$ approaches $\sigma^2/2n$.

4A. The range as an estimate of σ. As mentioned in the Introduction, section 1, it is now common practice in industry to estimate the standard deviation by means of a multiple of the range $R' = c_n(x_n - x_1)$, for small samples, where $c_n = 1/[E(y_n) - E(y_1)]$, y_n and y_1 being the greatest and least observations drawn from a sample of size n from a normal distribution $N(y, a, 1)$. Although we are principally interested in large sample statistics, for the sake of completeness, we shall include a few remarks about the use of the range in small samples.

Now R' is an unbiased estimate of σ, and its variance may be computed for small samples, see for example Hartley [7, 1942]. In the present case, although both R' and s' are unbiased estimates of σ, they are not normally distributed,

nor are we considering their asymptotic properties; therefore the previously defined concept of efficiency does not apply. We may however use the ratio of the variances as an arbitrary measure of the relative precision of the two statistics. The following table lists the ratio of the variances of the two statistics, as well as the variances themselves expressed as a multiple of the population variance for samples of size $n = 2, 3, \cdots, 10$.

4B. Quasi ranges for estimating σ. The fact that the ratio $\sigma^2(s')/\sigma^2(R')$ falls off in Table IV as n increases makes it reasonable to inquire whether it might not be worthwhile to change the systematic estimate slightly by using the statistic $c_{1|n}[x_{n-1} - x_2]$, or more generally $c_{r|n}[x_{n-r} - x_{r+1}]$ where $c_{r|n}$ is the multiplicative constant which makes the expression an unbiased estimate of σ (in particular $c_{r|n}$ is the constant to be used when we count in $r + 1$ observations from each end of a sample of size n, thus $c_{r|n} = 1/[E(y_{n-r} - y_{r+1})]$ where the

TABLE IV

Relative precision of s' and R', and their variances expressed as a multiple of σ^2, the population variance

n	$\sigma^2(s')/\sigma^2(R')$	$\sigma^2(s')/\sigma^2$	$\sigma^2(R')/\sigma^2$
2	1.000	.570	.570
3	.990	.273	.276
4	.977	.178	.182
5	.962	.132	.137
6	.932	.104	.112
7	.910	.0864	.0949
8	.889	.0738	.0830
9	.869	.0643	.0740
10	.851	.0570	.0670

y's are drawn from $N(y, a, 1)$). This is certainly the case for large values of n, but with the aid of the unpublished tables mentioned at the close of section 2, we can say that it seems not to be advantageous to use $c_{1|n}[x_{n-1} - x_2]$ for $n \leq 10$. Indeed the variance $c_{1|10}[x_9 - x_2]$, for the unit normal seems to be about 10, as compared with $\sigma^2(R')/\sigma^2 = .067$ as given by Table IV, for $n = 10$. The uncertainty in the above statements is due to a question of significant figures.

Considerations which suggest constructing a statistic based on the difference of two order statistics which are not extreme values in small samples, weigh even more heavily in large samples. A reasonable estimate of σ for normal distributions, which could be calculated rapidly by means of punched-card equipment is

$$\hat{\sigma} = \frac{1}{c}\,[x(n_2) - x(n_1)], \tag{13}$$

where the $x(n_i)$ satisfy Condition 1, and where $c = u_2 - u_1$, u_2 and u_1 are the expected values of the n_2 and n_1 order statistics of a sample of size n drawn from a unit normal. Without loss of generality we shall assume the x_i are drawn from a unit normal. Furthermore we let $\frac{n_2}{n} = \lambda_2 = 1 - \lambda_1 = 1 - \frac{n_1}{n}$. Of course σ will be asymptotically normally distributed, with variance

$$\sigma^2(\hat{\sigma}) = \frac{2}{nc^2}\left[\frac{\lambda_1(1-\lambda_1)}{[f(u_1)]^2} + \frac{\lambda_2(1-\lambda^2)}{[f(u_2)]^2} - \frac{2\lambda_1(1-\lambda_2)}{f(u_1)f(u_2)}\right]. \tag{14}$$

Because of symmetry $f(u_1) = f(u_2)$; using this and the fact that $\lambda_1 = 1 - \lambda_2$, we can reduce (14) to

$$\sigma^2(\hat{\sigma}) = \frac{2}{nc^2}\frac{\lambda_1(1-2\lambda_1)}{[f(u_1)]^2}. \tag{15}$$

We are interested in optimum spacing in the minimum variance sense. The minimum for $\sigma^2(\hat{\sigma})$ occurs when $\lambda_1 \doteq .0694$, and for that value of λ_1, $\sigma^2(\hat{\sigma}) \doteq .767\ \sigma^2/n$. Asymptotically s' is also normally distributed, with $\sigma^2(s') = \sigma^2/2n$. Therefore we may speak of the efficiency of $\hat{\sigma}$ as an estimate of σ as .652. It is useful to know that the graph of $\sigma^2(\hat{\sigma})$ is very flat in the neighborhood of the minimum, and therefore varying λ_1 by .01 or .02 will make little difference in the efficiency of the estimate $\hat{\sigma}$ (providing of course that c is appropriately adjusted). K. Pearson [13] suggested this estimate in 1920. It is amazing that with punched-card equipment available it is practically never used when the appropriate conditions described in the Introduction are present.

The occasionally used semi-interquartile range, defined by $\lambda_1 = .25$ has an efficiency of only .37 and an efficiency relative to $\hat{\sigma}$ of only .56.

As in the case of the estimate of the mean by systematic statistics, it is pertinent to inquire what advantage may be gained by using more order statistics in the construction of the estimate of σ. If we construct an estimate based on four order statistics, and then minimize the variance, it is clear that the extreme pair of observations will be pushed still further out into the tails of the distribution. This is unsatisfactory from two points of view in practice: i) we will not actually have an infinite number of observations, therefore the approximation concerning the normality of the order statistics may not be adequate if λ_1 is too small, even in the presence of truly normal data; ii) the distribution functions met in practice often do not satisfy the required assumption of normality, although over the central portion of the function containing most of the probability, say except for the 5% in each tail normality may be a good approximation. In view of these two points it seems preferable to change the question slightly and ask what advantage will accrue from holding two observations at the optimum values just discussed (say $\lambda_1 = .07$, $\lambda_2 = .93$) and introducing two additional observations more centrally located.

We define a new statistic

$$\hat{\sigma}' = \frac{1}{c'}\left[x(n_4) + x(n_3) - x(n_2) - x(n_1)\right], \tag{16}$$

$c' = E[x(n_4) + x(n_3) - x(n_2) - x(n_1)]$, where the observations are drawn from a unit normal. We take $\lambda_1 = 1 - \lambda_4$, $\lambda_2 = 1 - \lambda_3$, $\lambda_1 = .07$. It turns out that $\sigma^2(\hat{\sigma}')$ is minimized for λ_2 in the neighborhood of .20, and that the efficiency compared with s' is a little more than .75. Thus an increase of two observations in the construction of our estimate of σ increases the efficiency from .65 to .75. We get practically the same result for $.16 \leq \lambda_2 \leq .22$.

Furthermore, it turns out that using $\lambda_1 = .02$, $\lambda_2 = .08$, $\lambda_3 = .15$, $\lambda_4 = .25$, $\lambda_5 = .75$, $\lambda_6 = .85$, $\lambda_7 = .92$, $\lambda_8 = .98$, one can get an estimate of σ based on eight order statistics which has an efficiency of .896. This estimate is more efficient than either the mean deviation about the mean or median for estimating σ The estimate is of course

$$\hat{\sigma}'' = [x(n_8) + x(n_7) + x(n_6) + x(n_5) - x(n_4) - x(n_3) - x(n_2) - x(n_1)]/C,$$

where $C = 10.34$.

To summarize: in estimating the standard deviation σ of a normal distribution from a large sample of size n, an unbiased estimate of σ is

$$\hat{\sigma} = \frac{1}{c}(x_{n-r+1} - x_r),$$

where $c = E(y_{n-r+1} - y_r)$ where the y's are drawn from $N(y, a, 1)$. The estimate $\hat{\sigma}$ is asymptotically normally distributed with variance

$$\sigma^2(\hat{\sigma}) = \frac{2}{nc^2}\frac{\lambda_1(1 - 2\lambda_1)}{[f(u_1)]^2},$$

where $\lambda_1 = r/n$, $f(u_1) = N(E(x_r), 0, \sigma^2)$. We minimize $\sigma^2(\hat{\sigma})$ for large samples when $\lambda_1 \doteq .0694$, and for that value of λ_1,

$$\sigma^2_{\text{opt}}(\hat{\sigma}) \doteq \frac{.767\sigma^2}{n}.$$

The unbiased estimate of σ

$$\hat{\sigma}' = \frac{1}{c'}(x_{n-r+1} + x_{n-s+1} - x_s - x_r)$$

may be used in lieu of $\hat{\sigma}$. If $\lambda_1 = r/n$, $\lambda_2 = s/n$ we find

$$\sigma^2(\hat{\sigma}' \mid \lambda_1 = .07, \lambda_2 = .20) \doteq \frac{.66\sigma^2}{n}.$$

4C. The mean deviations about the mean and median. The next level of computational difficulty we might consider for the construction of an estimate of σ is the process of addition. The mean deviation about the mean is a well known, but not often used statistic. It is defined by

$$\text{m.d.} = \sum_{i=1}^{n} | x_i' - \bar{x} |/n. \tag{17}$$

For large samples from a normal distribution the expected value of m.d. is $\sqrt{\frac{2}{\pi}}\,\sigma$, therefore to obtain an unbiased estimate of σ we define the new statistic $A = \sqrt{\frac{\pi}{2}}$ m.d. Now for large samples A has variance $\sigma^2[\frac{1}{2}(\pi - 2)]/n$, or an efficiency of .884. However there are slight awkwardnesses in the computation of A which the mean deviation about the median does not have.

It turns out that for samples of size $n = 2m + 1$ drawn from a normal distribution $N(y, a, 0)$ the statistic

$$M' = \sqrt{\frac{\pi}{2}}\frac{\sum |x_i - x_{m+1}|}{2m} \tag{18}$$

asymptotically has mean σ and variance

$$\sigma^2(M') = \frac{1}{2m}\left(\frac{\pi - 2}{2}\right)\sigma^2. \tag{19}$$

Thus in estimating the standard deviation of a normal distribution from large samples we can get an efficiency of .65 by the judicious selection of two observations from the sample, an efficiency of .75 by using four observations, and an efficiency of .88 by using the mean deviation of all the observations from either the mean or the median of the sample, and an efficiency of .90 by using eight order statistics.

5. Estimation of the correlation coefficient. In the present section we consider the estimation of the correlation coefficient of a normal bivariate population:

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \\ &\quad \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{(x-a)^2}{\sigma_x^2} + \frac{(y-b)^2}{\sigma_y^2} - \frac{2\rho(x-a)(y-b)}{\sigma_x\sigma_y}\right)\right]. \end{aligned} \tag{20}$$

The efficient estimate of ρ in a sample $O_n : (x_1', y_1'), (x_2', y_2'), \cdots, (x_n', y_n')$ drawn from the density (20) is

$$r = \frac{\sum(x_i' - \bar{x}')(y_i' - \bar{y})}{[\sum(x_i' - \bar{x})^2 \sum(y_i' - \bar{y})^2]^{\frac{1}{2}}}. \tag{21}$$

There are numerous other techniques in the literature for estimating ρ, among them i) the tetrachoric correlation coefficient which depends on a four-fold table, ii) the adjusted rank correlation coefficient which depends on assigning ranks to the x and y observations. These and other estimates of the correlation coefficient are discussed by Kendall [10].

We shall be concerned with the construction of some estimates of the correlation coefficient which are particularly adapted for use with punched-card equipment. A counting sorter is adequate for the first two cases discussed; in line with our previous development we shall then consider a technique which uses simple addition of the observed values, but does not require sums of squares or products (in the special case where variances of x and y are equal).

5A. Estimation of ρ when means and standard deviations are known. Let us suppose that the means and variances of the variables x and y, distributed according to (20) are given, and consider the problem of estimating the correlation coefficient ρ from a sample of size n. There will be no generality lost by assuming $a = b = 0$, $\sigma_x^2 = \sigma_y^2 = 1$. The technique used will be to construct lines $y = 0$, $x = \pm k$, which cut the xy-plane into six parts. We will form an estimate of ρ based upon the number of observations falling in the four corners. Figure 1 represents the lines laid out in the manner suggested in connection with a scatter diagram of 25 observations; naturally the method is recommended for

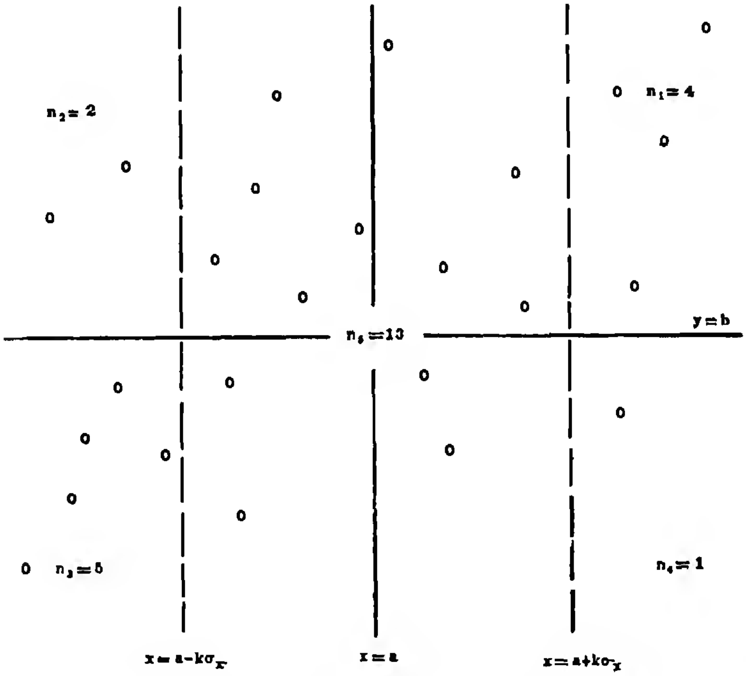

FIG. 1 DIAGRAM OF THE CONSTRUCTION DESCRIBED IN PARAGRAPH 5A WITH A SAMPLE OF 25 OBSERVATIONS SUPERIMPOSED

use only with large samples, the 25 observations are for purposes of illustration only. More specifically after assigning the special values mentioned immediately above to the means and variances in (20), we define

$$
\begin{aligned}
p_1 &= \int_0^\infty \int_k^\infty f(x, y)\, dx\, dy, & p_3 &= \int_{-\infty}^0 \int_{-\infty}^{-k} f(x, y)\, dx\, dy, \\
p_2 &= \int_0^\infty \int_{-\infty}^{-k} f(x, y)\, dx\, dy, & p_4 &= \int_{-\infty}^0 \int_k^\infty f(x, y)\, dx\, dy, \\
p_5 &= \int_{-\infty}^\infty \int_{-k}^k f(x, y)\, dx\, dy = \int_{-k}^k N(x, 0, 1)\, dx. & &
\end{aligned}
\tag{22}
$$

We denote by n_i the number of observations falling into the region containing probability density p_i. Of course $\sum_{i=1}^{5} n_i = n$. Now we may write the joint probability distribution of the n_i as

$$(23) \qquad g(n_1, n_2, n_3, n_4) = \frac{n!}{\prod_1^5 n_i!} \prod_{i=1}^{5} p_i^{n_i}.$$

remembering that $n_5 = n - \sum_1^4 n_i$.

We shall now derive the maximum likelihood estimate of ρ from (23). Taking the logarithm of (23) we have

$$(24) \qquad \log g = \log c + \sum_{i=1}^{5} n_i \log p_i,$$

where c is the multinomial coefficient on the right of (23). Differentiating (24) with respect to ρ gives

$$(25) \qquad \frac{d(\log g)}{d\rho} = \sum_{i=1}^{4} \frac{n_i \dot{p}_i}{p_i}.$$

where $\dot{p}_i = \frac{dp_i}{d\rho}$, of course $\frac{dp_5}{d\rho} = 0$ because p_5 is functionally independent of ρ. To get $\bar{\rho}$, the maximum likelihood estimate of ρ, under our restrictions, we must equate the right of (25) to zero and solve for ρ. Before proceeding it will be useful to note the following relations:

$$p_1 = p_3 \,; \; p_2 = p_4$$

$$(26) \qquad \dot{p}_1 = -\dot{p}_4 \,; \; \dot{p}_2 = -\dot{p}_3 \,; \; \dot{p}_1 = p_3 \,; \; p_2 = \dot{p}_4$$

$$p_1 + p_4 = \int_k^{\infty} N(x, 0, 1)dx = \lambda; \qquad p_2 + p_3 = \int_{-\infty}^{-k} N(x, 0, 1)dx = \lambda.$$

If after making appropriate substitutions from (26) we set the right of (25) equal to zero we get

$$\frac{n_1 \dot{p}_1}{p_1} - \frac{n_2 \dot{p}_1}{\lambda - p_1} + \frac{n_3 \dot{p}_1}{p_1} - \frac{n_4 \dot{p}_1}{\lambda - p_1} = 0,$$

and since in general $\dot{p}_1 \neq 0$, the condition is that

$$(27) \qquad \frac{n_1 + n_3}{n_2 + n_4} = \frac{p_1}{\lambda - p_1}.$$

Unless all four of the n_i are zero (which is unlikely for reasonable values of λ because n is large), it is possible to find a value of ρ which will make the right side of (27) equal to the ratio formed from the observations on the left, and the value of ρ so determined is the maximum likelihood estimate ρ under the restrictions we have imposed. In practice this equation may be solved by con-

sulting a table of the bivariate normal distribution—see for example K. Pearson [14]. Alternatively [27] may be solved by referring to Figure 3. Truman Kelley [9, 1939] has considered a closely related problem in connection with the validation of test items.

It may be inquired whether it would not be preferable to reduce the present design to a tetrachoric case by using only the cutting lines $x = 0$, $y = 0$. An investigation of the variance of $\bar{\rho}$ reveals that such is not the case. We proceed to determine the asymptotic variance by means of the usual maximum likelihood technique. Differentiating (25) once more we have

$$\frac{d^2(\log g)}{d\rho^2} = \sum_{i=1}^{4} \frac{n_i(p_i \ddot{p}_i - \dot{p}_1^2)}{p_1^2}, \tag{28}$$

where $\ddot{p}_1 = \dfrac{d^2 p_1}{d\rho^2}$. We note that $E(n_i) = np_i$, therefore

$$E\left(\frac{d^2(\log g)}{d\rho^2}\right) = n\left(\sum_{i=1}^{4} \ddot{p}_i - \sum_{i=1}^{4} \frac{\dot{p}_1^2}{p_i}\right). \tag{29}$$

but since the derivative of a sum is equal to the sum of its derivatives, and $p_1 + p_4 = \lambda$, $p_2 + p_3 = \lambda$, the first sum in the square brackets vanishes. Suitable substitutions from (26) will reduce the second sum so that we get

$$-E\left[\frac{d^2(\log g)}{d\rho^2}\right] = \frac{2n\dot{p}_1^2\lambda}{p_1(\lambda - p_1)}. \tag{30}$$

Therefore asymptotically $\bar{\rho}$ is normally distributed with variance

$$\sigma^2(\bar{\rho}) = \frac{p_1(\lambda - p_1)}{2n\lambda\dot{p}_1^2}. \tag{31}$$

In general the optimum value (in the minimum variance sense) of λ which determines the cutting lines $x = \pm k$ will depend on the true value of ρ. To carry out the minimization process in general will require fairly extensive computations, which we feel would be justified. For the present we shall restrict ourselves to minimizing $\sigma^2(\bar{\rho})$ for the case $\rho = 0$.

We have

$$\dot{p}_1 = \frac{1}{2\pi}\exp\left[-\tfrac{1}{2}k^2\right] = \frac{1}{\sqrt{2\pi}} f(k).$$

when $\rho = 0$, and $p_1 = \frac{1}{2}\lambda$. This gives

$$\sigma^2(\bar{\rho} \mid \rho = 0) = \frac{\pi\lambda}{4n[f(k)]^2}. \tag{32}$$

We wish to minimize the expression on the right. We recall that a similar expression λ_1/f_1^2 was to be minimized in section 3 when the optimum pair of

observations for estimating the mean of a normal distribution was found. Using the previous results we have $\lambda \doteq .2702$, $k = .6121$; which gives us finally

$$\sigma^2_{\text{opt}}(\bar{\rho} \mid \rho = 0) \doteq \frac{1.939}{n}. \tag{33}$$

To summarize: if a sample of size n is drawn from a normal bivariate population with known means a_x, a_y and variances σ_x^2 and σ_y^2, but unknown correlation ρ, the maximum likelihood estimate of ρ based on the number of observations falling in the four corners of the plane determined by the lines $x = a_x \pm k\sigma_x$, $y = a_y$ is found by solving for ρ the equation

$$\frac{n_1 + n_3}{n_1 + n_2 + n_3 + n_4} = \frac{p_1}{\lambda}.$$

where n_1 is the number of observations falling in the upper right, n_2 in the upper left, n_3 in the lower left, n_4 in the lower right hand corner, and p_i is the probability density in the region into which the n_i fall, $\lambda = p_1 + p_4$. The variance of this estimate $\bar{\rho}$ is given by

$$\sigma^2(\bar{\rho}) = \frac{p_1(\lambda - p_1)}{2n\lambda p_1^2},$$

which is minimized for $\rho = 0$ by setting $k = .6121$, $\lambda = .2702$, giving

$$\sigma^2_{\text{opt}}(\bar{\rho} \mid \rho = 0) \doteq \frac{1.939}{n}.$$

On the other hand if the usual tetrachoric estimate is used with $x = 0$, $y = 0$ as the cutting lines we get $\sigma_T^2(\bar{\rho} \mid \rho = 0) = \pi^2/4n$. The relative efficiency of the tetrachoric compared with the optimum statistic is therefore .787. The variance of the efficient estimate r given in (25) when $\rho = 0$ is $1/n$. Consequently the efficiency of our estimate $\bar{\rho}$ compared to that of r is about .515 for the special case $\rho = 0$ under consideration. This means about twice as large a sample is required to get the same precision with $\bar{\rho}$ as with r. Doubling the sample and using the cruder statistic $\bar{\rho}$ may often be an economical procedure.

It may be surmised that a still better estimate of ρ could be constructed by employing four cutting lines, say $x = \pm k$, $y = \pm k$. The simplifications which we used to obtain the estimate $\bar{\rho}$ no longer hold when we use this new construction. However, it is still possible to compute the minimum variance of the new estimate which we will call $\bar{\rho}'$, for the special case $\rho = 0$. It again turns out that $k \doteq .6121$ minimizes and we get

$$\sigma^2_{\text{opt}}(\bar{\rho}' \mid \rho = 0) \doteq \frac{1.52}{n}, \tag{34}$$

which makes the efficiency of $\bar{\rho}'$ (compared with r) about .66 as compared with 515 for $\bar{\rho}$. This suggests that if some very simple technique can be found for obtaining $\bar{\rho}'$, $\bar{\rho}'$ would be worth using. Unfortunately the author has not been able to construct a rapid way of finding $\bar{\rho}'$.

5B. Estimation of ρ when the parameters are unknown. A more practical situation than the case treated in paragraph 5A, is the case in which all parameters of (20) are unknown. This case will be treated by means of order statistics. We construct an order statistic analogue of the estimate $\bar{\rho}$ which we will call $\bar{\rho}^*$. In general the procedure will be as follows: Each of the N observations in the sample has an x coordinate and a y coordinate

i) order the observations with respect to the x coordinate;

ii) discard all observations except the n with the largest x coordinates called the *right* set and the n with the smallest x coordinates called the *left* set, retaining, therefore, $2n$ observations,

iii) order the pooled $2n$ observations with respect to the y coordinate;

iv) break the $2n$ observations into two sets of n observations each; the *upper* set containing the n observations with the greatest y coordinates, and the *lower* set containing the n observations with the smallest y coordinates,

v) reorder the *upper* set of observations with respect to the x coordinate; the n observations will be divided into those whose x coordinates belong to the *right* set and those whose x coordinates belong to the *left* set,

vi) the estimate $\bar{\rho}^*$ will be obtained by solving the equation

$$\frac{n_1^*}{n - n_1^*} = \frac{p_1^*}{\lambda_1^* - p_1^*}. \tag{35}$$

where n_1^* is the number of observations in the *upper* set which are also numbers of the *right* set and p_1^* is $\int_0^\infty \int_{k^*}^\infty f(x, y)dx\, dy$, while $f(x, y)$ is the bivariate normal (20) with $\sigma_x = \sigma_y, = 1, a = b = 0$, and $\int_{k^*}^\infty N(x, 0, 1)\, dx = \frac{n}{N} = \lambda_1^*$.

Figure 2 represents graphically the construction described above for a scatter diagram composed of 25 observations. Of course the number 25 is only for purposes of illustration, as the method is only proposed for use with large samples.

The procedure of ordering the x's and choosing the right and left sets of observations is analogous to cutting the bivariate distribution by the two lines $x = \pm k$ as described in paragraph 5A, indeed $x = x_{n+1}$ and $x = x_{N-n}$ are the corresponding lines, but they vary from sample to sample. To continue the analogy, ordering the remaining observations with respect to y and dividing them into *upper* and *lower* sets of equal size is like cutting the plane with the line $y = 0$. Finally formula (35) is analogous to formula (27) Another similar change is that where formerly we had among relations (26) the equalities $p_1 =$

p_3, $p_2 = p_4$, we now have the corresponding relations amongst the number of observations in the four corners of the plane, namely $n_1^* = n_3^*$, $n_2^* = n_4^*$ which

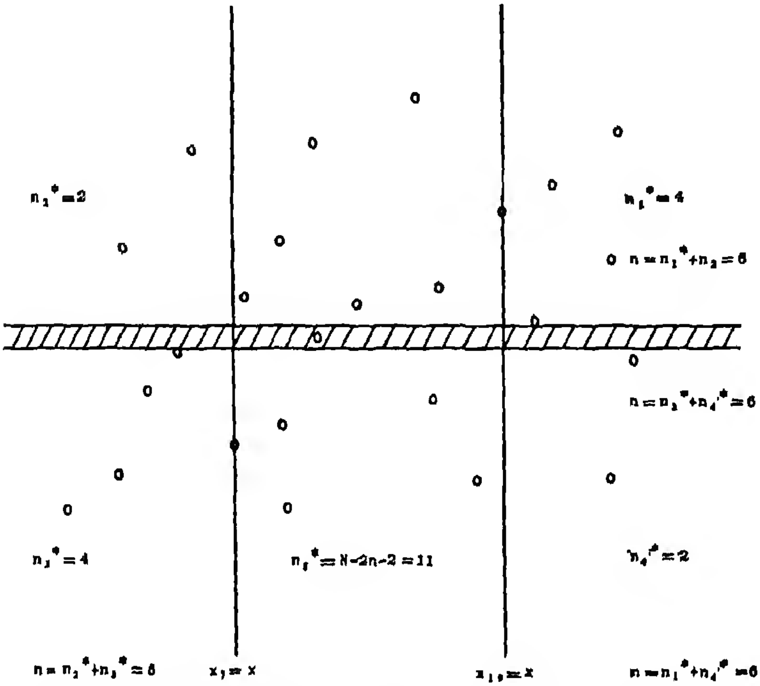

FIG. 2. DIAGRAM OF THE CONSTRUCTION DESCRIBED IN PARAGRAPH 5B ON THE BASIS OF 25 OBSERVATIONS $n = 6$

can readily be seen by inspection of the fourfold table we have constructed below (omitting all reference to $N - 2n$ pairs of observations we have discarded).

	Left set	Right set	Totals
Upper set	n_2^*	n_1^*	n
Lower set	n_3^*	n_4^*	n
Totals.	n	n	$2n$

We have dwelt at length upon the analogy between the two constructions because one of the principal difficulties in working with order statistics is to design a mathematically workable model. The author has found it fruitful when constructing systematic statistics to study a workable analogy which does not involve the order statistics directly, and then to build upon correspondences such as those described.

Some may not wish to read further in this paragraph when they are informed that asymptotically the variance of $\bar{\rho}^*$ is essentially the same as that of $\bar{\rho}$. They should proceed to page 404. For the others we proceed to the demonstration.

Suppose we draw a sample of N pairs of observations (x_i', y_i') from the bivariate normal (20). If we discard from these all pairs except those with the n largest x_i and the n smallest x_i, we are left with the right set and the left set. We shall need the joint distribution of x_{n+1} and x_{N-n}

$$(36) \qquad J(x_{n+1}, x_{N-n}) = \frac{N!}{(n!)^2(N-2n-2)!}\left(\int_{-\infty}^{x_{n+1}} g(x)\,dx\right)^n \left(\int_{x_{n+1}}^{x_{N-n}} g(x)\,dx\right)^{N-2n-2}\left(\int_{x_{N-n}}^{\infty} g(x)\,dx\right)^n g(x_{n+1}))g(x_{N-n}).$$

where $g(x)$ is the marginal distribution of x obtained from (20), $N(x, a, \sigma_x^2)$. We assume x_{n+1}, x_{N-n} satisfy Condition 1. Considering x_{n+1}, x_{N-n} as fixed and given for the moment we wish to look at the distribution of the y coordinates. We may consider the y coordinates of the observations in the right set as drawn from the distribution of y

$$\varphi'(y) = \frac{\int_{x_{N-n}}^{\infty} f(x, y)\,dx}{\int_{-\infty}^{\infty}\int_{x_{N-n}}^{\infty} f(x, y)\,dx\,dy} = \frac{\int_{x_{N-n}}^{\infty} f(x, y)\,dx}{\int_{x_{N-n}}^{\infty} g(x)\,dx}.$$

Similarly the y coordinates of the observations belonging to the left set may be considered as independently drawn from

$$\psi'(y) = \frac{\int_{-\infty}^{x_{n+1}} f(x, y)\,dx}{\int_{-\infty}^{\infty}\int_{-\infty}^{x_{n+1}} f(x, y)\,dx\,dy} = \frac{\int_{-\infty}^{x_{n+1}} f(x, y)\,dx}{\int_{-\infty}^{x_{n+1}} g(x)\,dx}.$$

To prevent confusion, in considering the y order statistics of the two sets, we shall designate those of the observations which are members of the right set by $u_1, u_2, \cdots, u_n$; while those observations belonging to the left set will have their ordered y coordinates designated $v_1, v_2, \cdots, v_n$. Of course the u's and v's separately satisfy an order relation like that given in (1).

The first question we answer is: given x_{n+1}, x_{N-n}, what is the probability that when we collate the u's and v's and split the observations into the upper set and lower set (see iv). there will be exactly c observations in the lower set whose y coordinates are designated by u's? In other words what is the probability that exactly c members of the lower set belong to the right set? An example for small values of n may clarify the problem. Suppose $n = 4$, and we observe $u_1 < v_1 < v_2 < v_3 < u_2 < u_3 < v_4 < u_4$; the y coordinates of the lower set of observations are u_1, v_1, v_2, v_3, and only the observation with u_1 for its y coordinate belongs to the right set, so for this case $c = 1$. To return

to our general problem, the probability that there are exactly c observations which are members of both the right set and the lower set is

$$(37) \quad P(c \mid x_{n+1}, x_{N-n}) = 1 - p(v_{n-c} > u_{c+1}) - p(u_c > v_{n-c+1}),$$

where $p(w > z)$ is the probability that w is greater than z. Now writing $\varphi(z) = \int_{-\infty}^{z} \varphi'(t)dt$, $\psi(z) = \int_{-\infty}^{z} \psi'(t)dt$ we may rewrite (37) as

$$(38) \quad \begin{aligned} &P(c \mid x_{n+1}, x_{N-n}) \\ &= 1 - \frac{n!}{c!(n-c-1)!} \int_{u_{c+1}}^{\infty} [\psi(v_{n-c})]^{n-c-1}[1 - \psi(v_{n-c})]^c \psi'(v_{n-c})\, dv_{n-c} \\ &- \frac{n!}{(n-c)!(c-1)!} \int_{-\infty}^{u_c} [\psi(v_{n-c+1})]^{n-c}[1 - \psi(n_{n-c+1})]^{c-1} \psi'(v_{n-c+1})\, dv_{n-c+1} \end{aligned}$$

After integrating the first integral of (38) by parts and simplifying we can rewrite (38) as

$$(39) \quad \begin{aligned} P(c \mid x_{n+1}, x_{N-n}) = {} & \frac{n!}{c!(n-c)!} [\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^c \\ & + \frac{n!}{(n-c)!(c-1)!} \int_{\psi(u_c)}^{\psi(u_{c+1})} \alpha^{n-c}(1-\alpha)^{c-1}\, d\alpha. \end{aligned}$$

We approximate the integral term of (39) by

$$\frac{n!}{(n-c)!(c-1)!} [\psi(u_{c+1}) - \psi(u_c)][\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^{c-1}$$

which leads us to the approximation

$$(40) \quad \begin{aligned} &P(c \mid x_{n+1}, x_{N-n}) \\ &= \frac{n!}{(n-c)!c!} [\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^{c-1}[1 + (c-1)\psi(u_{c+1}) - c\psi(u_c)]. \end{aligned}$$

The joint distribution of u_c, u_{c+1} is given by

$$(41) \quad \begin{aligned} &Q(u_c, u_{c+1} \mid x_{N-n}) \\ &= \frac{n!}{(c-1)!(n-c-1)} \varphi(u_c)^{c-1}(1 - \varphi(u_{c+1}))^{n-c-1} \varphi'(u_c)\varphi'(u_{c+1}). \end{aligned}$$

Next we multiply P as given by (40) by Q from (41) and integrate out u_c. This gives us except for terms of $O\left(\frac{1}{n}\right)$ and higher

$$(42) \quad \begin{aligned} &\frac{n!n!}{c!(n-c-1)!c!(n-c)!} [\varphi(u_{c+1})]^c \\ &\qquad \cdot [1 - \varphi(u_{c+1})]^{n-c-1}[\psi(u_{c+1})]^{n-c}[1 - \psi(u_{c+1})]^c \varphi'(u_{c+1}). \end{aligned}$$

When expression (42) is multiplied by (36), we finally get the approximate joint distribution of c, x_{n+1}, x_{N-n}, u_{c+1}.

Before proceeding further we let

$$
(43) \quad
\begin{aligned}
\varphi(u_{c+1}) &= \frac{\int_{-\infty}^{u_{c+1}} \int_{x_{N-n}}^{\infty} f(x, y)\, dx\, dy}{1 - \lambda_2^*} = \frac{p_4^*}{1 - \lambda_2^*}, \\
\psi(u_{c+1}) &= \frac{\int_{-\infty}^{u_{c+1}} \int_{-\infty}^{x_{n+1}} f(x, y)\, dx\, dy}{\lambda_1^*} = \frac{p_3^*}{\lambda_1^*},
\end{aligned}
$$

where $\lambda_1^* = \int_{-\infty}^{x_{n+1}} g(x)dx$, $\lambda_2^* = \int_{-\infty}^{x_{N-n}} g(x)dx$. If we also let $p_1^* = 1 - \lambda_2^* - p_4^*$, $p_2^* = \lambda_1^* - p_3^*$ we can write

$$
(44) \quad
\begin{aligned}
&R(c,\ x_{n+1},\ x_{N-n},\ u_{c+1}) \\
&\qquad = K(\lambda_2^* - \lambda_1^*)^{N-2n-2} p_1^{*\,n-c-1} p_2^{*\,c} p_3^{*\,n-c} p_4^{*\,c} p_4^{*\prime} \lambda_1^{*\prime} \lambda_2^{*\prime},
\end{aligned}
$$

where the primes indicate derivatives of p_4^*, λ_1^*, λ_2^* with respect to the appropriate suppressed variables, u_{c+1}, x_{n+1}, x_{N-n}, respectively.

We now proceed to the maximum likelihood estimate of ρ. We take the logarithm of (44) and then take partial derivatives with respect to a the mean of x, b the mean of y, and ρ the correlation coefficient. After equating these partial derivatives to zero we have the following three maximum likelihood equations which must be solved simultaneously to obtain the estimates $\hat{a}^*$, $\hat{b}^*$, and $\hat{\rho}^*$:

$$
(45) \quad
\begin{aligned}
\frac{1}{N}\biggl[\frac{N - 2n - 2}{\lambda_2^* - \lambda_1^*} \frac{\partial(\lambda_2^* - \lambda_1^*)}{\partial a} &+ \frac{n - c - 1}{p_1^*} \frac{\partial p_1^*}{\partial a} \\
&+ \frac{c}{p_2^*} \frac{\partial p_2^*}{\partial a} + \frac{n - c}{p_3^*} \frac{\partial p_3^*}{\partial a} + \frac{c}{p_4^*} \frac{\partial p_4^*}{\partial a}\biggr] = 0,
\end{aligned}
$$

$$
(46) \quad \frac{1}{N}\left[\frac{n - c - 1}{p_1^*} \frac{\partial p_1^*}{\partial b} + \frac{c}{p_2^*} \frac{\partial p_2^*}{\partial b} + \frac{n - c}{p_3^*} \frac{\partial p_3^*}{\partial b} + \frac{c}{p_4^*} \frac{\partial p_4^*}{\partial b}\right] = 0,
$$

$$
(47) \quad \frac{1}{N}\left[\frac{n - c - 1}{p_1^*} \frac{\partial p_1^*}{\partial \rho} + \frac{c}{p_2^*} \frac{\partial p_2^*}{\partial \rho} + \frac{n - c}{p_3^*} \frac{\partial p_3^*}{\partial \rho} + \frac{c}{p_4^*} \frac{\partial p_4^*}{\partial \rho}\right] = 0.
$$

where terms $O\left(\frac{1}{N}\right)$ have been neglected. Equations (45) and (46) are satisfied, again except for terms $O\left(\frac{1}{N}\right)$, when $\hat{a}^* = \frac{1}{2}(x_{n+1} + x_{N-n})$, $\hat{b}^* = u_{c+1}$. Using this information we examine (47) and find it satisfied when

$$
(48) \quad \frac{n - c}{c} = \frac{p_1^*}{\lambda_1^* - p_1^*},
$$

which is directly analogous to equation (27), and is the form promised in (35), if $n_1^* = n - c$. The estimate $\hat{\rho}^*$ is obtained by solving (48) for ρ, where $p_1^* =$

$\int_0^\infty \int_{k^*}^\infty f(x, y)\,dx\,dy$, and $f(x, y)$ is given by (20) with variances equal unity and means equal zero, and $\int_{k^*}^\infty g(x)\,dx = \lambda_1^* = 1 - \lambda_2^* = n/N$.

We shall not go through the derivation of $\sigma^2(\bar{\rho}^*)$ here. The usual maximum likelihood technique may be used. It turns out that the covariances between $\hat{a}^*$ and $\bar{\rho}^*$ and between $\hat{b}^*$ and $\bar{\rho}^*$ are $O\left(\frac{1}{N^2}\right)$. Neglecting such terms we find that the variance is

$$\sigma^2(\bar{\rho}^*) = \frac{p_1^*(\lambda_1^* - p_1^*)}{2N\lambda_1^* \dot{p}_1^{*2}}. \tag{49}$$

To summarize: if a sample of size N is drawn from a normal bivariate population with unknown parameters, the maximum likelihood estimate of ρ based on the $2n$ observations composed of those observations with the n largest x coordinates and the n smallest x coordinates, may be obtained by solving for ρ the equation

$$\frac{n - c}{n} = \frac{p_1^*}{\lambda^*},$$

where $\frac{1}{2} > \lambda^* = n/N > 0$, $p_1^* = \int_0^\infty \int_{k^*}^\infty f(x, y \mid \sigma_x = 1, a_x = a_y = 0)\,dx\,dy$, $\int_{k^*}^\infty N(x, 0, 1)\,dx = \lambda^*$, *and $n - c$ is the number of the $2n$ observations with largest y coordinates, which also have largest x coordinates. The variance of this estimate $\bar{\rho}^*$ is given by*

$$\sigma^2(\bar{\rho}^*) = \frac{p_1^*(\lambda_1^* - p_1^*)}{2N\lambda^* \dot{p}_1^{*2}},$$

and for $\rho = 0$ the variance is minimized by choosing $\lambda^* = .2702$, that is by choosing that 27 per cent of the observations with largest x coordinates, and that 27 per cent with smallest x coordinates, and for this value of λ^*

$$\sigma^2_{\text{opt}}(\bar{\rho}^* \mid \rho = 0) \doteq \frac{1.939}{N}.$$

Equation (49) is of course exactly analogous to the expression given in (31) for the case of known means and variances. Therefore if the variance minimization problem is solved in general for the case of paragraph 5A, the large sample solution of the problem for unknown means and variances will also be solved.

Figure 3 may be used to obtain the estimates $\bar{\rho}$ or $\bar{\rho}^*$ in case the methods of paragraphs 5A or 5B are used. Essentially the figure solves equations (27) and (48). The procedure for the problem of paragraph 5A is

i) when $n_1 + n_3 > n_2 + n_4$ evaluate the ratio $\frac{n_1 + n_3}{n_1 + n_2 + n_3 + n_4} = x_0$ and

find the intersection of the line $x = x_0$ with the curve for the particular λ being used;

ii) through the point of intersection of the vertical line $x_0 = x$ and the λ curve draw a horizontal line;

iii) the value of $\tilde{\rho}$ is indicated on the vertical axis at the point of intersection of the horizontal line and the vertical axis,

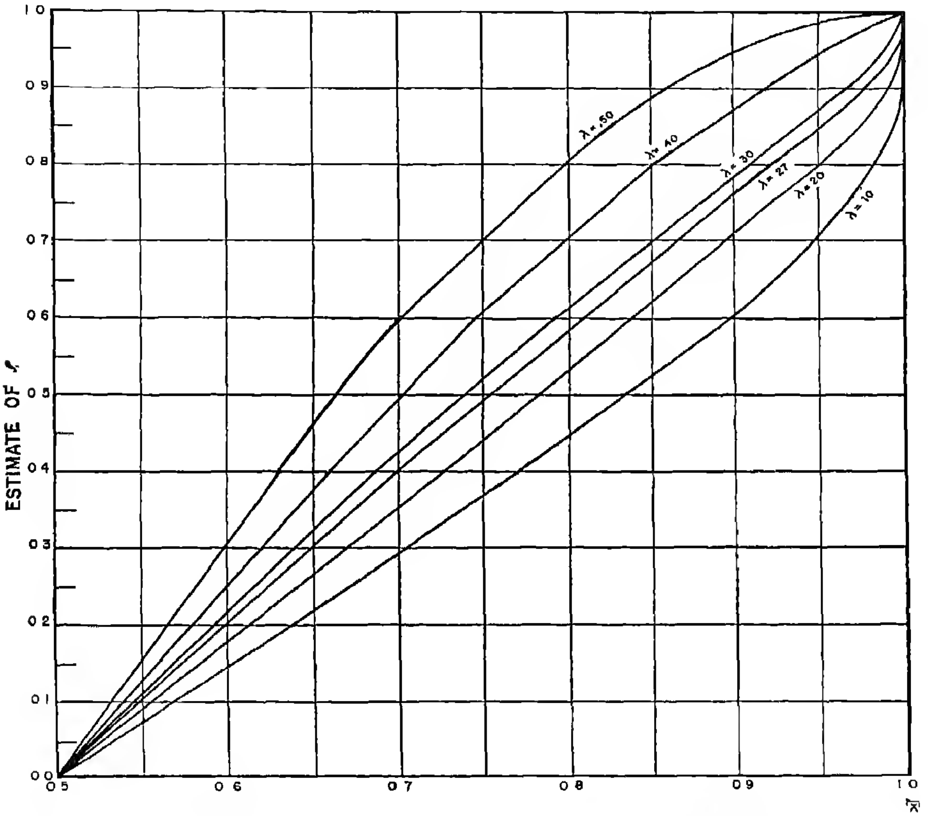

FIG. 3 CURVES FOR ESTIMATING THE CORRELATION COEFFICIENT ρ

iv) when $n_1 + n_3 < n_2 + n_4$ use the ratio $x_0 = \dfrac{n_2 + n_4}{n_1 + n_2 + n_3 + n_4}$ and follow the same procedure, $\tilde{\rho}$ will be the negative of the number appearing on the vertical axis.

Example. Suppose a sample of 1000 is drawn from a normal bivariate population for which the mean of x is a, and the mean of y is b, and the variance of x is σ_x^2, all three parameters known (it is not necessary to know σ_y^2). The xy

plane is cut by the three lines $x = a \pm k\sigma_x$, $y = a$, where, say, $k = .612$, so that $\lambda = .27$. Suppose we find the observations are distributed as follows:

in the upper right-hand corner: $160 = n_1$

in the lower left-hand corner: $170 = n_3$

in the upper left-hand corner: $110 = n_2$

in the lower right-hand corner: $110 = n_4$.

To estimate $\bar{\rho}$ we set up $x_0 = (n_1 + n_3)/(n_1 + n_2 + n_3 + n_4) = 330/550 = .6$. Referring to Figure 3 we find that the estimate of ρ, $\bar{\rho} = .20$.

In using Figure 3 for this case it is useful to know that for

$$\lambda = .50 \quad k = 0.000 \qquad \lambda = .27 \quad k = 0.612$$

$$\lambda = .40 \quad k = 0.253 \qquad \lambda = .20 \quad k = 0.841$$

$$\lambda = .30 \quad k = 0.524 \qquad \lambda = .10 \quad k = 1.282$$

If the means and variances of the variables are unknown, we may use the method of paragraph 5B:

i) when $n - c > c$ evaluate the ratio $(n - c)/n = x_0$, and find the intersection of the line $x = x_0$ with the curve for the particular λ_1 being used;

ii) through the point of intersection of the vertical line $x_0 = x$ and the λ_1 curve draw a horizontal line;

iii) the value of $\bar{\rho}^*$ is indicated on the vertical axis at the point of intersection of the horizontal line and the vertical axis;

iv) when $n - c < c$, use the ratio $c/n = x_0$ and follow the same procedure, $\bar{\rho}^*$ will be the negative of the number appearing on the vertical axis.

Example: Suppose a sample of 1000 is drawn from a normal bivariate population with all parameters unknown. Suppose we set $n = 200$, and follow the procedure given in paragraph 5B of this section, and suppose we find the observations are distributed as follows:

in the upper right-hand corner: $50 = n - c$

then of course

in the lower left-hand corner: $50 = n - c$

in the upper left-hand corner: $150 = c$

in the lower right-hand corner: $150 = c$

The estimate this time is clearly negative, so we set $x_0 = c/n = 150/200 = .75$. Referring to Figure 3 we find using the curve corresponding to $\lambda = .20$ that the estimate of ρ, $\bar{\rho} = -.44$.

5C. The use of averages for estimating ρ when the variance ratio is known. Nair and Shrivastava [12, 1942] have considered the use of means for estimating

regression coefficients when one observation is taken at each of n equally spaced fixed variates, x_i ($i = 1, 2, \cdots, n$), and y is normally distributed. Their procedure was essentially to consider the ordered fixed variates, and to discard a group of observations in the interior, much as we discarded the set of observations whose x coordinates were $x_{n+1}, x_{n+2}, \cdots, x_{N-n}$ in paragraph 5B. The resulting estimates depended essentially on the averages of the y's on the right and left sets of observations, and on the averages of the fixed x's in the two sets.

In an unpublished manuscript George Brown has considered a problem even more closely related to the one considered in paragraph 5A Suppose x and y normally distributed according to (20) with equal variances σ^2, and means equal to zero. (The ratio of variances must be known, equality is unnecessary.) Retain only those observations for which $|x_i| \geq k\sigma$, and from them form the statistic

$$\rho_B = \frac{\bar{y}_+ - \bar{y}_-}{\bar{x}_+ - \bar{x}_-}, \tag{50}$$

where $\bar{y}_+$ and $\bar{x}_+$ are the average of the n_1 x's and y's for which $x_i > k\sigma$ and $\bar{y}_-$ and $\bar{x}_-$ are similarly defined for the n_2 observations for which $x_i < -k\sigma$. Then ρ_B is an unbiased estimate of ρ. Regarding the x's as fixed variates it turns out that

$$\sigma^2(\rho_B) = \frac{(1 - \rho^2)\sigma^2}{(\bar{x}_+ - \bar{x}_-)^2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right). \tag{51}$$

If we approximate by substituting expected values for observed values (55) turns out to be $(1 - \rho^2)\sigma^2\lambda/2N[g(k)]^2$, where $\lambda = \int_{-\infty}^{-k} g(x)\,dx$, $g(x) = N(x, 0, 1)$. The value of k which minimizes this expression is our old friend $k = .6121$, which gives $\lambda = .2702$ Therefore for $\rho = 0$ and large samples, the minimum variance is approximately $1.23\ \sigma^2/N$, for an efficiency of about .81. The relative efficiency of the methods of paragraphs 5A and 5B are .635 compared with the present technique.

We presume that the analogous order statistics construction would produce much the same result. Our interest in the present technique is to supply an approximate answer to the question of what is to be gained by going from the counting technique proposed in paragraph 5B to the next level of computational difficulty—addition.

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THE NON-CENTRAL WISHART DISTRIBUTION AND CERTAIN PROBLEMS OF MULTIVARIATE STATISTICS[1]

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1. Summary. The non-central Wishart distribution is the joint distribution of the sums of squares and cross-products of the deviations from the sample means when the observations arise from a set of normal multivariate populations with constant covariance matrix but expected values that vary from observation to observation. The characteristic function for this distribution is obtained from the distribution of the observations (Theorem 1). By using the characteristic functions it is shown that the convolution of several non-central Wishart distributions is another non-central Wishart distribution (Theorem 2) A simple integral representation of the distribution in the general case is given (Theorem 3). The integrand is a function of the roots of a determinantal equation involving the matrix of sums of squares and cross-products of deviations of observations and the matrix of sums of squares and cross-products of deviations of corresponding expected values.

The knowledge of the non-central Wishart distribution is applied to two general problems of multivariate normal statistics. The moments of the generalized variance, which is the determinant of sums of squares and cross-products multiplied by a constant, are given for the cases of the expected values of the variates lying on a line (Theorem 4) and lying on a plane (Theorem 5) The likelihood ratio criterion for testing linear hypotheses can be expressed as the ratio of two determinants or as a symmetric function of the roots of a determinantal equation In either case there is involved a matrix having a Wishart distribution and another matrix independently distributed such that the sum of these two matrices has a non-central Wishart distribution When the null hypothesis is not true the moments of this criterion are given in the non-central planar case (Theorem 6).

2. Introduction. The well-known Wishart distribution is the distribution of the sums of squares and cross-products of deviations from the sample means of observations from a multivariate normal distribution. If the expected values of the variates change from observation to observation (with the covariance matrix constant), the distribution of sums of squares and cross-products is the *non-central* Wishart distribution. This distribution has been given explicitly [1] for the simple cases of the non-central problem. If we think of the expected values of each observation as defining a point in a space of dimensionality equal to the number of variates, we can say that the cases handled are those in which the points corresponding to a sample lie on a line or

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[1] Part of a thesis submitted to the Mathematics Department of Princeton University in partial fulfillment of the requirements for the degree of Doctor of Philosophy, June, 1945

a plane. Although the explicit formulas for the distribution of higher rank are extremely complicated and have not been derived, the characteristic function is relatively simple. The distribution in general can be given in terms of a simple multiple integral

The Wishart distribution is the basis of much of the sampling theory associated with the multivariate normal distribution. It plays a role similar to that of the χ^2-distribution in univariate normal theory. It can be used in deriving the distributions of the generalized T^2 and of the multiple correlation coefficient when all variates have a normal distribution; it is used in deriving the moments of the likelihood ratio criterion for testing the general linear hypothesis (including the test of the means of several populations being equal) as well as deriving the moments of other such criteria[2]). For the problems of the T^2 and the test of the linear hypothesis and many other problems, the non-central Wishart distribution must be substituted for the central Wishart distribution when the null hypothesis is not true. That is, the non-central distribution can be the basis of obtaining the power function for many tests in multivariate normal statistics. As an example of the application of the non-central Wishart distribution to these problems, in this paper we obtain the moments of the generalized variance and the moments of the criterion for linear hypotheses when the population means lie on a line or a plane. Applications to other problems such as testing collinearity, comparing scales of measurement, and multiple regression in time series analysis will be published in a later paper [3]. Another problem to which this non-central theory can be applied is a method of estimating the parameters of a single equation of a complete system of linear stochastic difference equations (developed by T. W. Anderson, M. A. Girshick and H. Rubin).

In [1] it was shown that one can make linear transformations on the observations which simplify the derivation of the non-central Wishart distribution in the linear and planar cases. Consider a set of N multivariate normal populations, each of p variates. Let the i-th ($i = 1, 2, \cdots, p$) variate of the α-th ($\alpha = 1, 2, \cdots, N$) population be $x_{i\alpha}$; let the mean of the variate be

$$E(x_{i\alpha}) = \mu_{i\alpha} \qquad (i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N). \tag{1}$$

Let the covariance matrix (of rank p) common to all N populations be

$$\| E(x_{i\alpha} - \mu_{i\alpha})(x_{j\alpha} - \mu_{j\alpha}) \| = \| \sigma_{ij} \| \qquad (\alpha = 1, 2, \cdots, N).$$

The probability element of the $x_{i\alpha}$ can be written as

$$| \sigma^{ij} |^{\frac{1}{2}N} (2\pi)^{-\frac{1}{2}pN} \exp [- \tfrac{1}{2} \sum_{i,j,\alpha} \sigma^{ij} (x_{i\alpha} - \mu_{i\alpha})(x_{j\alpha} - \mu_{j\alpha})] \prod_{i,\alpha} dx_{i\alpha}, \tag{2}$$

where

$$\| \sigma^{ij} \| = \| \sigma_{ij} \|^{-1}.$$

[2] See Wilks [2] for example.

The sum of squares and cross-products of deviations from the means in a sample $\{x_{i\alpha}\}$ are

$$(3) \qquad a_{ij} = \sum_{\alpha=1}^{N} (x_{i\alpha} - \bar{x}_i)(x_{j\alpha} - \bar{x}_j),$$

where

$$\bar{x}_i = \frac{1}{N} \sum_{\alpha=1}^{N} x_{i\alpha}.$$

The dimensionality, say t, of the space spanned by $\| \mu_{i\alpha} \|$ is equal to the rank of

$$(4) \qquad \| \tau_{ij} \| = \| \sum_{\alpha=1}^{N} (\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j) \|,$$

where

$$\bar{\mu}_{i\alpha} = \frac{1}{N} \sum_{\alpha=1}^{N} \mu_{i\alpha}.$$

As a result of a linear transformation it was demonstrated that the distribution of a_{ij} is the same as that of $\sum_{\alpha=1}^{N-1} x'_{i\alpha} x'_{j\alpha}$ where the $x_{i\alpha}$ have a normal multivariate distribution with covariance matrix $\| \sigma_{ij} \|$ and expected values

$$E(x'_{i\alpha}) = \mu'_{i\alpha}$$
$$(i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N - 1),$$

such that

$$\| \tau_{ij} \| = \| \sum_{\alpha=1}^{N-1} \mu'_{j\alpha} \|.$$

The joint distribution of a_{ij} is given for three cases:

(i) *Case* $t = 0$:

$$(5) \; W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N - 1, 0) = K_0 \, | \sigma^{ij} |^{\frac{1}{2}(N-1)} \, | a_{ij} |^{\frac{1}{2}(N-p-2)} \exp [- \tfrac{1}{2} \sum_{i,j} \sigma^{ij} a_{ij}];$$

(ii) *Case* $t = 1$:

$$(6) \quad W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N - 1, 1) = K_1 \exp [-\tfrac{1}{2} \sum_{i,j} \sigma^{ij} \tau_{ij}] \, | \sigma^{ij} |^{\frac{1}{2}(N-1)} \, | a_{ij} |^{\frac{1}{2}(N-p-2)}$$
$$\times \exp [- \tfrac{1}{2} \sum_{i,j} \sigma^{ij} a_{ij}] [\sum_{i,j} a_{ij} \tau_{ij}]^{-\frac{1}{4}(N-3)} \, \mathrm{I}_{\frac{1}{2}(N-3)}(\sqrt{\textstyle\sum_{i,j} a_{ij} \tau_{ij}});$$

(iii) *Case* $t = 2$:

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N - 1, 2) = K_2 \exp [- \tfrac{1}{2} \sum_{i,j} \sigma^{ij} \tau_{ij}] \, | \sigma^{ij} |^{\frac{1}{2}(N-1)}$$
$$(7) \quad \times \, | a_{ij} |^{\frac{1}{2}(N-p-2)} \exp [- \tfrac{1}{2} \sum_{i,j} \sigma^{ij} a_{ij}] \sum_{w=0}^{\infty} \frac{(u_1 u_2)^w}{2^{2w} w! \Gamma(\frac{1}{2}[N - 2] + w)}$$
$$\times \, (u_1 + u_2)^{-\frac{1}{2}(\frac{1}{2}[N-3]+2w)} \, \mathrm{I}_{\frac{1}{2}(N-3)+2w} (\sqrt{u_1 + u_2}),$$

where

$$K_0^{-1} = 2^{\frac{1}{2}p(N-1)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}[N - i]),$$

$$K_1^{-1} = 2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p-1} \Gamma(\tfrac{1}{2}[N - 1 - i]),$$

$$K_2^{-1} = 2^{\frac{1}{2}p(N-1)-\frac{1}{2}(N-3)} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p-2} \Gamma(\tfrac{1}{2}[N - 2 - i]),$$

$I_n(x)$ is the Bessel function of purely imaginary argument, and u_1 and u_2 are the two non-zero roots of

$$| \mathrm{T} - \lambda A^{-1} | = 0 \tag{8}$$

(here $\mathrm{T} = \| \tau_{ij} \|$ and $A = \| a_{ij} \|$) The number $N - 1$ is the *number of degrees of freedom* and t is the *rank*. The matrix $\| \sigma_{ij} \|$ we shall call the *sigma matrix*, and $\| \tau_{ij} \|$ we shall call the *means sigma matrix*.

Let $\kappa_1^2, \kappa_2^2, \cdots, \kappa_p^2$ be the real, non-negative roots of the determinantal equation

$$| \mathrm{T} - \lambda \Sigma | = 0 \tag{9}$$

(where $\Sigma = \| \sigma_{ij} \|$). There is a non-singular $p \times p$ matrix Ψ $(= \| \psi_{ij} \|)$ such that

$$\Psi \Sigma \Psi' = I \tag{10}$$

and

$$\Psi T \Psi' = \| \kappa_i^2 \delta_{ij} \| \tag{11}$$

(where I is the identity, Ψ' is the transpose of Ψ and $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$). Then the quantities

$$b_{ij} = \sum_{h,k=1}^{p} \psi_{ih} \psi_{jk} a_{hk} \tag{12}$$

have the distribution $W(b_{ij}, \delta_{ij}, \kappa_i^2 \delta_{ij}; p, n, t)$ where $n = N - 1$ and $\kappa_i^2 = 0$ for $i = t + 1, t + 2, \cdots, p$). This is the same distribution that would be derived if the b_{ij} were defined by

$$b_{ij} = \sum_{\alpha=1}^{n} y_{i\alpha} y_{j\alpha}, \tag{13}$$

where the distribution of the $y_{i\alpha}$ is

$$(2\pi)^{-\frac{1}{2}pn} \exp [- \tfrac{1}{2} \sum_{i=1}^{p} \sum_{\alpha=1}^{n} (y_{i\alpha} - \kappa_i \delta_{i\alpha})]^2 . \tag{14}$$

This simplified distribution of the observations has been called the canonical form

3. The characteristic function of the non-central Wishart distribution. We shall find the characteristic function of the a_{ii} and $2a_{ij}$ $(i \neq j)$ as defined in (3). We first obtain the characteristic function of the b_{ii} and $2b_{ij}$ $(i \neq j)$ as defined in (13) and then perform a linear transformation to obtain the characteristic function of the a_{ij}. The characteristic function of the b_{ii} and $2b_{ij}$ $(i \neq j)$ is defined as

$$E\left(\exp\left[i \sum_{i,j=1}^{p} b_{ij}\theta_{ij}\right]\right), \tag{15}$$

where

$$\theta_{ij} = \theta_{ji}$$

and i in the exponent is the imaginary quantity
We can write (15) as

$$\begin{aligned} &E\left(\exp\left[i \sum_{i,j=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}\, y_{j\alpha}\, \theta_{ij}\right]\right) \\ &\quad = (2\pi)^{-\frac{1}{2}pn} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=1}^{n} (y_{i\alpha} - \kappa_i\,\delta_{i\alpha})^2 + i \sum_{i,j=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}\, y_{j\alpha}\,\theta_{ij}\right] \\ &\qquad\qquad \times \prod_{i=1}^{p}\prod_{\alpha=1}^{n} dy_{i\alpha}\,. \end{aligned}$$

Let us first integrate the $y_{i\alpha}$ for $i = 1, 2, \cdots, p$ and $\alpha = t+1, t+2, \cdots, n$, that is, make the integration

$$\begin{aligned} &(2\pi)^{-\frac{1}{2}p(n-t)} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \\ &\qquad \cdot \exp\left[-\frac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=t+1}^{n} y_{i\alpha}^2 + i \sum_{i,j=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}\, y_{j\alpha}\,\theta_{ij}\right] \prod_{i=1}^{p}\prod_{\alpha=t+1}^{n} dy_{i\alpha}. \end{aligned}$$

This is, however, the characteristic function of a Wishart distribution with $n - t$ degrees of freedom [4], namely

$$| \delta_{ij} - 2i\theta_{ij} |^{-\frac{1}{2}(n-t)}. \tag{16}$$

Now we must make the integration

$$\begin{aligned} &(2\pi)^{-\frac{1}{2}pt} \exp\left[-\frac{1}{2}\sum_{\eta=1}^{t} \kappa_\eta^2\right] \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \\ &\quad \cdot \exp\left[-\frac{1}{2}\sum_{i=1}^{p}\sum_{\eta=1}^{t} y_{i\eta}^2 + i \sum_{i,j=1}^{p}\sum_{\eta=1}^{t} y_{i\eta}\, y_{j\eta}\,\theta_{ij} + \sum_{\eta=1}^{t} y_{\eta\eta}\,\kappa_\eta\right] \times \prod_{i=1}^{p}\prod_{\eta=1}^{t} dy_{i\eta}. \end{aligned} \tag{17}$$

There is a $p \times p$ matrix $G = \| g_{ij} \|$ such that

$$\sum_{h,k=1}^{p} g_{hi}\, d_{kh}\, g_{kj} = \delta_{ij},$$

where

$$d_{kh} = \delta_{kh} - 2i\theta_{kh}.$$

Let us make the transformation

$$y_{i\eta} = \sum_{h=1}^{p} g_{ih} z_{h\eta} + d^{i\eta} \kappa_{\eta},$$

where

$$\| d^{ih} \| = \| d_{ih} \|^{-1}.$$

Then the exponent of (17) within the integral sign is

$$-\frac{1}{2}\left\{\sum_{i=1}^{p} \sum_{\eta=1}^{t} z_{i\eta}^2 - \sum_{\eta=1}^{t} d^{\eta\eta} \kappa_{\eta}^2\right\},$$

and the Jacobian of the transformation is

$$| d_{i\eta} |^{-\frac{1}{2}t}.$$

Hence, the integral of (17) is

$$| d^{ij} |^{\frac{1}{2}t} \exp\left[-\frac{1}{2}\left(\sum_{\eta=1}^{t} \kappa_{\eta}^2 - \sum_{\eta=1}^{t} d^{\eta\eta} \kappa_{\eta}^2\right)\right]. \tag{18}$$

This result is obviously true if the θ_{ij} are pure imaginary and sufficiently small so that $\| d_{ij} \| = \| \delta_{ij} - 2i\theta_{ij} \|$ (which is real in this case) is positive definite. For all complex θ_{ij} in a neighborhood of the origin (17) converges because the real part of $\| d_{ij} \|$ is positive definite. Similarly the integral of the derivative with respect to θ_{ij} of the integrand converges for θ_{ij} in this neighborhood. It follows that the (complex) derivative of the characteristic function exists in this neighborhood because the derivative of the integrand is measurable and is absolutely integrable. Therefore, the characteristic function is analytic in a neighborhood of the origin. From this it follows that the characteristic function is analytic in an open set containing the flat space of real θ_{ij}. By analytic continuation, then, (18) is the value of (17) in the open set containing real θ_{ij}. The characteristic function (15) is the product of (16) and (18). Accordingly, we have the result that the characteristic function of the b_{ii} and $2b_{ij}$ $(i \neq j)$ defined by (13) is

$$| d^{ij} |^{\frac{1}{2}n} \exp\left[-\frac{1}{2}\left(\sum_{\eta=1}^{t} \kappa_{\eta}^2 - \sum_{\eta=1}^{t} d^{\eta\eta} \kappa_{\eta}^2\right)\right]. \tag{19}$$

It is clear that if $\kappa_{\eta} = 0$ (for all η), this function reduces to the characteristic function of the Wishart distribution with n degrees of freedom, namely,

$$| \delta_{ij} - 2i\theta_{ij} |^{-\frac{1}{2}n}. \tag{20}$$

It is interesting to note that (19) factors into two parts, one of which is (20) and the other is

$$\exp\left[-\frac{1}{2}\left(\sum_{\eta=1}^{t} \kappa_{\eta}^2 - \sum_{\eta=1}^{t} d^{\eta\eta} \kappa_{\eta}^2\right)\right]. \tag{21}$$

The distribution function similarly factors into two parts, one of which is the Wishart distribution, whose characteristic function is (20). Thus the non-central Wishart distribution function is the convolution of a function (central Wishart distribution) and another (the transform of (21) the first of which is a factor of this same non-central Wishart distribution.

In the planar case the characteristic function can be written as

$$\frac{\exp\left[-\frac{1}{2}(\kappa_1^2 - d^{11}\kappa_1^2)\right]}{|\delta_{ij} - 2i\theta_{ij}|^{\frac{1}{2}n_1}} \cdot \frac{\exp\left[-\frac{1}{2}(\kappa_2^2 - d^{22}\kappa_2^2)\right]}{|\delta_{ij} - 2i\theta_{ij}|^{\frac{1}{2}n_2}},$$

where $n_1 + n_2 = n$. From this fact it is clear that the distribution for the planar case (if $n \geq 2p + 2$) is a convolution of two distributions each of the linear case.

This deduction can also be made from the distribution (14). Let

$$b'_{ij} = y_{i1}y_{j1} + \sum_{\alpha=3}^{n_1+1} y_{i\alpha}y_{j\alpha},$$

$$b''_{ij} = y_{i2}y_{j2} + \sum_{\alpha=n_1+2}^{n} y_{i\alpha}y_{j\alpha}.$$

Then it is clear that the b'_{ij} has the non-central Wishart distribution with n_1 degrees of freedom and parameter κ_1^2 in the direction of the first coordinate axis, while the b''_{ij} has the non-central Wishart distribution with n_2 degrees of freedom and parameter κ_2^2 in the direction of the second coordinate axis. Since

$$b_{ij} = b'_{ij} + b''_{ij},$$

the distribution of the b_{ij} is a convolution of the distributions of b'_{ij} and b''_{ij}. In general the non-central distribution is the convolution of t distributions of the linear case (provided $n \geq tp + t$).

It is easy to show that if one has two (or more) non-central Wishart distributions of rank 1 with parameters in the same direction, the convolution is again a non-central Wishart distribution with parameter in the same direction. Suppose b'_{ij} and b''_{ij} have non-central Wishart distributions with parameter κ'^2_1 and κ''^2_1 in the direction of the first coordinate axes and n_1 and n_2 degrees of freedom respectively. The characteristic functions are

$$|d^{ij}|^{\frac{1}{2}n_1} \exp\left[-\tfrac{1}{2}(\kappa_1'^2 - d^{11}\kappa_1'^2)\right]$$

and

$$|d^{ij}|^{\frac{1}{2}n_2} \exp\left[-\tfrac{1}{2}(\kappa_1''^2 - d^{11}\kappa_1''^2)\right].$$

The product is

$$|d^{ij}|^{\frac{1}{2}n} \exp\left[-\tfrac{1}{2}(\kappa_1^2 - d^{11}\kappa_1^2)\right],$$

where $n = n_1 + n_2$ and $\kappa_1^2 = \kappa'^2_1 + \kappa''^2_1$.

Now let us deduce the characteristic function of the a_{ii} and $2a_{ij}$ ($i \neq j$). Since by (12) the b's are transforms of the a's we can write $a_{ij} = \sum_{h,k=1}^{p} \psi^{ih}\psi^{jk} b_{hk}$.

Then

$$E\left(\exp\left[i\sum_{i,j=1}^{p} a_{ij}\phi_{ij}\right]\right) = E\left(\exp\left[i\sum_{i,j=1}^{p}\phi_{ij}\psi^{ih}\psi^{jk}b_{hk}\right]\right), \tag{22}$$

where $\phi_{ij} = \phi_{ji}$. If we define

$$\theta_{hk} = \sum_{i,j=1}^{p}\phi_{ij}\psi^{ih}\psi^{jk}, \tag{23}$$

then (22) can be derived by substituting (23) in (19).

Let

$$\Phi = \| \phi_{ij} \|.$$

Then

$$\| d_{ij} \| = D = \Psi'^{-1}(\Sigma^{-1} - 2i\Phi)\Psi^{-1}$$

and

$$D^{-1} = \Psi(\Sigma^{-1} - 2i\Phi)^{-1}\Psi'.$$

The characteristic function of the a's then can be written as

$$\frac{\exp\left[-\frac{1}{2}\{tr(\Psi T\Psi') - tr[\Psi(\Sigma^{-1} - 2i\Phi)^{-1}\Psi'\Psi T\Psi']\}\right]}{\{|\Psi'^{-1}| \cdot |\Sigma^{-1} - 2i\Phi| \cdot |\Psi^{-1}|\}^{\frac{1}{2}n}}$$

using (10) and (11) The denominator is

$$\{|\Psi'| \quad |\Psi|\}^{-\frac{1}{2}n}|\Sigma^{-1} - 2i\Phi|^{\frac{1}{2}n}$$

and the numerator can be written as[3]

$$\exp\left[-\tfrac{1}{2}\{\mathrm{tr}\,(M'\Psi'\Psi M) - \mathrm{tr}\,[M'\Psi'\Psi(\Sigma^{-1} - 2i\Phi)^{-1}\Psi'\Psi M]\}\right]$$

where

$$M = \| \mu_{i\alpha} - \mu_i \|$$

and

$$M'M = T.$$

We may summarize in the following theorem:

THEOREM 1. *Given a_{ij} $(i, j = 1, 2, \cdots, p)$ defined by* (3) *where the $x_{i\alpha}$ $(i = 1, 2, \cdots, p, \alpha = 1, 2, \cdots, N)$ are distributed according to* (2), *the characteristic function of a_{ij} and $2a_{ij}$ $(i \neq j)$ is*

$$\begin{aligned} E\left(\exp\left[i\sum_{i,j=1}^{p}a_{ij}\phi_{ij}\right]\right) &= \frac{|\sigma^{ij}|^{\frac{1}{2}(N-1)}\exp\left[-\dfrac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{N}\sigma^{ij}(\mu_{i\alpha} - \bar{\mu}_i)(\mu_{j\alpha} - \bar{\mu}_j)\right]}{|\bar{\sigma}^{ij}|^{\frac{1}{2}(n-1)}} \\ &\quad \cdot \exp\left[\frac{1}{2}\sum_{\substack{i,j=1\\h,k=1}}^{p}\sum_{\alpha=1}^{N}\sigma^{hi}\bar{\sigma}_{ij}\sigma^{jh}(\mu_{h\alpha} - \bar{\mu}_h)(\mu_{k\alpha} - \bar{\mu}_k)\right], \end{aligned} \tag{24}$$

[3] The result follows from the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

where

$$\| \tilde{\sigma}_{ij} \|^{-1} = \| \tilde{\sigma}^{ij} \| = \| \sigma^{ij} - 2i\phi_{ij} \|$$

and

$$\phi_{ij} = \phi_{ji} .$$

Suppose we have two sets of quantities a'_{ij} and a''_{ij} each set of which is distributed according to a non-central Wishart distribution with sigma matrix $\| \sigma^{ij} \|$, one having n' degrees of freedom (or n''), means sigma matrix τ'_{ij} (or τ''_{ij}) of rank t' (or t''). Consideration of the characteristic functions (24) shows that

$$a_{ij} = a'_{ij} + a''_{ij}$$

has a non-central Wishart distribution with matrix $\| \sigma^{ij} \|$, $n' + n''$ degrees of freedom and a matrix

$$\| \tau_{ij} \| = \| \tau'_{ij} \| + \| \tau''_{ij} \|.$$

The rank of the distribution is equal to the rank of $\| \tau_{ij} \|$. This result can also be deduced from the representation of a'_{ij} and a''_{ij} in terms of observations from non-central normal populations. It is a straightforward generalization of the same result for central Wishart distributions.

THEOREM 2. *The convolution of two or more non-central Wishart distributions with identical sigma matrices is a non-central Wishart distribution with means sigma matrix equal to the sum of the means sigma matrices of the components.*

4. An integral representation of the non-central Wishart distribution in the general case. It was shown in [1] that

$$W(b_{ij}, \delta_{ij}, \kappa_i^2 \delta_{ij}; p, n, t) = Ce^{-\frac{1}{2}trB} \int | B - YY' |^{\frac{1}{2}(n-p-t-1)} e^{tr(K'Y)} \, dY$$

where

$$C^{-1} = e^{\frac{1}{2}tr(K'K)} 2^{\frac{1}{2}pn} \pi^{\frac{1}{4}p(p-1)+\frac{1}{2}pt} \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}[n - t + 1 - i]),$$

$$dB = \prod_{i=1}^{p} \prod_{j=i}^{p} db_{ij},$$

$$dY = \prod_{i=1}^{p} \prod_{\eta=1}^{t} dy_{i\eta},$$

$$B = \| b_{ij} \|,$$

$$Y = \| y_{i\eta} \| \qquad (\eta = 1, 2, \cdots, t),$$

$$K = \| \kappa_i \delta_{i\eta} \| \qquad (\eta = 1, 2, \cdots, t),$$

and the integration is on Y over the range $\| B - YY' \|$ positive semi-definite. This is equivalent to

$$(25) \qquad Ce^{-\frac{1}{2}trB} | B |^{\frac{1}{2}(n-p-t-1)} \int | I - Y'B^{-1}Y |^{\frac{1}{2}(n-p-t-1)} e^{tr(K'Y)} \, dY.$$

The integration is over the range of Y for which $|| I - Y'B^{-1}Y ||$ is positive semi-definite.

There is a p by p matrix $H = || h_{ij} ||$ such that

$$H'B^{-1}H = I$$

$$H'K = W = || w_i \delta_{i\eta} ||,$$

where w_i^2 are the roots of

$$| \kappa_i^2 \delta_{ij} - \lambda b^{ij} | = 0,$$

$$|| b^{ij} || = || b_{ij} ||^{-1}.$$

Then make the transformation to $Z = || z_{i\eta} ||$ by

$$Y = HZ.$$

The Jacobian of the transformation is

$$| H |^t = | B |^{\frac{1}{2}t}.$$

Then (25) can be written as

$$(26) \qquad Ce^{-\frac{1}{2}trB} | B |^{\frac{1}{2}(n-p-1)} \int | I - Z'Z |^{\frac{1}{2}(n-p-t-1)} e^{trW('Z)} \, dZ.$$

Partition

$$Z = \begin{Vmatrix} Z^1 \\ Z_1 \end{Vmatrix}$$

such that Z_1 is square ($t \times t$). Let $I - Z_1'Z_1 = E'E$, (in terms of Z_1), where E is specified uniquely and consider the transformation of variables from Z_2 to V defined by

$$Z_2 = VE.$$

Then (26) can be written as

$$Ce^{-\frac{1}{2}trB} | B |^{\frac{1}{2}(n-p-1)} dB \int | I - Z_1'Z_1 |^{\frac{1}{2}(n-2t-1)} e^{tr(W_1'Z_1)} \, dZ_1$$

$$\cdot \int | I - V'V |^{\frac{1}{2}(n-p-t-1)} \, dV_1$$

where

$$W_1 = || w_\eta \delta_{\eta\xi} || \qquad (\eta, \xi = 1, 2, \cdots, t).$$

The first integration is over the range $(I - Z_1'Z_1)$ positive semi-definite and the second is over $(I - V'V)$ positive semi-definite. The value of the second integral is

$$\int |I - V'V|^{\frac{1}{2}(n-p-t-1)}\, dV = \frac{\pi^{\frac{1}{2}t(p-t)} \prod_{i=1}^{p-t} \Gamma(\tfrac{1}{2}[n - 2t + 1 - i])}{\prod_{i=1}^{p-t} \Gamma(\tfrac{1}{2}[n - t + 1 - i])}.$$

Hence (26) can be written as

$$C_1 e^{-\frac{1}{2}trB} |B|^{\frac{1}{2}(n-p-1)} \int |I - Z_1'Z_1|^{\frac{1}{2}(n-2t-1)} e^{tr(W_1'Z_1)}\, dZ_1 \tag{27}$$

with

$$C_1^{-1} = e^{\frac{1}{2}tr(K'K)} 2^{\frac{1}{2}pn} \pi^{\frac{1}{2}p(p-1)+\frac{1}{2}t^2} \times \prod_{i=1}^{t} \Gamma(\tfrac{1}{2}[n - t + 1 - i]) \prod_{i=1}^{p-t} \Gamma(\tfrac{1}{2}[n - t + 1 - i]). \tag{28}$$

The first part of (27) is, except for a constant factor, a central Wishart distribution with n degrees of freedom. The integral of the second part is obviously a symmetric function of the w_i'. In terms of the a_{ij} the w_i^2 are simply the roots of (8). We can sum these results in a theorem.

THEOREM 3. *Given a sample of observations* $\{x_{i\alpha}\}$ $(i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N)$ *distributed according to* (2), *the probability density function of the sums of squares and cross products of deviations from the sample means defined by* (3) *is*

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, t) = C_1 |\sigma^{ij}|^{\frac{1}{2}(N-1)} |a_{ij}|^{\frac{1}{2}(N-p-2)} \cdot \exp\left[-\frac{1}{2}\sum_{i,j} \sigma^{ij} a_{ij}\right] \int \left|\delta_{\eta\xi} - \sum_{i=1}^{t} z_{i\eta} z_{i\xi}\right|^{\frac{1}{2}(N-2t-2)} \cdot \exp \sum_{i=1}^{t} w_i z_{ii} \prod_{\eta,\xi=1}^{t} dz_{\eta\xi}$$

integrated over

$$\left\| \delta_{\eta\xi} - \sum_{i=1}^{t} z_{i\eta} z_{i\xi} \right\|$$

positive semi-definite where C_1 $(n = N - 1)$ *and* τ_{ij} *are defined by* (28) *and* (4), *respectively, and where* w_η^2 *are the* t *non-zero roots of* (8).

5. The moments of the generalized variance in the linear and planar non-central cases.

5.1. *The linear case.* The generalized variance, which is the determinant of the variances and covariances,[4] is a measure of the spread of the observations. If one thinks of the N observations of each variate as a vector in N-space with

[4] This definition of Wilks [5] was made in terms of variances and covariances defined by a_{ij}/N (from equation (3)). Since we consider $a_{ij}/(N-1)$ to be the variances and covariances we define $|a_{ij}/(N-1)|$ as the generalized variance.

origin at the sample mean, the generalized variance is proportional to the square of the volume of the p dimensional parallelotope which is defined by these vectors as principal edges. Another geometric interpretation can be given in terms of the p-dimensional variate space. The generalized variance is proportional to the sum of the squared volumes of all possible parallelotopes that can be joined by choosing as the p principal edges p of the N sample vectors (origin at the sample mean).

In this section we consider the moments of the generalized variance when the distributions of the observations are non-central multivariate normal. In terms of the first geometric representation this means that the center of one or more of the vector distributions is different from the others. For convenience we shall assume that the distribution of the observations $\{y_{i\alpha}\}$ is according to (14). This will give as much generality as if we treated observations $\{x_{i\alpha}\}$ having the distribution (2). Moreover, we shall consider the determinant of sums of squares and cross-products instead of the determinant of variances and covariances. It is clear that the determinantant $|b_{ij}|$, defined by (13), is simply a multiple (by $|\Sigma|(N-1)^p$) of $\left|\frac{a_{ij}}{N-1}\right|$, defined by (3).

Let us first consider the linear case, i.e., $\kappa = \kappa_1 \neq 0$ and $\kappa_i = 0$ $(i = 2, \cdots, p)$ in (14). The first of the p vectors is centered on the first coordinate axis, not at the origin. Then the probability density function of the b_{ij} is

$$\frac{e^{-\frac{1}{2}\kappa^2}\,|b_{ij}|^{\frac{1}{2}(n-p-1)}\exp\left[-\frac{1}{2}\sum_{i=1}^{p} b_{ii}\right]}{2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n-i])}\sum_{\alpha=0}^{\infty}\frac{(\kappa^2 b_{11})^{\alpha}}{2^{2\alpha}\alpha!\Gamma(\frac{1}{2}n+\alpha)}. \tag{29}$$

We wish to find the moments $E(|b_{ij}|^h)$. Let

$$b_{ij} = s_i s_j r_{ij}.$$

Then s_i^2 is the sum of squares of the i-th variate and $||r_{ij}||$ is the matrix of sample correlation coefficients. The Jacobian of this transformation (to s_i^2, r_{ij}) is

$$(s_1^2)^{\frac{1}{2}(p-1)}(s_2^2)^{\frac{1}{2}(p-1)}\cdots(s_p^2)^{\frac{1}{2}(p-1)}.$$

The probability element of the s^2's and r's is

$$\frac{\exp\left[-\frac{1}{2}\kappa^2-\frac{1}{2}\sum_{i=1}^{p}s_i^2\right]\prod_{i=1}^{p}(s_i^2)^{\frac{1}{2}n-1}}{2^{\frac{1}{2}pn}\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n-i])}\,|r_{ij}|^{\frac{1}{2}(n-p-1)} \tag{30}$$
$$\times\sum_{\alpha=0}^{\infty}\frac{(\kappa^2 s_1^2)^{\alpha}}{2^{2\alpha}\alpha!\Gamma(\frac{1}{2}n+\alpha)}\prod_{i=1}^{p}d(s_i^2)\prod_{i=1}^{p}\prod_{j=i+1}^{p}dr_{ij}.$$

It is clear from (30) that the s_i^2 are distributed independently and that the set r_{ij} have a joint distribution independent of the s_i^2's. Hence

$$E(|b_{ij}|^h) = E(|s_i s_j r_{ij}|^h) = \prod_{i=1}^{p} E[(s_i^2)^h]\cdot E(|r_{ij}|^h).$$

The probability element of s_i^2 $(i = 2, 3, \cdots, p)$ is

$$\frac{e^{-\frac{1}{2}s_i^2}(s_i^2)^{\frac{1}{2}n-1}}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n)}\, d(s_i^2),$$

which is simply the χ^2-distribution. The h-th moment of s_i^2 $(i = 2, 3, \cdots, p)$ is

$$E[(s_i^2)^h] = \frac{2^h\Gamma(\frac{1}{2}n + h)}{\Gamma(\frac{1}{2}n)}.$$

The probability element of s_1^2 is

$$\text{(31)}\qquad \frac{e^{-\frac{1}{2}\kappa^2}(s_1^2)^{\frac{1}{2}n-1}e^{-\frac{1}{2}s_1^2}}{2^{\frac{1}{2}n}} \sum_{\alpha=0}^{\infty} \frac{(\kappa^2 s_1^2)^\alpha}{2^{2\alpha}\alpha!\Gamma(\frac{1}{2}n + \alpha)}\, d(s_1^2).$$

This is the χ'^2-distribution (non-central χ^2-distribution) which was given by Fisher [6]. Applying term-by-term integration (the series converges properly) we get the h-th moment

$$E[(s_1^2)^h] = 2^h e^{-\frac{1}{2}\kappa^2} \sum_{\alpha=0}^{\infty} \frac{(\kappa^2)^\alpha\Gamma(\frac{1}{2}n + h + \alpha)}{2^\alpha\alpha!\Gamma(\frac{1}{2}n + \alpha)}.$$

The probability element of the r_{ij} is the well known distribution of correlation coefficients,

$$\frac{\Gamma^{p-1}(\frac{1}{2}n)\,|r_{ij}|^{\frac{1}{2}(n-p-1)}}{\pi^{\frac{1}{4}p(p-1)}\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n - i])} \prod_{i=1}^{p}\prod_{j=i+1}^{p} dr_{ij}.$$

Since

$$\int \frac{|r_{ij}|^{\frac{1}{2}(n-p-1)}}{\pi^{\frac{1}{4}p(p-1)}} \prod_{i=1}^{p}\prod_{j=i+1}^{p} dr_{ij} = \frac{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n - i])}{\Gamma^{p-1}(\frac{1}{2}n)},$$

where the integration is over the entire (permissable) range o the r_{ij}, we have as a consequence the h-th moment of the determinant (since n is arbitrary)

$$\begin{aligned} E(|r_{ij}|^h) &= \frac{\Gamma^{p-1}(\frac{1}{2}n)}{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n - i])} \int \frac{|r_{ij}|^{\frac{1}{2}(n-p-1)+h}}{\pi^{\frac{1}{4}p(p-1)}} \prod_{i=1}^{p}\prod_{j=i+1}^{p} dr_{ij} \\ &= \frac{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n - i] + h)\Gamma^{p-1}(\frac{1}{2}n)}{\prod_{i=1}^{p-1}\Gamma(\frac{1}{2}[n - i])\Gamma^{p-1}(\frac{1}{2}n + h)}. \end{aligned}$$

Hence, the h-th moment of $| s_i s_j r_{ij} |$ is

$$(32) \qquad 2^{ph} \frac{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n + 2h - i])}{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[n - i])} e^{-\frac{1}{2}\kappa^2} \sum_{\alpha=0}^{\infty} \frac{\kappa^{2\alpha}\Gamma(\frac{1}{2}n + h + \alpha)}{2^\alpha \alpha! \Gamma(\frac{1}{2}n + \alpha)}.$$

Let us summarize this in a theorem for the a_{ij}.

THEOREM 4. *If the quantities a_{ij} $(i, j = 1, 2, \cdots, p)$ have the distribution $W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N - 1, 1)$ defined by* (6), *then the moments of $| a_{ij} |$ are given by*

$$E(| a_{ij} |^h) = | \sigma^{ij} |^h 2^{ph} e^{-\frac{1}{2}\kappa^2} \frac{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[N - 1 - i] + h)}{\prod_{i=1}^{p-1} \Gamma(\frac{1}{2}[N - 1 - i])} \sum_{\alpha=0}^{\infty} \frac{\kappa^{2\alpha}\Gamma(\frac{1}{2}[N - 1] + h + \alpha)}{2^\alpha \alpha! \Gamma(\frac{1}{2}[N - 1] + \alpha)},$$

where κ^2 is the non-zero root of (9).

The h-th moment of the generalized variance $| a_{ij}/(N - 1) |$ is obtained by dividing the above expression by $(N - 1)^{ph}$.

If $\kappa^2 = 0$, expression (32) clearly reduces to the moment given by Wilks [5]

$$(33) \qquad 2^{ph} \frac{\prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i] + h)}{\prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])}.$$

The expression (32) gives the moments of the generalized variance when the means of the observations are not fixed, but lie on a line. The distribution of $| b_{ij} |$ is not a simple function even in the central case. However, in any particular case one could find the first few moments of $| b_{ij} |$ and fit a distribution function. It is to be noted that the convergence of the series is nearly as rapid as that for $e^{\frac{1}{2}\kappa^2}$.

5.2. *The planar case.* Next we shall treat the planar case for two dimensions. Suppose that $\kappa_i^2 \neq 0$ $(i = 1, 2)$. The probability density function of b_{11}, b_{12}, and b_{22} is

$$(34) \qquad \frac{\exp\left[-\frac{1}{2}(\kappa_1^2 + \kappa_2^2) - \frac{1}{2} \sum_{i=1}^{2} b_{ii} \right]}{2^n \sqrt{\pi}} (b_{11} b_{22} - b_{12}^2)^{\frac{1}{2}(n-3)} \times \sum_{\alpha,\beta=0}^{\infty} \frac{[\kappa_1^2 \kappa_2^2 (b_{11} b_{22} - b_{12}^2)]^\alpha (\kappa_1^2 b_{11} + \kappa_2^2 b_{22})^\beta}{2^{4\alpha+2\beta} \alpha! \beta! \Gamma(\frac{1}{2}[n - 1] + \alpha) \Gamma(\frac{1}{2}n + 2\alpha + \beta)}.$$

Let $b_{11} = s_1^2$, $b_{22} = s_2^2$, and $b_{12} = s_1 s_2 r$. The Jacobian is $s_1 s_2$. The probability element of s_1^2, s_2^2 and r is

$$(35) \qquad \frac{e^{-\frac{1}{2}(\kappa_1^2+\kappa_2^2)}}{2^n \sqrt{\pi}} (s_1^2)^{\frac{1}{2}n-1} (s_2^2)^{\frac{1}{2}n-1} (1 - r^2)^{\frac{1}{2}(n-3)} e^{-\frac{1}{2}(s_1^2+s_2^2)} \times \sum_{\alpha,\beta=0}^{\infty} \frac{(\kappa_1^2 \kappa_2^2 s_1^2 s_2^2)^\alpha (1 - r^2)^\alpha (\kappa_1^2 s_1^2 + \kappa_2^2 s_2^2)^\beta \, d(s_1^2) \, d(s_2^2) \, dr}{2^{4\alpha+2\beta} \alpha! \beta! \Gamma(\frac{1}{2}[n - 1] + \alpha) \Gamma(\frac{1}{2}n + 2\alpha + \beta)}.$$

We wish to find $E\{[s_1^2 s_2^2(1 - r^2)]^h\}$. Let us first multiply (35) by $(1 - r^2)^h$ and integrate from -1 to $+1$. We then obtain

$$2^{-n} e^{-\frac{1}{2}(\kappa_1^2+\kappa_2^2)} (s_1^2)^{\frac{1}{2}n-1} (s_2^2)^{\frac{1}{2}n-1} e^{-\frac{1}{2}(s_1^2+s_2^2)}$$

$$\times \sum_{\alpha,\beta=0}^{\infty} \frac{(\kappa_1^2 \kappa_2^2 s_1^2 s_2^2)^\alpha (\kappa_1^2 s_1^2 + \kappa_2 s_2^2)^\beta \Gamma(\frac{1}{2}[n-1] + h + \alpha)\, d(s_1^2)\, d(s_2^2)}{2^{4\alpha+2\beta} \alpha! \beta! \Gamma(\frac{1}{2}[n-1] + \alpha) \Gamma(\frac{1}{2}n + 2\alpha + \beta) \Gamma(\frac{1}{2}n + h + \alpha)}.$$

Next we multiply by $(s_1^2)^h (s_2^2)^h$, set $(\kappa_1^2 s_1^2 + \kappa_2^2 s_2^2)^\beta / \beta!$ equal to

$$\sum_{\beta_1+\beta_2=\beta} \frac{(\kappa_1^2 s_1^2)^{\beta_1} (\kappa_2^2 s_2^2)^{\beta_2}}{\beta_1! \beta_2!},$$

and integrate s_1^2 and s_2^2 from 0 to ∞. We obtain

$$E([b_{11}b_{22} - v_{12}^2]^h)$$

$$= 2^{2h} \exp\left[-\tfrac{1}{2}(\kappa_1^2 + \kappa_2^2)\right] \tag{36}$$

$$\times \sum_{\alpha,\beta_1,\beta_2=0}^{\infty} \frac{(\kappa_1^2)^{\alpha+\beta_1} (\kappa_2^2)^{\alpha+\beta_2} \Gamma(\frac{1}{2}n + h + \alpha + \beta_1) \Gamma(\frac{1}{2}n + h + \alpha + \beta_2)}{2^{2\alpha+\beta_1+\beta_2} \alpha! \beta_1! \beta_2! \Gamma(\frac{1}{2}[n-1] + \alpha) \Gamma(\frac{1}{2}n + 2\alpha + \beta_1 + \beta_2)}$$

$$\times \frac{\Gamma(\frac{1}{2}[n-1] + h + \alpha)}{\Gamma(\frac{1}{2}n + h + \alpha)},$$

which is the expected value we are seeking.

Clearly this reduces to a special case of (32) if κ_2^2 is set equal to zero.

Now we consider the planar case in p dimensions. Geometrically we have p vectors in n-space. If the $\{y_{i\alpha}\}$ are distributed according to (14) the mean point (i.e., center of distribution) of the first two vectors is different from the origin, but the mean point of each of the other $p - 2$ vectors is the origin. The vectors are distributed independently. The determinant

$$|b_{ij}| = \left| \sum_{\alpha=1}^{n} y_{i\alpha} y_{j\alpha} \right|$$

is the square of the volume of the parallelopiped which can be expressed as

$$v_1 v_2 \cdots v_p \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{p-1},$$

where v_i is the length of the i-th vector and θ_i is the angle between the $(i + 1)$-st vector and the flat space determined by the first i vectors. The distribution of $v_3, \cdots, v_p$ and $\theta_2, \cdots, \theta_{p-1}$ is statistically independent of v_1, v_2, and θ_1; for no matter what the plane of the first two vectors is, the conditional distribution of the other variables is the same. Hence

$$E(|b_{ij}|^h) = E[(v_1 v_2 \sin\theta_1)^{2h}] \cdot E[(v_3 v_4 \cdots v_p \sin\theta_2 \cdots \sin\theta_{p-1})^{2h}].$$

If the y's had simply the distribution

$$(37) \qquad \frac{1}{(2\pi)^{\frac{1}{2}pn}} \exp\left[-\tfrac{1}{2}\sum_{i=1}^{p}\sum_{\alpha=1}^{n} y_{i\alpha}^2\right],$$

then the h-th moment of $|b_{ij}|$ would be (33), and the h-th moment of

$$v_1^2 v_2^2 \sin^2\theta_1 = \begin{vmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{vmatrix}$$

would be

$$2^{2h}\frac{\prod_{i=1}^{2}\Gamma(\frac{1}{2}[n+1-i]+h)}{\prod_{i=1}^{2}\Gamma(\frac{1}{2}[n+1-i])}.$$

Since the distribution of $v_3, v_4, \cdots, v_p$ and $\theta_2, \cdots, \theta_{p-1}$ is the same whether the y's are distributed according to (14) or (37), we have

$$(38) \qquad E[(v_3 \cdots v_p \sin\theta_2 \cdots \sin\theta_{p-1})^{2h}] = 2^{h(p-2)}\frac{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+2h+1-i])}{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+1-i])}.$$

Multiplying (36) by (38) we obtain the h-th moment of $|b_{ij}|$, namely,

$$E(|b_{ij}|^h) = 2^{hp}\exp[-\tfrac{1}{2}(\kappa_1^2+\kappa_2^2)]\frac{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+2h+1-i])}{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[n+1-i])}$$

$$\times \sum_{\alpha,\beta_1,\beta_2=0}^{\infty}\frac{(\kappa_1^2)^{\alpha+\beta_1}(\kappa_2^2)^{\alpha+\beta_2}\Gamma(\frac{1}{2}n-h+\alpha+\beta_1)}{2^{2\alpha+\beta_1+\beta_2}\,\alpha!\,\beta_1!\,\beta_2!\,\Gamma(\frac{1}{2}[n-1]+\alpha)}$$

$$\frac{\Gamma(\frac{1}{2}n+h+\alpha+\beta_2)\,\Gamma(\frac{1}{2}[n-1]+h+\alpha)}{\Gamma(\frac{1}{2}n+2\alpha+\beta_1+\beta_2)\Gamma(\frac{1}{2}n+h+\alpha)}.$$

This result may be summarized as follows:

THEOREM 5. *Let the probability density function of the quantities a_{ij} $(i, j = 1, 2, \cdots, p)$ be*

$$W(a_{ij}, \sigma_{ij}, \tau_{ij}; p, N-1, 2)$$

defined by (7). *Then the h-th moment of $|a_{ij}|$ is*

$$E(|a_{ij}|^h) = |\sigma^{ij}|^h\, 2^{hp}\exp[-\tfrac{1}{2}\kappa_1^2+\kappa_2^2]\frac{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[N-i]+h)}{\prod_{i=3}^{p}\Gamma(\frac{1}{2}[N-i])}$$

$$(39) \quad \sum_{\alpha,\beta_1,\beta_2=0}^{\infty} \frac{(\kappa_1^2)^{\alpha+\beta_1}(\kappa_2^2)^{\alpha+\beta_2}\Gamma(\frac{1}{2}[N-1]+h+\alpha+\beta_1)\Gamma(\frac{1}{2}[N-1]+h+\alpha+\beta_2)}{2^{2\alpha+\beta_1+\beta_2}\alpha!\beta_1!\beta_2!\Gamma(\frac{1}{2}[N-2]+\alpha)\Gamma(\frac{1}{2}[N-1]+2\alpha+\beta_1+\beta_2)} \times \frac{\Gamma(\frac{1}{2}[N-2]+h+\alpha)}{\Gamma(\frac{1}{2}[N-1]+h+\alpha)},$$

with κ_i^2 *defined by* (9).

The h-th moment of the generalized variance $|a_{ij}/(N-1)|$ is obtained by dividing the above expression by $(N-1)^{ph}$. This formula holds for all $h > -\frac{1}{2}(N-p)$.

6. The moments of the criterion for testing linear hypothesis in the linear and planar non-central cases.

6.1. *The moments of the criterion.* There are several linear hypotheses concerning the means of multivariate normal populations that can be included in a general formulation of the problem. We shall first of all consider a simple case of a linear hypothesis and find the moments of the criterion under linear and planar alternatives. In Section 6.2 we shall indicate some linear hypotheses that can be reduced to this simple case. Regression problems and the problem of equality of means in several populations (studied by Wilks) are included.

Suppose the variates $z_{i\alpha}$ ($i = 1, 2, \cdots, p$; $\alpha = 1, 2, \cdots, n$) and $y_{i\gamma}$ ($i = 1, 2, \cdots p$; $\gamma = 1, 2, \cdots, q$) have the probability element

$$(40) \quad \frac{|\sigma^{ij}|^{\frac{1}{2}(n+q)}}{(2\pi)^{\frac{1}{2}p(n+q)}} \exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{p}\sum_{\alpha=1}^{n}\sigma^{ij}z_{i\alpha}z_{j\alpha}\right] \exp\left[-\tfrac{1}{2}\sum_{i,j=1}^{p}\sum_{\gamma=1}^{q}\sigma^{ij}(y_{i\gamma}-\mu_{i\gamma})(y_{j\gamma}-\mu_{j\gamma})\right]\prod_{i=1}^{p}\prod_{\alpha=1}^{n}dz_{i\alpha}\prod_{i=1}^{p}\prod_{\gamma=1}^{q}dy_{i\gamma}.$$

Let us consider the hypothesis H_0 that the means of mp y's are zero, namely,

$$H_0: \mu_{i\gamma} = 0 \quad (i = 1, 2, \cdots, p; \gamma = 1, 2, \cdots, m)$$

Let

$$(41) \quad a_{ij} = \sum_{\gamma=1}^{m} y_{i\gamma}\, y_{j\gamma},$$

$$(42) \quad b_{ij} = \sum_{\alpha=1}^{n} z_{i\alpha}z_{j\alpha},$$

$$(43) \quad c_{ij} = a_{ij} + b_{ij}.$$

Then the likelihood ratio criterion for testing H_0, called by Hsu [8] the Wilks-Lawley hypothesis, is the $\frac{1}{2}(n+q)$ power of

$$(44) \quad W = \frac{|b_{ij}|}{|c_{ij}|}.$$

Under the null hypothesis the b_{ij} have a Wishart distribution with n degrees of freedom, and the a_{ij} are distributed independently of b_{ij} such that c_{ij} has a Wishart distribution with $n + m$ degrees of freedom. Wilks [5] has given the moments of W and in some special cases the distribution of W.

We shall now obtain the moments of W for distributions specified by (40) where the rank of $\| \mu_{i\gamma} \|$ $(\gamma = 1, 2, \cdots, m)$ is 2, i.e., the planar case. Under this assumption the b_{ij} have a Wishart distribution with n degrees of freedom, the a_{ij} are independently distributed in such a way that the c_{ij} have a non-central Wishart distribution with $n + m$ degrees of freedom. Let κ_1^2 and κ_2^2 be the non-zero roots of

$$(45)\qquad \left| \sum_{\gamma=1}^{m} \mu_{i\gamma} \mu_{j\gamma} - \lambda \sigma_{ij} \right| = 0.$$

It is clear that the distribution of W is unchanged if σ^{ij} is set equal to δ_{ij}. Furthermore, we can take $\tau_{ij} = \kappa_i^2 \delta_{ij}$ then the c_{ij} are distributed according to $W(c_{ij}, \delta_{ij}, \kappa_j^2 \delta_{ij}; p, n, 2)$ with $n + m$ degrees of freedom. The moments will be obtained by a method similar to that used by Wilks [5].

Let the expected value given by (39) be

$$(46)\qquad E(|c_{ij}|^h) = K(n + m, h, p, \kappa_i^2),$$

which is a constant depending on $n + m, h, p, \kappa_1^2$, and κ_2^2. If $D(a_{ij})$ represents the distribution function of the a_{ij}, one can write (46) as

$$(47)\qquad K(n+m, h, p, \kappa_i^2) = \frac{1}{2^{\frac{1}{2}pn} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])}$$
$$\int | c_{ij} |^h | b_{ij} |^{\frac{1}{2}(n-p-1)} \exp [- \tfrac{1}{2} \sum b_{ii}] D(a_{ij})] \prod_{i=1}^{p} \prod_{j=i}^{p} db_{ij}\, dA$$

where dA is the volume element of the a_{ij}, and where the integration is over the entire (permissable) ranges of the b_{ij} and a_{ij}. Equation (47) holds since the c's are functions of the b's and a's. Multiplying (47) by

$$(48)\qquad 2^{\frac{1}{2}pn} \prod_{i=1}^{p} \Gamma(\tfrac{1}{2}[n + 1 - i]),$$

then replacing n by $g + 2$ and dividing by (48) again, we obtain

$$(49)\qquad K(n + m + 2g, h, p, \kappa_i^2) \frac{2^{\frac{1}{2}p(n+2g)} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i] + g)}{2^{\frac{1}{2}pn} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])}$$
$$= \frac{1}{2^{\frac{1}{2}pn} \pi^{\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])}$$
$$\cdot \int | c_{ij} |^h | b_{ij} |^{\frac{1}{2}(n+2g-p-1)} \exp \left[- \tfrac{1}{2} \sum_{i=1}^{p} b_{ii}] D(a_{ij}) \prod_{i=1}^{p} \prod_{i=j}^{p} db_{ij}\, dA] \right].$$

By definition the right hand side of (49) is the expected value of $| c_{ij} |^h | b_{ij} |^g$. Hence

$$E(| c_{ij} |^h | b_{ij} |^g) = K(n + m + 2g, h, p, \kappa_i^2) \frac{2^{pg} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i] + g)}{\prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])}$$

In this expression it is permissible to set h equal to $-g$ (n could have been replaced by $n + 2g$ in (47) to insure the argument of each Γ function being positive). Then we have

$$\begin{aligned} E(W^g) &= E(| c_{ij} |^{-g} | b_{ij} |^g) \\ &= K(n + m + 2g, -g, p, \kappa_i^2) \frac{2^{gp} \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i] + g)}{\prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])}. \end{aligned}$$

Finally, the g-th moment is

$$\begin{aligned} E(W^g) = \exp[-\tfrac{1}{2}(\kappa_1^2 + \kappa_2^2)] &\frac{\prod_{i=3}^{p} \Gamma(\frac{1}{2}[n + m + 1 - i]) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i] + g)}{\prod_{i=3}^{p} \Gamma(\frac{1}{2}[n + m + 1 - i] + g) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])} \\ \cdot \sum_{\alpha,\beta_1,\beta_2=0}^{\infty} &\left[\frac{(\kappa_1^2)^{\alpha+\beta_1} (\kappa_2^2)^{\alpha+\beta_2} \Gamma(\frac{1}{2}[n + m] + \alpha + \beta_1) \Gamma(\frac{1}{2}[n + m] + \alpha + \beta_2)}{2^{2\alpha+\beta_1+\beta_2} \alpha! \beta_1! \beta_2! \Gamma(\frac{1}{2}[n + m - 1] + g + \alpha) \Gamma(\frac{1}{2}[n + m] + \alpha)} \right. \\ &\left. \cdot \frac{\Gamma(\frac{1}{2}[n + m - 1] + \alpha)}{\Gamma(\frac{1}{2}[n + m] + g + 2\alpha + \beta_1 + \beta_2)} \right]. \end{aligned} \tag{50}$$

We can summarize in the following theorem:

THEOREM 6 *Let* $z_{i\alpha}$ $(i = 1, 2, \cdots, p: \alpha = 1, 2, \cdots, n)$ *and* $y_{i\gamma}$ $(i = 1, 2, \cdots, p; \gamma = 1, 2, \cdots, q)$ *have* (40) *as a joint distribution. Define* a_{ij}, b_{ij} *and* c_{ij} *by* (41), (42), *and* (43), *respectively. Let* κ_1^2 *and* κ_2^2 *be the non-zero roots of* (45). *Then the g-th moment of W, defined by* (44), *is* (50). Expression (50) gives the moments of W in the planar case. The linear case is a special case of the planar case, that is, it is the planar case for $\kappa_2^2 = 0$. The g-th moment of W in the linear case is given by

$$\begin{aligned} E(W^g) = \exp[-\tfrac{1}{2}\kappa_1^2] &\frac{\prod_{i=2}^{p} \Gamma(\frac{1}{2}[n + m + 1 - i]) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i] + g)}{\prod_{i=2}^{p} \Gamma(\frac{1}{2}[n + m + 1 - i] + g) \prod_{i=1}^{p} \Gamma(\frac{1}{2}[n + 1 - i])} \\ &\times \sum_{\beta_1=0}^{\infty} \frac{(\kappa_1^2)^{\beta_1} \Gamma(\frac{1}{2}[n + m] + \beta_1)}{2^{\beta_1} \beta_1! \Gamma(\frac{1}{2}[n + m] + g + \beta_1)}. \end{aligned} \tag{51}$$

For $\kappa_1^2 = 0$, (51) reduces to the expresion given for the moments under the null hypothesis.

Wilks [7] has given the distribution of W under the null hypothesis for several special cases (i.e., certain pairs of n and p). In general, however, the distribution function is too complicated to write down explicitly. When the null hypothesis is not satisfied (i.e., at least one $\kappa_1^2 \neq 0$) the distribution functions are yet more involved. Hence, we shall not write any explicitly.

Hsu [8] has given the asymptotic distribution of W. Suppose that

$$\Psi_n = \sum_{i,j=1}^{p} \sum_{\gamma=1}^{m} \mu_{i\gamma}\mu_{j\gamma}$$

tends to the limit Ψ_0 as n tends to infinity (if the μ's are functions of n). Then the limiting distribution of $x = -(n + q) \log W$ (which equals $-2 \log \Lambda$, where Λ is the likelihood ratio criterion) is

$$2^{-\frac{1}{2}pm} e^{-\frac{1}{2}\Psi_0} x^{\frac{1}{2}pm-1} e^{-\frac{1}{2}x^2} \sum_{\alpha=0}^{\infty} \frac{\Psi_0^\alpha x^\alpha}{2^{2\alpha}\alpha!\,\Gamma(\frac{1}{2}pm + \alpha)}. \tag{52}$$

That is, it is the χ'^2 distribution with pm degrees of freedom and parameter Ψ_0.

For most purposes, alternative hypotheses of the means being on a line (i.e., of rank one) are sufficiently general. In any particular case, one can compute from (51) mumerical values for several moments and then fit an appropriate distribution function. If one wishes to consider alternative hypotheses of rank two, one can use (50) and similarly compute numerical values for moments. The series in either (51) or (50) converge rapidly. To construct an approximate power function for linear alternatives, say, one would fit distribution functions for several values of κ_1^2 and find the desired percentage levels.

There is a matrix $\|d_{ij}\|$ such that

$$\|b_{ij}\| = \|d_{ij}\| \cdot \|d_{ij}\|'$$

and

$$\|a_{ij}\| = \|d_{ij}\| \cdot \|\lambda_j\delta_{ij}\| \cdot \|d_{ij}\|',$$

where the λ's are roots of

$$|a_{ij} - \lambda b_{ij}| = 0 \tag{53}$$

It follows that

$$\|c_{ij}\| = \|d_{ij}\| \cdot \|(1 + \lambda_j)\delta_{ij}\| \cdot \|d_{ij}\|'.$$

Then W can be written as

$$\begin{aligned} W &= \frac{|d_{ij}| \cdot |d_{ij}|'}{|d_{ij}| \cdot |(1 + \lambda_j)\delta_{ij}| \cdot |d_{ij}|'} \\ &= \frac{1}{\prod_{j=1}^{p}(1 + \lambda_j)}. \end{aligned} \tag{54}$$

The distribution of the roots of (53) in the linear case has been given by Roy [9] for a_{ij} of dimensionality p.[5] The distribution in the planar case has been indicated by Anderson [3]. One could obtain the probability of W not exceeding a given value by integrating the λ's over the proper range.

6.2. *Examples of the general linear hypothesis.* A number of hypotheses concerning the expected values of variates with multivariate normal distributions can be put into the form of H_0. The equivalence of the hypotheses is demonstrated by means of linear transformations.

As an example consider the hypothesis H_1 that the means of several normal multivariate populations are equal when the respective covariance matrices are equal. Let $x_{i\alpha}^u$ be the α-th ($\alpha = 1, 2, \cdots, N^u$) observation of the i-th ($i = 1, 2, \cdots, p$) variate in the u-th ($u = 1, 2, \cdots, U$) population. Let

$$E(x_{i\alpha}^u) = \mu_i^u \qquad \begin{array}{l}(i = 1, 2, \cdots, p)\\(u = 1, 2, \cdots, U),\end{array} \tag{55}$$

and let the covariance matrix be $\| \sigma_{ij} \|$. Then the hypothesis is

$$H_1 : \mu_i^u = \mu_i \qquad \begin{array}{l}(i = 1, 2, \cdots, p)\\(u = 1, 2, \cdots, U).\end{array} \tag{56}$$

For testing this hypothesis let

$$b_{ij} = \sum_{u=1}^{U} \sum_{\alpha=1}^{N^u} (x_{i\alpha}^u - \bar{x}_i^u)(x_{j\alpha}^u - \bar{x}_j^u), \tag{57}$$

$$a_{ij} = \sum_{u=1}^{U} N^u(\bar{x}_i^u - \bar{x}_i)(\bar{x}_j^u - \bar{x}_j), \tag{58}$$

where

$$\begin{aligned} \bar{x}_i^u &= \frac{1}{N^u} \sum_{\alpha=1}^{N^u} x_{i\alpha}^u, \\ \bar{x}_i &= \frac{1}{N} \sum_{u=1}^{u} \sum_{\alpha=1}^{N^u} x_{i\alpha}^u, \\ N &= \sum_{u=1}^{u} N^u. \end{aligned} \tag{59}$$

The b_{ij} have $n = N - U$ degrees of freedom and $c_{ij} = a_{ij} + b_{ij}$, have $N - 1$ degrees of freedom. Then the $N/2$ root of the likelihood ratio criterion for H_1 is W defined by (44). For this case equation (45) is

$$\left| \sum_{u=1}^{u} N^u(\mu_i^u - \bar{\mu}_i)(\mu_j^u - \bar{\mu}_j) - \lambda\sigma_{ij} \right| = 0,$$

where

$$\bar{\mu}_i = \frac{1}{N} \sum_{u=1}^{u} N^u \mu_i^u.$$

[5] Roy erroneously claims his distribution to hold for the planar case and higher rank.

Hsu has demonstrated that the general regression problem can be put into the form of H_0. Suppose that $x_{i\alpha}$ ($i = 1, 2, \cdots, p$; $\alpha = 1, 2, \cdots, N$) follow a multivariate normal distribution with covariance matrix $\| \sigma_{ij} \|$, and let the expected value of $x_{i\alpha}$ be

$$E(x_{i\alpha}) = \sum_{r=1}^{q} \beta_{ir} w_{r\alpha} \qquad (q \leq N - p),$$

where the q by N matrix

$$W = \| w_{r\alpha} \|$$

is of rank q. Let H_2 be the hypothesis that

$$H_2 : B_1 = \| \beta_{iu} \| = 0 \qquad (i = 1, 2, \cdots, p; u = 1, 2, \cdots, m \leq q)$$

with the w's known. Let

$$\begin{aligned} W_1 &= \| w_{u\alpha} \| \qquad (u = 1, 2, \cdots, m; \alpha = 1, 2, \cdots, N) \\ W_2 &= \| w_{r\alpha} \| \\ &\qquad (r = m + 1, \cdots, 9; \alpha = 1, 2, \cdots, N), \\ X &= \| x_{i\alpha} \| \qquad (i = 1, 2, \cdots, p; \alpha = 1, 2, \cdots, N). \end{aligned}$$

Let

$$\| b_{ij} \| = XX' - XW'(WW')^{-1}WX$$

$$\| c_{ij} \| = XX' - XW_2'(W_2W_2')^{-1}W_2X'.$$

(with $\| c_{ij} \| = XX'$ if $W_2 = 0$). Then the likelihood ratio criterion for H_2 is the $N/2$-th power of W, defined by (44).

The equation (45) can be written in terms of Z, B_1, and W as

$$(60) \qquad | B_1W_1(I - W_2'(W_2W_2')^{-1}W_2)W_1'B_1' - \lambda\Sigma | = 0$$

for $m < q$. If $m = q$, (45) becomes

$$(61) \qquad | B_1WW'B_1' - \lambda\Sigma | = 0.$$

In (60) and (61) there are no more non-zero roots than the rank of B_1. It is clear that the roots of (60) (or (61)) depend on the matrix W as well as B_1. The distribution of Λ the likelihood ratio criterion under the null hypothesis does not depend on the distribution of the matrix W (if W is not constant). However, the distribution when the null hypothesis is not satisfied does depend on κ_1^2 or on κ_1^2 and κ_2^2, and hence, on the distribution of the elements of W as well as the value of B_1.

The special case of H_0 for $m = q = 1$ gives as the likelihood ratio criterion as a function of Hotelling's generalized T^2. From the moments indicated in (50) we can deduce the distribution of T^2 when the null hypothesis is not true [3]. This result has been obtained by Hsu [10] by another method.

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ON HOTELLING'S WEIGHING PROBLEM[1]

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1. Summary. The paper contains some solutions of the weighing problems proposed by Hotelling [1]. The experimental designs are applicable to a broad class of problems of measurement of similar objects. The chemical balance problem (in which objects may be placed in either of the two pans of the balance) is almost completely solved by means of designs constructed from Hadamard matrices. Designs are provided both for a balance which has a bias and for one which has no bias.

The spring balance problem (in which objects may be placed in only one pan) is completely solved when the balance is biased. For an unbiased spring balance, designs are given for small numbers of objects and weighing operations. Also the most efficient designs are found for the unbiased spring balance, but it is shown that in some cases these cannot be used unless the number of weighings is as large as the binomial coefficient $\binom{p}{\frac{1}{2}p}$ or $\binom{p}{\frac{1}{2}(p+1)}$ where p is the number of objects.

It is found that when p objects are weighed in $N \geq p$ weighings, the variances of the estimates of the weights are of the order of σ^2/N in the chemical balance case (σ^2 is the variance of a single weighing), and of the order of $4\sigma^2/N$ in the spring balance case.

2. Introduction. The problem is fully discussed by Hotelling [1] and refers to the design of a certain class of simple experiments. We may consider the typical example of the class to be that of weighing several small objects on a chemical balance or other weighing device. Hotelling and Yates [2] have shown that the individual weights may be determined more accurately by weighing the objects in combinations rather than weighing each one separately. The designs are applicable to a great variety of problems of measurement, not only of weights, but of lengths, voltages and resistances, concentrations of chemicals in solutions, in fact any measurements such that the measure of a combination is a known linear function of the separate measures with numerically equal coefficients. The designs should be particularly useful in biological and chemical laboratories engaged in routine chemical analyses. We shall, however, in the interest of simplicity, discuss the problem in the language of weighing operations.

A particular design is denoted by a matrix. The three objects to be weighed in four weighing operations may be weighed by the following design:

[1] Journal Paper No. J-1405 of the Iowa Agricultural Experiment Station, Ames, Iowa. Project No. 890.

$$X = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix}$$

where the rows refer to weighing operations and the columns refer to the objects. In the above design the first two objects are weighed together in the first weighing operation, the first and third objects are weighed together in the second weighing operation, etc. From the four resulting weights the individual weights are estimated by the method of least squares The design problem consists of finding matrices which will minimize the variances of these estimates.

There are two distinct though closely related problems here. One is to find efficient designs for the case in which the measure of a combination can only be the sum of the individual measures. This would be the case, for example, in weighing objects with a spring balance and we shall refer to it as the spring balance problem. The other problem is to find designs when an individual measurement may be either added or subtracted in a combination. This would be the case in weighting objects with a chemical balance (since an object may be put in either pan of the balance) and will be called the chemical balance problem In the latter problem the design matrix may contain 0's, 1's, and -1's, whereas in the spring balance problem the matrix may contain only 0's and 1's.

We shall use Hotelling's notation. There are p objects with weights b_1, b_2, $\cdots$, b_p to be weighed in $N \geq p$ weighing operations. The design matrix is denoted by

(1) $$X = \| x_{\alpha i} \| \quad \alpha = 1, \cdots, N; i = 1, \cdots, p.$$

Denoting the transpose of X by X', let

(2) $$X'X = \| a_{ij} \| = \| a^{ij} \|^{-1}$$

(3) $$g_i = \sum_\alpha x_{\alpha i} y_\alpha$$

where y_α is the observed result of the α-th weighing operation. The least squares estimates of the b_i are

(4) $$\hat{b}_i = \sum_j a^{ij} g_j$$

and the variances of these estimates are

(5) $$\sigma_i^2 = a^{ii} \sigma^2$$

where σ^2 is the error variance of a single weighing operation. The a^{ii} will be called variance factors.

Hotelling's main theorem states that or any design, $a^{ii} \geq 1/N$, hence the best possible design is one such the inverse of the product of the design matrix by its transpose has its main diagonal elements equal to $1/N$. We shall call such a design an optimum design. Examples show that optimum designs do not exist for all values of N and p.

When an optimum design does not exist, the question arises as to how a best design shall be defined. In the present paper a design will be called best if the determinant of the matrix $\| a^{ij} \|$ is minimized. A best design in this sense is, therefore, a design which gives the smallest confidence region in the $\hat{b}_i$ $(i = 1, 2, \cdots, p)$ space for the estimates of the weights.

In certain situations, other definitions of best designs may conceivably be preferred. Thus, problems may arise in which one might prefer:

(a) to minimize the variance factors subject to the restriction that they be equal, (b) to minimize some function of the variance factors, or (c) to minimize only a certain subset of the a^{ii} on a minor of the matrix $\| a^{ij} \|$ as might be the case when one wanted only rough estimates of the weights of some of the objects, but accurate estimates of the others.

When an optimum design exists, the confidence regions are not only minimized, but, as Hotelling has shown, the variance factors are also minimized. It is not true in general, however, that a best design as here defined (minimum confidence regions) will also minimize the variance factors. Examples illustrating this point are given in the last part of section 6 and the first part of section 7.

3. Hadamard Matrices. The problem of finding the best designs is closely related to the Hadamard determinant problem. Hadamard [3] proved the following result: If the elements $x_{\alpha\beta}$ of a square matrix X are restricted to the range $-1 \leq x_{\alpha\beta} \leq 1$, the maximum possible value of the determinant of X is $N^{\frac{1}{2}N}$, and when this maximum is achieved all $x_{\alpha\beta} = \pm 1$ and the matrix is orthogonal in the sense that $X'X$ is a diagonal matrix; the non-zero elements of $X'X$ are all equal to N. A matrix X which satisfies these conditions will be denoted by H_N. Obviously if H_N exists for a given N, it is the solution of the design problem in the chemical balance case when $N = p$.

With regard to the existence of H_N, it is known that a necessary condition is

$$N \equiv 0 \pmod 4$$

with the exception of $N = 2$. It is not known however whether the above condition is sufficient, although it is known (Paley [4]) that H_{4k} exists for the range

$$0 < 4k \leq 100$$

with the possible exception of $4k = 92$. Paley and Williamson [5] give methods of constructing H_{4k} in the given range (excepting 92) based on the theory of finite fields.

When N is a power of two, H_N is easily constructed by taking direct products of

$$H_2 = \left\| \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right\|$$

Thus

$$H_4 = H_2 \cdot H_2 = \left\| \begin{matrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{matrix} \right\|$$

Sylvester [6] first studied this class of matrices and Kishen [7] has described weighing designs based on this subset of the H_N.

The following examples of Hadamard matrices may be found in the literature: Paley [4] exhibits an H_8, H_{12}, and H_{28}: Kishen gives an H_{16}. From these examples H_{24} and H_{32} may be constructed at once from the direct products $H_2 \cdot H_{12}$ and $H_2 \cdot H_{16}$. The following is an H_{20}:

$$\left\| \begin{array}{cccccccccccccccccccc}
+ & - & - & - & - & + & - & - & - & - & + & + & - & - & + & + & - & + & + & - \\
- & + & - & - & - & - & + & - & - & - & + & + & + & - & - & - & + & - & + & + \\
- & - & + & - & - & - & - & + & - & - & - & + & + & + & - & + & - & + & - & + \\
- & - & - & + & - & - & - & - & + & - & - & - & + & + & + & + & + & - & + & - \\
- & - & - & - & + & - & - & - & - & + & + & - & - & + & + & - & + & + & - & + \\
- & + & + & + & + & + & - & - & - & - & - & + & - & - & + & + & + & - & - & + \\
+ & - & + & + & + & - & + & - & - & - & + & - & + & - & - & + & + & + & - & - \\
+ & + & - & + & + & - & - & + & - & - & - & + & - & + & - & - & + & + & + & - \\
+ & + & + & - & + & - & - & - & + & - & - & - & + & - & + & - & - & + & + & + \\
+ & + & + & + & - & - & - & - & - & + & + & - & - & + & - & + & - & - & + & + \\
- & - & + & + & - & + & - & + & + & - & + & - & - & - & - & - & + & + & + & + \\
- & - & - & + & + & - & + & - & + & + & - & + & - & - & - & + & - & + & + & + \\
+ & - & - & - & + & + & - & + & - & + & - & - & + & - & - & + & + & - & + & + \\
+ & + & - & - & - & + & + & - & + & - & - & - & - & + & - & + & + & + & - & + \\
- & + & + & - & - & - & + & + & - & + & - & - & - & - & + & + & + & + & + & - \\
- & + & - & - & + & - & - & + & + & - & + & - & - & - & - & + & - & - & - & - \\
+ & - & + & - & - & - & - & - & + & + & - & + & - & - & - & - & + & - & - & - \\
- & + & - & + & - & + & - & - & - & + & - & - & + & - & - & - & - & + & - & - \\
- & - & + & - & + & + & + & - & - & - & - & - & - & + & - & - & - & - & + & - \\
+ & - & - & + & - & - & + & + & - & - & - & - & - & - & + & - & - & - & - & +
\end{array} \right\|$$

where the signs represent ± 1. This example was constructed by Williamson's method [5]. Thus examples of H_{4k} for the range $4 \leq 4k \leq 32$ are immediately available and methods of construction exist for the range $36 \leq 4k \leq 88$.

4. Chemical Balance Problem. When $N = 0 \pmod 4$ an optimum design exists if H_N exists and is obtained by using any p columns of H_N. When $N \not\equiv 0 \pmod 4$ we may construct very efficient designs as follows: If $N \equiv 1$ we may add a row of ones to H_{N-1}; if $N \equiv 2$ we may add two rows of ones or a row of H_2's to H_{N-2}; and if $N \equiv 3$ we may delete one row from H_{N+1}. The worst of these designs will be obtained when two rows of ones are added to an H_{N-2}, and in this case the variance factors are

$$a^{ii} = \frac{1}{N-2} \frac{N+2p-4}{N+2p-2} < \frac{1}{N-2}. \tag{6}$$

Since it is known that these factors must be greater than $1/N$ for the best possible design in this case, the above design will be quite near the best design for large N.

For small values of N we shall consider only the case $N = p$, since if one

wanted to make $N > p$ weighings, he would normally choose N to be a multiple of four because the gain in efficiency by using optimum designs is rather large for small N. In general more than p weighings would be required because σ^2 is not usually known. Thus several additional weighings may be made in order to obtain several degrees of freedom for estimating σ^2.

When H_N does not exist we have already defined the best design as one which minimizes the confidence region for estimating the weights; that is equivalent to maximizing $|a_{ij}|$ or minimizing $|a^{ij}|$. There may be several designs with the same minimum, but we shall not give all of them. Thus when $p = 3$ the best designs are

$$X = \begin{Vmatrix} + & + & 0 \\ + & - & + \\ - & + & + \end{Vmatrix}, \quad \begin{Vmatrix} + & + & + \\ + & - & + \\ - & + & + \end{Vmatrix} \text{ and } \begin{Vmatrix} + & + & - \\ + & - & + \\ - & + & + \end{Vmatrix}$$

all of which have $A = 16$ (which is considerably smaller than the value 27 that A would have if an optimum design existed). Using the notation

$$(a^{ii}) = (a^{11}, a^{22}, \cdots, a^{pp}),$$

the first of the above designs for $p = 3$ gives

$$(a^{ii}) = (\tfrac{3}{8}, \tfrac{3}{8}, \tfrac{1}{2})$$

while the second and third give

$$(a^{ii}) = (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}).$$

For $N = p = 5$, two best designs are

$$X = \begin{Vmatrix} + & + & + & + & - \\ + & + & + & - & + \\ + & + & - & + & + \\ + & - & + & + & + \\ - & + & + & + & + \end{Vmatrix} \text{ and } \begin{Vmatrix} + & - & - & - & - \\ + & + & + & - & - \\ + & - & + & - & + \\ + & - & + & + & - \\ + & + & - & + & + \end{Vmatrix}$$

both of which have

$$A = 3^2 2^8 \text{ and } (a^{ii}) = (2/9, 2/9, 2/9, 2/9, 2/9)$$

For $N = p = 6$, a best design is

$$X = \begin{Vmatrix} + & - & - & - & - & - \\ + & - & - & - & + & + \\ + & - & - & + & + & - \\ + & - & + & + & - & + \\ + & + & + & - & + & - \\ + & + & - & + & - & + \end{Vmatrix}$$

which has

$$A = 5^2 2^{10} \text{ and all } a^{ii} = 1/5.$$

For $N = p = 7$, a best design is

$$X = \left\| \begin{matrix} + & - & - & - & - & - & - \\ + & + & + & + & - & - & - \\ + & + & - & - & - & + & + \\ + & + & - & - & + & + & - \\ + & - & - & + & + & - & + \\ + & - & + & + & - & + & - \\ + & - & + & - & + & - & + \end{matrix} \right\|$$

which has

$$A = 2^{12}3^4 \text{ and all } a^{ii} = 1/6$$

These designs were constructed by a method due to Williamson [8] which will be described in sections 5 and 7 It is interesting to note that no minor of an H_8 is a best design for $N = p = 7$, for any minor of an H_8 gives $A = 2^{18} < 2^{12}3^4$ and all $a^{ii} = \frac{1}{4}$

5. Spring Balance Problem. $N = p = 4k + 3$ When $N = p$ and $N \equiv 3 \pmod 4$ the best possible design for the spring balance case is determined by H_{N+1} if it exists Let K_{N+1} denote a matrix formed from H_{N+1} by adding or subtracting the elements of the first row of H_{N+1} from the corresponding elements of the other rows in such a way as to make the first element of each of the remaining rows zero Obviously

$$| K_{N+1} | = \pm | H_{N+1} |$$

and excepting the first row, the elements of K_{N+1} are 0 and ± 2 with the signs of the non-zero elements the same for elements in the same row. Let L_N be the matrix obtained by omitting the first row and column of K_{N+1}, by changing all non-zero elements to $+1$, and by permitting two rows if necessary to make the determinant of L_N positive. Then

$$| H_{N+1} | = 2^N | L_N |$$

and it is clear that, given L_N, one could reverse the procedure and determine an H_{N+1}. In the same manner, there is a correspondence in general between square matrices with elements ± 1 and square matrices of one less order with elements 0 and 1 The ratio of the values of corresponding determinants is always 2^N if their determinants do not vanish, hence the 0,1 determinant will have its maximum value when its corresponding $+1$ determinant has a maximum value. Thus $| L_N |$ is the maximum value possible for a determinant of 0's and 1's of order N, and the value is

$$| L_N | = (N + 1)^{\frac{1}{2}(N+1)}/2^N. \tag{7}$$

The variance factors are

$$a^{ii} = 4N/(N + 1)^2.$$

We knew in advance, of course, that the a^{ii} would be greater than $1/N$ since an optimum design cannot exist unless the design matrix has its elements equal to ± 1, and we must here restrict the design to have only 0 and 1 as its elements. Since L_N is a best possible design for the spring balance case, it follows that designs for the spring balance problem can be no more than about $\frac{1}{4}$ as efficient as designs for the chemical balance problem.

6. Spring Balance $N > p$. When $N > p$ the device used in the chemical balance case to get optimum designs cannot be used. For if we select p columns from an L_N we may get rows of zeros which would waste weighing operations. A different approach is necessary and a clue is given by the designs L_N. In these designs p is odd and the objects are weighed $\frac{1}{2}(p+1)$ at a time in each weighing operation. We shall show in general that objects should be weighed $\frac{1}{2}(p+1)$ at a time when p is odd, and we shall obtain a corresponding result for p even.

Let P_r be a matrix whose rows are all the arrangements of r ones and $p - r$ zeros $(0 \leq r \leq p)$. (The symbol should also have a subscript p but that is omitted because any specific value for p will always be clear from the context.) The matrix will have p columns and $\binom{p}{r}$ rows. Let Q be a matrix made up of matrices P_r arranged in vertical order. Let n_r be the number of times P_r is used in constructing Q. Q is a weighing design for p objects and

$$N = \sum_r n_r \binom{p}{r}$$

weighing operations. The matrix $Q'Q$ will have diagonal elements

$$a = \sum n_r \binom{p-1}{r-1} \tag{9}$$

and non-diagonal elements

$$b = \sum n_r \binom{p-2}{r-2}. \tag{10}$$

The determinant of $Q'Q$ is

$$A = (a-b)^{p-1}[a + (p-1)b$$

and we may write A in the form

$$A = c^{p-1}d$$

where

$$c = a - b, \text{ and } d = a + (p-1)b. \tag{11}$$

We shall prove the following theorem:

If $p = 2k - 1$ *where* k *is a positive integer, and if* N *contains the factor* $\binom{p}{k}$, *then* A *will be maximized when* $n_k = N \Big/ \binom{p}{k}$ *and all other* $n_r = 0$.

We shall demonstrate this statement by showing that if any n_s $(s \neq k)$ is decreased and n_k is increased in such a way that N remains unchanged, then A will be increased. Let n_s be reduced by an amount m so chosen that

$$m' = m\binom{p}{s} \Big/ \binom{p}{k}$$

is an integer; we may then increase n_k by m' leaving N unchanged. It is readily found that these changes in n_s and n_k produce the following changes in c and d:

$$\Delta c = m\binom{p}{s}\frac{(k-s)(k-s-1)}{p(p-1)}$$

$$\Delta d = m\binom{p}{s}\frac{(k-s)(k+s)}{p}$$

both of which are positive on zero when $s < k$ and A is necessarily increased.

When $s > k$, Δc is positive but Δd is negative and it must be shown that the net effect of these changes is to increase A, we shall assume now that $n_r = 0$ when $r < k$.

$$\Delta A = (c + \Delta c)^{p-1}(d + \Delta d) - c^{p-1}d < [c^{p-1} + (p-1)c^{p-2}\Delta c](d + \Delta d) - c^{p-1}d < c^{p-2}[c\Delta d + (p-1)d\Delta c + (p-1)\Delta c\Delta d]$$

where in the second line we have omitted terms in Δc of higher order than the first. These terms are all positive since all their fctors are positive. The bracket in the last expression on substituting from (9), (10), and (11), may be reduced to

$$m\binom{p}{s}\Bigg[(k-s)\sum_{r\geq k} n_r\left\{\binom{p-2}{r-1} + (k-s-1)\binom{p-2}{r-1}\right. + \frac{m}{p^2}\binom{p}{s}(k-s)^2(k+s)(k-s-1)\Bigg],$$

and then to

$$m\binom{p}{s}\Bigg[\sum_{r\geq k} n_r\binom{p-1}{r-1}(k-s)\frac{r(k-s) + (k+s+1-2r)}{p-1} + \frac{m}{p^2}\binom{p}{s}(k-s)^2(k+s)(k-s-1)\Bigg].$$

Each term of the sum in the bracket is greater than or equal to zero when $k > 1$, $r \geq k$, $s > k$ since the fraction is readily seen to be negative or zero under these

circumstances. The fraction vanishes only when $k = 2$, $r = k$, $s = k + 1$. The other term in the bracket is negative but it is dominated by the term in the sum for which $r = s$, as may be shown as follows: The two terms in question may be written

$$n_s \binom{p-1}{s-1}(k-s)\,\frac{s(k-s)+k-s+1}{p-1} + \frac{m}{p}\binom{p-1}{s-1}\frac{(k-s)^2(k+s)(k-s-1)}{s}$$

and since $n_s \geq m$, this expression is less than or equal to

$$m\binom{p-1}{s-1}(k-s)\left[\frac{s(k-s)+k-s+1}{p-1} + \frac{(k^2-s^2)(k-s-1)}{ps}\right]$$

which is positive for $s > k$ since the bracket is negative as may be seen by factoring out $\dfrac{1}{p(p-1)s}$ and putting the result in the form

$$(k-s+1)(s^2+(p-1)k^2) - pk(p-s) + (2s+1)(k-s).$$

Thus ΔA has been shown to be positive and the theorem is proved

The above argument has shown that P_k or repetitions of P_k give more efficient designs than any other combination of the designs $P_1, P_2, \cdots, P_k$ The question now arises as to whether these are the best possible designs. We shall show that they are by considering the matrices L_N of section 5 which are known to give the greatest efficiency in the spring balance case. Let $p = 4t + 3$ and let $N = \binom{p}{2t+2}$, and suppose L_p exists (i.e. H_{p+1} exists). Using P_{2t} as the weighing design we find the a_{ij} are

$$a_{ii} = 2N(t+1)/p$$
$$a_{ij} = N(t+1)/p \qquad i \neq j.$$

A single application of the design L_p gives

$$a_{ii} = 2(t+1)$$
$$a_{ij} = t+1 \qquad i \neq j$$

and N/p repetitions of L_p gives an a_{ij} matrix with elements equal to N/p times the given elements for one application of the design. The two designs are therefore equivalent and P_{2t} is a best design.

The variance factors for repetitions of the design P_k are

$$a^{ii} = \frac{4}{N}\frac{p^2}{(p-1)^2} \qquad N \equiv 0 \text{ Mod} \binom{p}{k} \tag{12}$$

and these are minimum variance factors[2] as may be shown by an argument entirely analogous to that used in proving the theorem. Thus P_k is a design which not only minimizes the confidence region for estimating the weights, but also minimizes the individual variance factors.

Efficient sub-matrices of the P_k have not been studied except for small p, but we may point out that square sub-matrices of order p which are as efficient as P_k do not exist unless H_{p-1} exists, for by the argument of section 4, it is possible to construct H_{p+1} from such sub-matrices Hence we cannot obtain variance factors as small as those given by equation (12) when $N = p$ unless H_{p+1} exists.

The situation here is analogous to that in the chemical balance case. By a proper selection of N we can obtain a design with the maximum possible efficiency for any odd value of p. But here we are much more restricted in our choice of N. In the chemical balance case N could be any multiple of 4 for which an H_N existed; in the present case N must be a multiple of p even in the most favorable instance ($p = 4t + 3$), and for some values of p it may be necessary that N be a multiple of $\binom{p}{\frac{1}{2}(p+1)}$.

We now turn to the case in which p is even. The theorem corresponding to the one given at the beginning of this section is:

If $p = 2k$ where k is a positive integer, and if N contains the factor $\binom{p+1}{k+1}$, then A will be maximized when

$$n_k = n_{k+1} = N \Big/ \binom{p+1}{k+1}$$

and all other $n_r = 0$.

We shall not prove this theorem in detail. By arguments analogous to those used earlier, it may be shown that A is increased when either n_s ($s < k$) is decreased and n_k is increased, or n_s ($s > k + 1$) is decreased and n_{k+1} is increased with N fixed. This done, we may put all $n_r = 0$ except n_k and n_{k+1} and then maximize A with respect to these two variables subject to the condition that

$$n_k \binom{p}{k} + n_{k+1} \binom{p}{k+1} = N.$$

The values of n_k and n_{k+1} which maximize A may be found by treating them as continuous variables and using the calculus.

The variance factors for these designs are

$$a^{ii} = \frac{4}{N}\frac{p}{p+2} \qquad N \equiv 0 \bmod \binom{p+1}{k+1}, \tag{13}$$

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[2] The author is indebted to a referee for suggesting this property of the design, and for several other valuable suggestions and corrections to the paper.

but these are not minimum variance factors. In fact one can obtain smaller variance factors than these by using only P_k in the design (omitting P_{k+1} entirely). In this case

$$a^{ii} = \frac{4}{N} \frac{(p-1)^2+1}{p^2} \qquad N \equiv 0 \bmod \binom{p}{k} \tag{14}$$

and

$$\frac{(p-1)^2+1}{p^2} < \frac{p}{p+2} \quad \text{when} \quad p > 2.$$

We have not found explicitly the design which minimizes the variance factors for p even, but it appears that the design would be made up largely from P_k with a small proportion of the design devoted to P_{k+1}. Thus (14) is very nearly the minimum possible variance factor.

7. Spring Balance Designs for Small p. When $p = 2$, each object may be weighed r times by itself, and the two objects may be weighed together s times to give

$$\| a_{ij} \| = \begin{Vmatrix} r+s & s \\ s & r+s \end{Vmatrix}$$

and if A is maximized subject to $2r + s = N$ we find

$$r = s = N/3$$

$$a^{ii} = 2/N$$

provided N is a multiple of 3. The most efficient basic design is therefore

$$X = \begin{Vmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

in accordance with the previous section. When N is not a multiple of 3 the best design is obtained by using the first row of X for the odd weighing when $N = 3t + 1$, and the last two rows when $N = 3t + 2$.

The case $p = 2$ is notable in that there is almost nothing to be gained by weighing the objects in combination. For the variance factors $2/N$ would be obtained by simply weighing each object separately $N/2$ times. The advantage of weighing in combination is only that square confidence regions in the b_1, b_2 space are replaced by ellipses with somewhat smaller area. If $a^{ii} = (r + s)/(r^2 + 2rs)$ is minimized subject to $2r + s = N$, we find

$$r = N(3 - \sqrt{3})/3, \; a^{ii} = 1.866/N$$

so that the a^{ii} are reduced slightly from $2/N$ but at the expense of increasing the area of the elliptical confidence regions.

For $p = 3$ the most efficient design when $N = 3$ is

$$X = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix}$$

as given by L_3 or P_2. It is easily shown that for $N > 3$, the most efficient design is given by repeating X even when $N \not\equiv 0 \pmod 3$. Thus for $N = 4$ we would repeat one row of X, for $N = 5$ we would repeat two rows of X, and so forth. The variance factors are

$$\begin{aligned} a^{ii} &= \frac{9}{4N} && N = 3t \\ &= \frac{9(N+1)}{4(N-1)(N+2)} && N = 3t+1 \\ &= \frac{9(N+1)}{4(N-2)(N+1)} && N = 3t+2. \end{aligned}$$

For $p = 4$ we may attempt to find by trial and error a sub-matrix of the design given by using P_2 once and P_3 once, but this would be a tedious process and the labor would soon become prohibitive for larger values of p. Hence another method must be found for obtaining the best designs when $N = p$ except when L_p exists A method is provided by Williamson [8]. Let D_p be the best design for $N = p$. Williamson shows that when $p < 7$, D_{p-1} is a minor of D_p, hence D_p may be found by adding a row and column of variables to D_{p-1} and expanding the determinant of the result by the bordered expansion. For small values of p it is easy to determine by inspection what values the variables should have in order to maximize the resulting expansion Williamson determined D_4 and D_5 by this method

There are two types of D_4 which give a maximum value of $A = 9$

$$D_4 = \begin{Vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{Vmatrix} \text{ and } \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{Vmatrix}.$$

The variance factors are all 7/9 for the first of these, and for the second

$$(a^{ii}) = (7/9, 7/9, 7/9, 4/9).$$

When $N = 5$, $p = 4$, there are a number of designs which give a maximum A of 19. None of these however has all a^{ii} equal, and we shall give only one example:

$$X = \begin{Vmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{Vmatrix}$$

which has

$$(a^{ii}) = \left(\frac{12}{19}, \frac{12}{19}, \frac{13}{19}, \frac{8}{19}\right).$$

When $N = 6$, there appears to be no design superior to P_2. It has variance factors all equal to 5/12 and $A = 48$,—a very large gain in efficiency over $N = 5$ at the expense of one additional observation.

When $p = 5$ there are three types of D_5 which give A a maximum value of 25, none of which has all variance factors equal. An example is

$$D_5 = \begin{Vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{Vmatrix}$$

with

$$(a^{ii}) = \left(\frac{19}{25}, \frac{19}{25}, \frac{16}{25}, \frac{11}{25}, \frac{16}{25}\right).$$

For $p = 6$, an example of a D_6 with all a^{ii} equal which maximizes A is

$$D_6 = \begin{Vmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{Vmatrix}$$

with $A = 81$ and $a^{ii} = 17/27$. This example was constructed by the bordered expansion method from D_5 and it turns out to be a sub-matrix of P_3. It is not as efficient as P_3, however, since substitution of $N = p = 6$ in equation (14) gives $a^{ii} = 13/27$. Hence we have shown that there does not exist a minor of P_3 (for $p = 6$) of order 6 which is as efficient as P_3 itself.

For $p = 7$, there is a most efficient design given by L_7.

$$L_7 = \begin{Vmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{Vmatrix}$$

with $A = 2^{10}$ and all $a^{ii} = 7/16$.

D_p for $p = 8$, 9, and 10 could presumably be constructed from L_7 in the same way and the designs for $p = 4$, 5, and 6 were constructed from L_8, but the computations become very tedious for these larger values of p.

The designs given in section 3 were constructed from the above designs by the method described in section 4.

8. Bias in Measuring Devices. In some kinds of experiments it may be necessary to estimate a bias in the measuring scale in order to estimate the measures of the objects. Such a bias may simply be regarded as an additional object to be measured except that it is an object which must be included in all the measuring operations. In the chemical balance case the bias presents no difficulty, for if an H_N exists, then there exists an H_N with a column whose elements are all $+1$. Such an H_N may be constructed from any given H_N by merely changing the signs of all elements in rows which begin with a minus sign. The result will be an H_N with $+1$'s in the first column and that column may be assigned to the bias. We note that the gain in efficiency by measuring objects in combinations is even greater in the case of a biased measuring scale than when there is no bias. For if the objects were measured separately, their measures would be estimated by the difference of two scale readings and would have variance $2\sigma^2$; hence the variance factors a^{ii} are to be compared with 2 (rather than 1) in the case of bias.

In the spring balance case, the additional restriction that all the elements of one column be one necessarily reduces the efficiency of the designs in the sense that the variance factors for p objects and a bias will be larger than the variance factors for $p + 1$ objects without bias. When the measures of p objects and a bias are to be estimated from $N = p + 1$ measuring operations, a best design may be obtained by adding a row of zeros and a column of ones (in that order) to the best design for $N = p$ without bias. This can be seen by recalling that there are two determinantal exressions for the volume of a simplex with one vertex at the origin in a Euclidean p space. (A simplex (Sommerville, [9]) is a polytope with $p + 1$ vertices bounded by $p + 1$ $(p - 1)$-dimensional hyperplanes.) The determinant of the best design for $N = p$ (without bias) is proportional to the volume of the largest simplex with one vertex at the origin and the other vertices restricted to be selected from the vertices of the unit cube. A determinant of order $p + 1$ with a column of ones and the other elements zero or one also gives the volume of a simplex with vertices selected from the vertices of the unit cube. Hence the two determinants (one of order p and one of order $p + 1$) must have the same maximum value, and as one of the vertices may be selected arbitrarily in the case of bias, we may select the origin.

In general, for $N \geq p$, similar geometrical reasoning will show that the best designs for the spring balance problem in the case of bias are easily constructed from Hadamard matrices as described in the following theorem:

If X is a best design for the chemical balance problem in the case of bias and if X contains a row of $+1$'s, then a best design for the spring balance problem in the case of bias is given by replacing the -1's in X by zeros.

We have seen that the best design in the chemical balance case is obtained from a Hadamard matrix with a column of $+1$'s. Obviously the matrix may be also made to contain a row of $+1$'s by changing the signs of certain columns. The design X consists of the column of ones together with any other p columns. The determinant of $X'X$ is $1/p!^2$ times the sum of squares of the volumes of a set of simplexes in a p space. There are $\binom{N}{p+1}$ of these simplexes deter-

mined by the different combinations of the rows of X taken $p + 1$ at a time, and the coordinates of their vertices are the last p elements of the rows of X. The vertices are therefore selected from the vertices of a cube in the p space which has its edges parallel to the coordinate axes, the origin at its center, and the lengths of its edges equal to two. Since X is a best design, the vertices are selected so as to maximize the sum of squares of the volumes of the simplexes. Now in the spring balance case we must maximize the sum of squares of the volumes of a set of simplexes which have their vertices selected from the vertices of the unit cube. Obviously this may be done by selecting vertices corresponding to the selection given by X. Thus it is necessary only to set up a correspondence beteen the vertices of the two cubes Since X contains the vertex $(1, 1, 1, 1, \cdots, 1)$ which is common to both cubes, the natural correspondence which identifies a vertex such as $(1, -1, -1, 1, -1, 1, \cdots)$ with $(1, 0, 0, 1, 0, 1, \cdots)$ may be used.

The variance factors for these spring balance designs are $4/N$ (for any $p \leq N$) when N is a multiple of four and H_N exists; when N is not a multiple of four and modifications of H_N as described in section 3 are used, the variance factors will differ from $4/N$ by terms of order $1/N^2$.

9. Addendum. After this paper was written, the paper of Plackett and Burman on "The Design of Multifactorial Experiments" appeared in *Biometrika*. Volume 33 (1946), pages 305-325. A part of this paper discusses Hadamard matrices much more completely than we have done in section 3 In particular Plackett and Burman have constructed all Hadamard matrices of order less than or equal to 100 (excepting 92).

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THE APPROXIMATE DISTRIBUTION OF STUDENT'S STATISTIC

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Summary. It is well known that various statistics of a large sample (of size n) are approximately distributed according to the normal law. The asymptotic expansion of the distribution of the statistic in a series of powers of $n^{-\frac{1}{2}}$ with a remainder term gives the accuracy of the approximation. H. Cramér [1] first obtained the asymptotic expansion of the mean, and recently P. L. Hsu [2] has obtained that of the variance of a sample. In the present paper we extend the Cramér-Hsu method to Student's statistic. The theorem proved states essentially that if the population distribution is non-singular and if the existence of a sufficient number of moments is assumed, then an asymptotic expansion can be obtained with the appropriate remainder. The first four terms of the expansion are exhibited in formula (35).

1. In a fundamental paper[1] P. L. Hsu [2] has devised a method for obtaining the asymptotic expansion of the distribution of various statistics. The present paper deals with the so-called Student statistic.

Let

$$\xi_1, \xi_2, \cdots, \xi_n$$

be n independent random variables having the same probability distribution represented by a distribution function $P(x)$. The rth moment and rth absolute moment are denoted by α_r and β_r respectively. It is assumed that $\alpha_1 = 0$ and that for a certain $k \geqq 3$, $\beta_k < \infty$ and that $\alpha_2 > 0$. Hence there is no loss of generality in assuming that $\alpha_2 = 1$.

Student's statistic is defined as

$$\bar{\xi}\left(\frac{\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2}{n(n-1)}\right)^{-\frac{1}{2}} \quad \text{where} \quad \bar{\xi} = \frac{1}{n}\sum_{r=1}^{n} \xi_r .$$

For brevity, we consider

$$n\bar{\xi}\left(\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}}.$$

Let its distribution function be denoted by $F(z)$, i.e ,

$$Pr\left\{n\bar{\xi}\left(\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}} \leqq z\right\} = F(z).$$

[1] The definitions of the various constants A, A_k, Q_k, Λ_k, ϑ, Θ, Θ_k, are the same as in Hsu's paper.

Discarding the case $k = 3$ where we can prove a more precise result and the singular case which can be shown to admit no asymptotic expansion in the sense of Cramér [1], we shall prove in this paper the following theorem:

THEOREM. *Let $P(x)$ be non-singular and $\alpha_{2k} < \infty$ for some integer $k \geqq 4$. Then*

$$(1) \qquad F(z) = \Phi(z) + \chi(z) + R(z), \qquad \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-\frac{1}{2}y^2}\, dy,$$

where $\chi(z)$ is a linear combination of the derivatives $\Phi'(z), \cdots, \Phi^{(3k-10)}(z)$ with each coefficient of the form $n^{-\frac{1}{2}\nu}(1 \leqq \nu \leqq k-3)$ times a quantity depending only on α_3, $\cdots$, α_{k-1} whose beginning terms are given in (35) *and where*

$$(2) \qquad |R(z)| \leqq Q_k(1 + |z|^{2k-3})n^{-\alpha_0}, \qquad \alpha_0 = \frac{(k-1)\left[\frac{k}{2}\right]}{2\left(\left[\frac{k}{2}\right] + 1\right)},$$

where Q_k is a constant depending on k and $P(x)$.[1]

We shall need some of Hsu's lemmas, i.e., his lemma 3, lemma 7 (both for the particular case $m = 2$) and lemma 8. These we shall quote with this numbering. The application of Hsu's method to Student's statistic depends on the following lemma.

2. LEMMA A. *For $u \geqq -1$, $l \geqq 1$, we have*

$$1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)} u^j - \left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)}\right) u^{2l}$$

$$\leqq \sqrt{1+u} \leqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)} u^j.$$

PROOF. By Taylor's expansion of $\sqrt{1+u}$, we have

$$\sqrt{1+u} = 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)} u^j$$

$$+ \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - 2l\right)\Gamma(2l+1)} (1 + \vartheta u)^{-\frac{1}{2}(4l-1)} u^{2l},$$

whence it follows that $(1 + \vartheta u)^{-\frac{1}{2}(4l-1)}$ is finite, and positive. The right-hand side inequality follows.

Similarly, if $u \geqq 0$,

$$\sqrt{1+u} \geqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j + \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-2l\right)\Gamma(2l+1)} u^{2l}$$

$$\geqq 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j - \left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\right) u^{2l}$$

since by a well-known result on the binomial theorem we have

$$1 + \sum_{j=1}^{\infty} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} = \sqrt{1-1} = 0.$$

For $-1 \leqq u < 0$, we have

$$1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j - \sqrt{1+u} = \frac{N}{D}, \quad \text{say.}$$

For $-1 \leqq u < 0$, we have

$$D = 1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j + \sqrt{1+u}$$

$$\geqq 1 + \sum_{j=1}^{2l-1} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}.$$

Next,

$$N = \left(1 + \sum_{j=1}^{2l-1} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)} u^j\right)^2 - (1+u)$$

is a polynomial in u of the form

$$u^{2l}(a_0 + a_1u + \cdots + a_{2l}u^{2l})$$

where $a_0 > 0$ and the successive coefficients have alternating signs; hence for $-1 \leq u \leqq 0$, $a_0 + a_1u + \cdots + a_{2l}u^{2l}$ assumes its maximum at $u = -1$. This

maximum is obtained by putting $u = -1$ in the numerator, hence for $-1 \leqq u < 0$,

$$N \leqq u^{2l}\left(1 + \sum_{j=1}^{2l-1} \frac{(-1)^j\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2} - j\right)\Gamma(j+1)}\right)^2$$

The left-hand side inequality in the lemma now follows.

For brevity we write the inequalities as

$$(3) \quad 1 + P_{2l}(u) = 1 + P_{2l-1}(u) - b_{2l}\,u^{2l} \leqq \sqrt{1+u} \leqq 1 + P_{2l-1}(u), \quad b_{2l} > 0.$$

3. We write

$$\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2 = \sum_{r=1}^{n} \xi_r^2 - n\bar{\xi}^2 = n + \sqrt{n(\alpha_4 - 1)}\,X - Y^2,$$

where

$$X = \sum_{r=1}^{n} \frac{\xi_r^2 - 1}{\sqrt{n(\alpha_4 - 1)}}, \quad Y = \sqrt{n}\bar{\xi}.$$

Then Student's statistic may be written as

$$n\bar{\xi}\left(\sum_{r=1}^{n} (\xi_r - \bar{\xi})^2\right)^{-\frac{1}{2}} = Y\left(1 + \sqrt{\frac{\alpha_4 - 1}{n}}\,X - \frac{Y^2}{n}\right)^{-\frac{1}{2}}.$$

Then, for every z, we have

$$F(z) = Pr\left\{Y\left(1 + \sqrt{\frac{\alpha_4 - 1}{n}}\,X - Y^2\right)^{-\frac{1}{2}} \leqq z\right\}$$

$$= Pr\left\{\sqrt{1 + \frac{z^2}{n}}\,Y \leqq z\sqrt{1 + \sqrt{\frac{\alpha_4 - 1}{n}}\,X}\right\}.$$

For brevity let

$$\sqrt{1 + \frac{z^2}{n}}\,Y = V, \qquad \sqrt{\frac{\alpha_4 - 1}{n}}\,X = U.$$

Suppose $z \leqq 0$; then we have by (3),

$$(4) \quad \begin{aligned} z + zP_{2l-1}(U) &\leqq z\sqrt{1+U} \leqq z + zP_{2l}(U) \\ Pr\{V \leqq z + zP_{2l-1}(U)\} &\leqq F(z) \leqq Pr\{V \leqq z + zP_{2l}(U)\} \end{aligned}$$

Suppose $z > 0$; then we have by Lemma A a similar inequality with the extreme terms exchanged.

Now we take $l = \left[\frac{k}{2}\right]$, and fix it henceforth.

Our next step is to obtain an asymptotic expansion for

$$Pr\{V \leqq z + zP_m(U)\} = Pr\left\{Y \leq z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} + z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} P_m\left(\sqrt{\frac{\alpha_4-1}{n}}\,X\right)\right\}$$

with $m = 2l - 1$ or $2l$, $l \geqq 1$.

Let b be any real number, and

$$z\left(1+\frac{z^2}{n}\right)^{-\frac{1}{2}} P_m\left(\sqrt{\frac{\alpha_4-1}{n}}\,X\right) = L_m(X).$$

Until section 12, we shall write simply $L(x)$ for either of the $L_m(x)$.

4. Let W be the probability function of the distribution of the random point (X, Y) and let $f(t_1, t_2)$ be the characteristic function.

$$W(S) = Pr\{(X, Y) \epsilon S\} \quad \text{for every Borel set } S \text{ in } \mathfrak{R}_2$$

$$f(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{it_1x+it_2y}\, dW = \left\{p\left(\frac{t_1}{\sqrt{n}}, \frac{t_2}{\sqrt{n}}\right)\right\}^n$$

$$p(t_1, t_2) = \int_{-\infty}^{\infty} e^{it_1(\alpha_4-1)^{-\frac{1}{2}}(x^2-1)+it_2x}\, dP.$$

Then

$$(5) \qquad Pr\{Y \leqq b + L(X)\} = \iint\limits_{y\leqq b+L(x)} dW = \iint\limits_{y\leqq b} dW + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x, y)\, dW$$

where

$$G(x, y) = \begin{cases} 1 & \text{if } b < y \leqq b + L(x), \\ -1 & \text{if } b + L(x) < y \leqq b, \\ 0 & \text{otherwise} \end{cases}$$

We approximate $G(x, y)$ by $H(x, y)$, where

$$H(x, y) = \begin{cases} e^{-\epsilon x^{2l}} & \text{if } b < y \leqq b + L(x) \\ -e^{-\epsilon x^{2l}} & \text{if } b + L(x) < y \leqq b \\ 0 & \text{otherwise} \end{cases}$$

We approximate dW by $(w(x, y) + \gamma(x, y))\, dx\, dy$, where

$$w(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\phi(t_1,t_2)\, dt_1 dt_2$$

$$\gamma(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\phi(t_1,t_2)\psi(it_1,it_2)\, dt_1 dt_2$$

$$\phi(t_1,t_2) = e^{-\frac{1}{2}(t_1^2+2\rho t_1t_2+t_2^2)},\ \rho = \mathrm{E}\left(\frac{(\xi^2-1)\xi}{\sqrt{\alpha_4-1}}\right) = \frac{\alpha_3}{\sqrt{\alpha_4-1}}$$

and $\psi(it_1, it_2)$ is given in Lemma 3 by taking therein $\zeta_1 = \dfrac{\xi^2 - 1}{\sqrt{\alpha_4 - 1}}$, $\zeta_2 = \xi$, ξ being any of the ξ_i's.

We write

$$
(6)\quad
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)\,dW \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))\,dW \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))(w(x,y) + \gamma(x,y))\,dy\,dx \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)\,dW \\
&- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx \\
&+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx
\end{aligned}
$$

5. We have

$$
(7)\quad | G(x, y-u) - H(x, y-u) | \leqq 1 - e^{-\epsilon x^{2l}} \leqq \epsilon x^{2l}
$$

$$
(8)\quad \left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))\,dW \right| \leqq \int_{-\infty}^{\infty} \epsilon x^{2l}\,dW = \epsilon E(X^{2l}) \leqq Q_k\epsilon
$$

since

$$
E(X^{2l}) \leqq A_k E\left(\frac{\xi^2 - 1}{\sqrt{\alpha_4 - 1}}\right)^{2l} \leqq Q_k
$$

where Q_k depends on $\alpha_3, \cdots, \alpha_{2k}$.

Similarly,

$$
(9)\quad \left| \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G(x,y-u) - H(x,y-u))(w(x,y) + \gamma(x,y))\,dy\,dx \right| \leqq Q_k\epsilon
$$

Next,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y-u)(w(x,y) + \gamma(x,y))\,dy\,dx \\
(10)\quad &= \iint\limits_{y \leqq u+b+L(x)} (w(x,y) + \gamma(x,y))\,dy\,dx - \iint\limits_{y \leqq u+b} (w(x,y) + \gamma(x,y))\,dy\,dx
\end{aligned}
$$

where the first term on the right-hand side, regarded as a function of $n^{-\frac{1}{2}}$, has a Taylor expansion in powers of $n^{-\frac{1}{2}}$, whose first few terms we shall compute in

section 9; for the present let us denote it by $B(u + b) + C(u + b)n^{-\frac{1}{2}(k-2)}$ where $C = C(u + b)$ is a constant depending on k, $P(x)$ and z, a more explicit estimate of which will be given in section 10.

Further, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y - u)\,dW = \int\int_{y \leqq u+b+L(x)} dW - \int\int_{y \leqq u+b} dW \tag{11}$$

by Cramér's asymptotic expansion for the mean $\sqrt{n}Y$, and as is also shown in Hsu's paper we have

$$\int\int_{y \leqq u+b} dW - \int\int_{y \leqq u+b} (w(x,y) + \gamma(x,y))\,dy\,dx = \Lambda_k n^{-\frac{1}{2}(k-2)} \tag{12}$$

Collecting all the results from (5)–(12), we get

$$\begin{aligned}\int\int_{y \leqq u+b+L(x)} & dW - B(u + b) - C(u + b)n^{-\frac{1}{2}(k-2)} \\ &= \Lambda_k(\epsilon + n^{-\frac{1}{2}(k-2)}) + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)\,dW \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)(w(x,y) + \gamma(x,y))\,dy\,dx\end{aligned}$$

Now we use A. C. Berry's weighting factor $\dfrac{1 - \cos Tu}{u^2}$ and obtain

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2}\Bigg(\int\int_{y \leqq u+b+L(x)} & dW - B(u + b) - C(u + b)\Bigg)\,du \\ &= \Lambda_k T(\epsilon + n^{-\frac{1}{2}(k-2)}) \\ &+ \int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2}\Bigg(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)\,dW \\ &- \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x,y - u)(w(x,y) + \gamma(x,y))\,dy\,dx\Bigg)\,du\end{aligned} \tag{13}$$

since

$$\int_{-\infty}^{\infty} \frac{1 - \cos Tu}{u^2}\,du = \pi T.$$

6. To transform the triple integral on the right-hand side of (13) we use the Fourier transform as Hsu did.

Let

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x - it_2y} H(x, y)\,dy\,dx = h(t_1,t_2);$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}H(x,y-u)\,dy\,dx = e^{-it_2u}h(t_1, t_2)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-it_1x-it_2y}\left(\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}H(x,y-u)\,du\right)dy\,dx$$
$$= \begin{cases}\pi(T-|t_2|)h(t_1,t_2) & \text{if } |t_2| \leq T\\ 0 & \text{otherwise}\end{cases}$$

since

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}e^{-it_2u}\,du = \begin{cases}\pi(T-|t_2|) & \text{if } |t_2| \leq T\\ 0, & \text{otherwise}\end{cases}$$

By Fourier inversion we have, almost everywhere,

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}H(x, y-u)\,du = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}e^{it_1x+it_2y}(T-|t_2|)h(t_1, t_2)\,dt_2\,dt_1$$

Hence

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x, y-u)\,dW\,du$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)f(t_1, t_2)h(t_1, t_2)\,dt_2\,dt_1 \tag{14}$$

Similarly we obtain

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x, y-u)(w(x, y)+\gamma(x, y))\,dy\,dx\right)du$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)\phi(t_1, t_2)\{1+\psi(it_1, it_2)\}h(t_1, t_2)\,dt_2\,dt_1 \tag{15}$$

From (14) and (15) we obtain

$$\int_{-\infty}^{\infty}\frac{1-\cos Tu}{u^2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x, y-u)\,dW\right.$$
$$\left.-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x, y-u)(w(x, y)+\gamma(x, y))\,dy\,dx\right)du$$
$$= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-T}^{T}(T-|t_2|)\{f(t_1, t_2)$$
$$-\phi(t_1, t_2)(1+\psi(it_1, it_2))\}h(t_1, t_2)\,dt_2\,dt_1. \tag{16}$$

7. To estimate the double integral on the right-hand side of (16) we break it up into parts and use the following estimates of $h(t_1, t_2)$.

LEMMA B. *We have for* $l = \left[\frac{k}{2}\right] \geq 1$,

(1) $$| h(t_1, t_2) | \leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l};$$

(2) $$| h(t_1, t_2) | \leqq Q_k t_1^{-2} z^2 N(| t_2 |, n^{-1/2}, \epsilon^{-1/2l})$$

where $N(| t_2 |, n^{-1/2}, \epsilon^{-1/2l})$ *is a polynomial with constant coefficients in the indicated arguments.*

PROOF.

$$| h(t_1, t_2) | = \iint_{b < y \leqq b+L(x)} e^{-it_1x - it_2y - \epsilon x^{2l}} \, dy \, dx - \iint_{b+L(x) < y \leqq b} e^{-it_1x - it_2y - \epsilon x^{2l}} \, dy \, dx$$

$$= \left(\int_{L(x) \geqq 0} \int_b^{b+L(x)} - \int_{L(x) < 0} \int_{b+L(x)}^{b} \right) e^{-it_1x - it_2y - \epsilon x^{2l}} \, dy \, dx$$

$$= (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} e^{-it_1x - \epsilon x^{2l}} (e^{it_2L(x)} - 1) \, dx.$$

Hence

$$| h(t_1, t_2) | \leqq | t_2 |^{-1} \int_{-\infty}^{\infty} | t_2 L(x) | e^{-\epsilon x^{2l}} \, dx.$$

Since

$$| L(x) | \leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} | x |^j$$

we obtain

$$| h(t_1, t_2) | \leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} \int_{-\infty}^{\infty} | x |^j e^{-\epsilon x^{2l}} \, dx$$

$$\leqq A_k | z | \sum_{j=1}^{2l} (\alpha_4 - 1)^{\frac{1}{2}j} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l}.$$

Next, we write

$$h(t_1, t_2) = (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} u''(x) v(x) \, dx$$

with

$$u''(x) = e^{-it_1x}, \qquad v(x) = e^{-\epsilon x^{2l}} (e^{-it_2L(x)} - 1).$$

Integrating by parts twice, we get

$$h(t_1, t_2) = (-it_2)^{-1} e^{-it_2b} \int_{-\infty}^{\infty} u(x) v''(x) \, dx$$

whence

$$|h(t_1, t_2)| \leqq A_k t_1^{-2} \int_{\infty}^{\infty} e^{-\epsilon x^{2l}} \{ |L''(x)| + \epsilon x^{2l-1} |L'(x)| + \epsilon |x|^{2l-2} |L(x)|$$
$$+ \epsilon^2 x^{2l-2} |L(x)| + |t_2| |L'^2(x)| \} dx \leqq Q_k t_1^{-2}(|z| + z^2) N(|t_2|, n^{-\frac{1}{2}}, \epsilon^{-\frac{1}{2l}})$$

The lemma is proved.

Now we write

$$\int_{-\infty}^{\infty} \int_{-T}^{T} (T - |t_2|)\{f - \phi(1 + \psi)\} h \, dt_2 \, dt_1$$

(17)
$$= \iint_{\substack{|t_1| \leqq Q_k n^{\frac{1}{2}} \\ |t_2| \leqq Q_k n^{\frac{1}{2}}}} + \iint_{\substack{|t_1| > Q_k n^{\frac{1}{2}} \\ |t_2| \leqq T}} + \iint_{\substack{|t_1| \leqq Q_k n^{\frac{1}{2}} \\ Q_k n^{\frac{1}{2}} < |t_2| \leqq T}} = I_1 + I_2 + I_3 .$$

On I_1 we use Lemma 3 and Lemma B, (1):

$$|I_1| \leqq Q_k |z| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} T \left(\sum_{j=1}^{l} n^{-\frac{1}{2}j} \epsilon^{-\frac{1}{2}(k-2)} \right) n^{-\frac{1}{2}(k-2)}$$
$$\cdot \left\{ \sum_{i=1}^{2} (|t_i|^k + \cdots + |t_i|^{3(k-2)} \right\} e^{-\frac{1}{2}(1-\rho^2)(t_1^2+t_2^2)} \, dt_2 \, dt_1$$
$$\leqq Q_k |z| T n^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l}.$$

On I_2 we use Lemma 7 and Lemma B (2). Since $|t_1| > Q_k n^{\frac{1}{2}}$, $|\Phi(1 + \psi)| \leqq e^{-nQ_k}$, and by Lemma 7, $p(t_1 n^{-\frac{1}{2}}, t_2 n^{-\frac{1}{2}}) = e^{-Q_k}$ so that $|f(t_1, t_2)| \leqq e^{-nQ_k}$, $|f - \Phi(1 + \psi)| \leqq e^{-nQ_k}$

$$I_2 \leqq Q_k z^2 \iint_{\substack{|t_1| > Q_k n^{\frac{1}{2}} \\ |t_2| \leqq T}} T t_1^{-2} e^{-nQ_k} N(|t_2|, n^{-\frac{1}{2}}, \epsilon^{-1/2l}) \, dt_2 \, dt_1 .$$

Let $\epsilon = n^{-\beta}$, $\beta > 0$, then it is evident that

$$|I_2| \leqq Q_k z^2.$$

Similarly using Lemma 7 and Lemma B, (1) on I_3 we see that

$$|I_3| \leqq Q_k |z| .$$

Therefore

(18)
$$\left| \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-T}^{T} (T - |t_2|)\{f(t_1, t_2) - \phi(t_1, t_2)(1 + \psi(it_1, it_2))\} h(t_1, t_2) \, dt_2 \, dt_1 \right|$$
$$\leqq Q_k \left(|z| + z^2 + |z| T n^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \right).$$

8. Combining (13), (16), (17) we obtain

$$(19)\quad \begin{aligned}&\left|\int_{-\infty}^{\infty} \frac{1-\cos Tu}{u^2}\left(\iint_{y \leqq u+b+L(x)} dW - B(u+b) - C(u+b)n^{-\frac{1}{2}(k-2)}\right) du\right| \\ &\leqq Q_k\left(T\epsilon + Tn^{-\frac{1}{2}(k-2)} + |z| + z^2 + |z|\, Tn^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l}\right).\end{aligned}$$

Now we shall choose T and ϵ suitably. Let

$$T = n^{\alpha}, \qquad \epsilon = n^{-\beta}, \qquad \alpha > 0, \qquad \beta > 0.$$

To make the right-hand side of (19) a constant depending on z only, we must have $\alpha \leqq \frac{1}{2}(k-2) \cdot \beta \leqq \alpha$. Then

$$\sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} = \sum_{j=1}^{2l} n^{((\beta-l)j+\beta)/2l}.$$

We must choose $\beta < k/2$, then

$$\sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \leqq A_k n^{(2\beta-l)/2l}.$$

To make the exponent as small as possible we choose $\beta = \alpha$, then

$$|z|\, Tn^{-\frac{1}{2}(k-2)} \sum_{j=1}^{2l} n^{-\frac{1}{2}j} \epsilon^{-(j+1)/2l} \leqq A_k |z|\, n^{\alpha-\frac{1}{2}(k-2)+(2\alpha-l)/2l} = A_k |z|\, n^{(l+1)\alpha/l-\frac{1}{2}(k-1)}$$

since α is to be as large as possible, we choose

$$\alpha = \alpha_0 = \min\left(\frac{k-2}{2}, \frac{(k-1)l}{2(l+1)}\right), \qquad l = \left[\frac{k}{2}\right].$$

Then we obtain

$$(20)\quad \left|\int_{-\infty}^{\infty} \frac{1-\cos Tu}{u^2}\left(\iint_{y \leqq u+b+L(x)} dW - B(u+b) - C(u+b)n^{-\frac{1}{2}(k-2)}\right) du\right| \leqq Q_k(1+z^2).$$

Let $F^*(u)$ be the distribution function of $Y - L(X)$, and let

$$F_1(u) = B(u) + C(u)n^{-\frac{1}{2}(k-2)}$$

Then we may write (20) as

$$(21)\quad \left|\int_{-\infty}^{\infty} \frac{1-\cos Tu}{u^2}\{F^*(u+b) - F_1(u+b)\}\, du\right| \leqq Q_k(1+z^2).$$

By the definition of $F_1(u)$ we see that the conditions in Lemma 8 are all satisfied with a certain constant D depending on k, $P(x)$, and z for the M therein. Then choosing b to be the a in Lemma 8, we obtain from Lemma 8 and (21),

$$(22)\quad DT\delta\left\{3\int_0^{T\delta} \frac{1-\cos x}{x^2}\, dx - \pi\right\} \leqq Q_k(1+z^2)$$

where

$$\delta = \frac{1}{2D} \text{ l.u.b. } | F^*(u) - F_1(u) |.$$

Now there exists A such that if $T\delta > A$, then

$$3 \int_0^{T\delta} \frac{1 - \cos x}{x^2} dx - \pi > 1,$$

hence it follows from (22) that

$$T\delta \leqq \max (A, D^{-1} Q_k(1 + z^2)).$$

Thus for another Q_k' exceeding both A and the above Q_k, we have

$$T\delta \leqq Q_k'(1 + z^2)$$

and so finally, dropping the prime,

(23) $$| F^*(u) - F_1(u) | \leqq Q_k(1 + z^2) D T^{-1} = Q_k(1 + z^2) D n^{-\alpha_0}.$$

In particular, taking b to be $z(1 + n^{-1} z^2)^{-\frac{1}{2}} = z'$, say:

(24) $$Pr \{Y - L(X) \leqq z'\} = B(z') + C(z') n^{-\frac{1}{2}(k-2)} + \Lambda_k(1 + z^2) D n^{-\alpha}$$

where

$B(z') + C(z') n^{-\frac{1}{2}(k-2)}$ = the Taylor expansion with a remainder of

$$\iint_{y - L(x) \leqq z'} (w(x, y) + \gamma(x, y)) \, dy \, dx$$

and D is an upper bound for

$$| B'(u) + C'(u) n^{-\frac{1}{2}(k-2)} |.$$

9. Let $\lambda = n^{-\frac{1}{2}}$, and rewrite the $z' + L_{2l-1}(x)$, $l \geqq 2$ there as $g(\lambda)$:

$$g(\lambda) = z' \left(1 + \frac{(\alpha_4 - 1)^{1/2}}{2} \lambda x - \frac{\alpha_4 - 1}{8} \lambda^2 x^2 + \frac{(\alpha_4 - 1)^{3/2}}{16} \lambda^3 x^3 + \cdots \right)$$

Then

$$g(0) = z'$$

$$g'(0) = \frac{(\alpha_4 - 1)^{\frac{1}{2}}}{2} z' x$$

$$g''(0) = -\frac{\alpha_4 - 1}{4} z' x^2$$

$$g'''(0) = \frac{3(\alpha_4 - 1)^{3/2}}{8} z' x^3.$$

Let $p \geqq 0,\ q \geqq 0$; $w_{pq}(x, y) = \dfrac{\partial^{p+q}}{\partial x^p \partial y^q} w(x,y)$ where $w(x, y)$ is defined in section 4 and we know that

$$w(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} e^{-(x^2 - 2\rho xy + y^2)/2(1-\rho^2)}.$$

Let

$$f_{pq}(\lambda) = \int_{-\infty}^{\infty} \int_{-\infty}^{g(\lambda)} w_{pq}(x, y)\, dy\, dx. \tag{26}$$

Then

$$\begin{aligned} f'_{pq}(\lambda) &= \int_{-\infty}^{\infty} g'(\lambda) w_{pq}(x, g(\lambda))\, dx \\ f''_{pq}(\lambda) &= \int_{-\infty}^{\infty} (g''(\lambda) w_{pq}(x, g(\lambda)) + g'^2(\lambda) w_{p,q+1}(x, g(\lambda)))\, dx \\ f'''_{pq}(\lambda) &= \int_{-\infty}^{\infty} (g'''(\lambda) w_{pq}(x, g(\lambda)) + 3g''(\lambda) g'(\lambda) w_{p,q+1}(x, g(\lambda)) \\ &\qquad + g'^3(\lambda) w_{p,q+2}(x, g(\lambda)))\, dx \end{aligned} \tag{27}$$

Let

$$\begin{aligned} \Phi = \Phi(z') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z'} e^{\frac{1}{2}y^2}\, dy, \quad \Phi^{(q)} = \frac{d^q}{dz^q} \Phi(z) \Big|_{z=z'} \\ I^r_{pq} &= \int_{-\infty}^{\infty} x^r w_{pq}(x, z')\, dx \end{aligned} \tag{28}$$

We have computed the following table of values of I^r_{pq}:

$r \backslash p$	0	1	2	3	$\geqq 4$
0	$\Phi^{(q)}$	0	0	0	0
1	$-\rho\Phi^{(q+1)}$	$-\Phi^{(q)}$	0	0	0
2	$\Phi^{(q)} + \rho^2\Phi^{(q+2)}$	$2\rho\Phi^{(q+1)}$	$2\Phi^{(q)}$	0	0
3	$-3\rho\Phi^{(q+1)} - \rho^3\Phi^{(q+3)}$	$-3\Phi^{(q)} - 3\rho^2\Phi^{(q+2)}$	$-6\rho\Phi^{(q+1)}$	$-6\Phi^{(q)}$	0

Next, we find, from (25)–(28),

$$\begin{aligned} f_{00}(0) &= \Phi; \\ f_{pq}(0) &= I^0_{p,q-1} \text{ for } q \geqq 1; \\ f'_{pq}(0) &= \frac{(\alpha_4 - 1)^{\frac{1}{2}}}{4} z' I^1_{pq} \\ f''_{pq}(0) &= -\frac{\alpha_4 - 1}{4} z' I^2_{pq} + \frac{\alpha_4 - 1}{4} z'^2 I^2_{p,q+1} \\ f'''_{pq}(0) &= \frac{3(\alpha_4 - 1)^{3/2}}{8} z' I^3_{pq} - \frac{3(\alpha_4 - 1)^{3/2}}{8} z'^2 I^3_{p,q+1} + \frac{(\alpha_4 - 1)^{3/2}}{8} z'^3 I^3_{p,q+2} \end{aligned} \tag{92}$$

Now we can expand

$$\iint\limits_{y \leq b+L(x)} w(x,y)\,dy\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w(x,y)\,dy\,dx = f_{00}(\lambda)$$

Write the Taylor's series for $f_{00}(\lambda)$:

$$f_{00}(\lambda) = f_{00}(0) + f_{00}'(0)\lambda + \frac{f_{00}''(0)}{2}\lambda^2 + \frac{f_{00}''(0)}{6}\lambda^3 + \cdots$$

Substituting from (29), we get

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} w(x,y)\,dy\,dx &= \Phi - \frac{(\alpha_4-1)^{1/2}}{2n^{1/2}}\rho z'\Phi^{(1)} \\ &+ \frac{\alpha_4-1}{8n}\{-z'(\Phi^{(0)} + \rho^2\Phi^{(2)}) + z'^2(\Phi^{(1)} + \rho^2\Phi^{(3)})\} \\ &+ \frac{(\alpha_4-1)^{3/2}}{48n^{3/2}}\{3z'(-3\rho\Phi^{(1)} - \rho^3\Phi^{(3)}) - 3z'^2(-3\rho\Phi^{(2)} - \rho^3\Phi^{(4)}) \\ &\qquad + z'^3(-3\rho\Phi^{(3)} - \rho^3\Phi^{(5)})\} + \cdots \end{aligned} \tag{30}$$

Further, we must obtain the beginning terms of $\gamma(x, y)$ as given in Lemma 3, for which purpose we refer to Hsu's paper. We have, in fact

$$\psi(it_1, it_2) = -\frac{iU^3}{6n^{1/2}} + \left(\frac{U_4}{24} - \frac{U_3^2}{36}\right)\frac{1}{n} + \left(\frac{U_5}{120} - \frac{U_3U_4}{72} + \frac{U_3^2}{216}\right)\frac{i}{n^{3/2}} + \cdots$$

where

$$\begin{aligned} U_3 &= E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^3 \\ &= \alpha_3 t_2^3 + 3\sqrt{\alpha_4-1}\,t_2^2 t_1 + 3\frac{\alpha_5-2\alpha_3}{\alpha_4-1}t_2 t_1^2 + \frac{\alpha_6-3\alpha_4+2}{(\alpha_4-1)^{3/2}}t_1^3 \\ U_4 &= E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^4 - 3\left(E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)\right)^2 \\ &= (\alpha_4-3)t_2^4 + 4\frac{\alpha_5-4\alpha_3}{\sqrt{\alpha_4-1}}t_2^3 t_1 + \cdots \\ U_5 &= E\left((t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^5 - 10E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^2 E\left(t_1\frac{\xi^2-1}{\sqrt{\alpha_4-1}} + t_2\xi\right)^3 \\ &= (\alpha_5 - 10\alpha_3)t_2^5 + \cdots \end{aligned}$$

To avoid the exhibition of very long expressions, let us separate the terms in $\psi(it_1, it_2)$ according to the powers of $n^{-1/2}$, and denote the terms of the power $n^{-1/2}\,n^{-1}, n^{-3/2}$ by ψ_1, ψ_2, ψ_3, respectively.

Thus $\psi_1 = -iU_3/6n^{1/2}$, and the corresponding $\gamma(x, y)$ is

$$\gamma_1(x,y) = -\frac{1}{6n^{1/2}}\left(\alpha_3 w_{03}(x,y) + 3\sqrt{\alpha_4 - 1}\, w_{12}(x,y) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\, w_{21}(x,y) + \cdots\right) \tag{31}$$

where, as hereafter, the terms omitted will yield nothing in the long run.

Now we have by (31) and (26),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\rho(\lambda)} \gamma_1(x, y)\, dy\, dx$$

$$= -\frac{1}{6n^{1/2}}\left(\alpha_3 f_{03}(\lambda) + 3\sqrt{\alpha_4 - 1} f_{12}(x, y) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f_{21}(x, y) + \cdots\right)$$

$$= -\frac{1}{6n^{1/2}}\left(\alpha_3 f_{03}(0) + 3\sqrt{\alpha_4 - 1} f_{12}(0) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f_{21}(0) + \cdots\right)$$

$$-\frac{1}{6n}\left(\alpha_3 f'_{03}(0) + 3\sqrt{\alpha_4 - 1} f'_{12}(0) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f'_{21}(0) + \cdots\right)$$

$$-\frac{1}{12n^{3/2}}\left(\alpha_3 f''_{03}(0) + 3\sqrt{\alpha_4 - 1} f''_{12}(0) + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1} f''_{21}(0) + \cdots\right)$$

$$= -\frac{1}{6n^{1/2}}(\alpha_3 I^0_{02}) - \frac{(\alpha_4 - 1)^{1/2}}{12n} z'(\alpha_3 I^1_{03} + 3\sqrt{\alpha_4 - 1}\, I^1_{12}) \tag{32}$$

$$+ \frac{\alpha_4 - 1}{48n^{3/2}}\left\{z'\left(\alpha_3 I^2_{03} + 3\sqrt{\alpha_4 - 1}\, I^2_{12} + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\, I^2_{21}\right)\right.$$

$$\left. - z'^2\left(\alpha_3 I^2_{04} + 3\sqrt{\alpha_4 - 1}\, I^2_{13} + 3\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\, I^2_{22}\right)\right\} + \cdots$$

$$= -\frac{\alpha_3}{6n^{1/2}}\Phi^{(2)} + \frac{(\alpha_4 - 1)^{1/2}}{12n} z'(\alpha_3 \rho\Phi^{(4)} + 3\sqrt{\alpha_4 - 1}\,\Phi^{(2)})$$

$$+ \frac{\alpha_4 - 1}{48n^{3/2}}\left\{z'\left[\alpha_3(\Phi^{(3)} + \rho^2\Phi^{(5)}) + 6\sqrt{\alpha_4 - 1}\,\rho\Phi^{(3)} + 6\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\,\Phi^{(1)}\right]\right.$$

$$\left. - z'^2\left[\alpha_3(\Phi^{(4)} + \rho^2\Phi^{(6)}) + 6\sqrt{\alpha_4 - 1}\,\rho\Phi^{(4)} + 6\,\frac{\alpha_5 - 2\alpha_3}{\alpha_4 - 1}\,\Phi^{(2)}\right]\right\} + \cdots.$$

Similarly, omitting the intermediate steps to save space, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\rho(\lambda)} \gamma_2(x, y)\, dy\, dx = \frac{1}{72n}\{3(\alpha_4 - 3)\Phi^{(3)} + 2\alpha_3^2\Phi^{(5)}\}$$

$$- \frac{(\alpha_4 - 1)^{1/2}}{144n^{3/2}} z'\left\{3(\alpha_4 - 3)\rho\Phi^{(5)} + 12\,\frac{\alpha_5 - 4\alpha_3}{\sqrt{\alpha_4 - 1}}\,\Phi^{(3)}\right. \tag{33}$$

$$\left. + 2\alpha_3^2\rho\Phi^{(7)} + 12\alpha_3\sqrt{\alpha_4 - 1}\,\Phi^{(5)}\right\} + \cdots.$$

$$
(34)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \gamma_3(x, y)\, dy\, dx = -\frac{1}{n^{3/2}}\left(\frac{\alpha_5 - 10\alpha_3}{120}\Phi^{(4)} + \frac{\alpha_3\alpha_4}{72}\Phi^{(6)} + \frac{\alpha_1^3}{216}\Phi^{(8)}\right) + \cdots .
$$

Combining (30), (32)–(34) and simplifying, we obtain, as the first four terms of the asymptotic expansion of $F(z)$:

$$
\begin{aligned}
\iint_{y - L(x) \leq z'} (w(x, y) + \gamma(x, y))\, dy\, dx &= \Phi - \frac{\alpha_3}{6n^{1/2}}(3z'\Phi^{(1)} + \Phi^{(2)}) \\
&+ \frac{1}{4n}\left\{\frac{\alpha_4 - 3}{6}\Phi^{(3)} + \frac{\alpha_3^2}{9}\Phi^{(9)}\right. \\
&+ z'\left(\frac{\alpha_3^2}{3}\Phi^{(4)} + \frac{2(\alpha_4 - 1) - \alpha_3^2}{2}\Phi^{(2)} - \frac{\alpha_4 - 1}{2}\Phi^{(0)}\right) \\
&\left. + z'^2\left(\frac{\alpha_4 - 1}{2}\Phi^{(1)} + \frac{\alpha_3^2}{2}\Phi^{(3)}\right)\right\} \\
(35)\qquad &+ \frac{1}{24n^{3/2}}\left\{-\frac{\alpha_5 - 10\alpha_3}{5}\Phi^{(4)} + \frac{\alpha_3\alpha_4}{3}\Phi^{(6)} + \frac{\alpha_3^3}{9}\Phi^{(8)}\right. \\
&+ z'\left[\frac{6\alpha_5 - 3\alpha_3 - 9\alpha_3\alpha_4}{2}\Phi^{(1)} + \frac{7\alpha_3(\alpha_4 - 1 - \alpha_3^2) - 2\alpha_5}{2}\Phi^{(3)}\right. \\
&\left. + \frac{\alpha_3(\alpha_3^2 - 5\alpha_4 + 7)}{2}\Phi^{(5)} - \frac{\alpha_3^3}{3}\Phi^{(7)}\right] \\
&+ z'^2\left[\frac{9\alpha_3\alpha_4 + 3\alpha_3 - 6\alpha_5}{2}\Phi^{(2)} + \frac{\alpha_3(3\alpha_3^2 - 7\alpha_4 + 7)}{2}\Phi^{(4)}\right] \\
&\left. + z'^3\left[\frac{-3\alpha_3(\alpha_4 - 1)}{2}\Phi^{(3)} - \frac{\alpha_3^2}{2}\Phi^{(5)}\right]\right\} + \cdots .
\end{aligned}
$$

10. In order to estimate the remainder $C(z')n^{-(k-2)/2}$ in the Taylor expansion we write, in accordance with Lemma 3,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} (w(x, y) + \gamma(x, y))\, dy\, dx \\
&\quad = \int_{-\infty}^{\infty}\int_{-\infty}^{g(\lambda)} \left\{w(x, y) + \sum_{\nu=1}^{k-3} \lambda^{\nu}\Sigma(-1)^{\nu_1+\nu_2} a_{\nu_1\nu_2} w_{\nu_1\nu_2}(x, y)\right\} dy\, dx \\
&\quad = f_{00}(\lambda) + \sum_{\nu=1}^{k-3}\lambda^{\nu}\Sigma(-1)^{\nu_1+\nu_2} a_{\nu_1\nu_2} f_{\nu_1\nu_2}(\lambda) = \sum_{j=0}^{k-3} f_{00}^{(j)}(0)\frac{\lambda^j}{j!} \\
&\qquad + f_{00}^{(k-2)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2)!} + \sum_{\nu=1}^{k-3}\lambda^{\nu}\Sigma(-1)^{\nu_1+\nu_2} a_{\nu_1\nu_2} \\
&\qquad \cdot\left\{\sum_{j=1}^{k-3-\nu} f_{\nu_1\nu_2}^{(j)}(0)\frac{\lambda^j}{j!} + f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\frac{\lambda^{k-2-\nu}}{(k-2-\nu)!}\right\}
\end{aligned}
$$

$$= B(z') + f_{00}^{(k-2)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2)!} + \sum_{\nu=1}^{k-3}\Sigma(-1)^{\nu_1+\nu_2}a_{\nu_1\nu_2}f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\frac{\lambda^{k-2}}{(k-2-\nu)!}$$

$$= B(z') + \Lambda_k\lambda^{k-2}\left(f_{00}^{(k-2)}(\theta\lambda) + \sum_{\nu=1}^{k-3} f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\right).$$

Thus

$$C(z') = \Lambda_k\left(f_{00}^{(k-2)}(\theta\lambda) + \sum_{\nu=1}^{k-3} f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)\right)$$

Now we may write

$$f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda) = \int_{-\infty}^{\infty}\Sigma(\Pi(g^{(s)}(\theta\lambda))^t)w_{pq}(x, g(\theta\lambda))\,dx$$

where, if we attach a weight s to $g^{(s)}(\theta\lambda)$, the polynomial under the integral sign is isobaric of weight $k - 2 - \nu$ in these $g^{(s)}$'s, and the coefficient of each term is a constant multiple of a certain $w_{pq}(x, g(\theta\lambda))$. Further, it is easily seen by induction that we have

$$g^{(s)}(\theta\lambda) = P_{1+2s}(z)(1 + \theta^2\lambda^2z^2)^{-\frac{1}{2}-s}$$

where $P_{1+2s}(z)$ is a polynomial of the three variables z, x, $\theta\lambda$ which is of at most the $(1 + 2s)$th degree in z and of the $(2l - 1)$st degree in x, and whose coefficients are all Λ_k.

Therefore,

$$\begin{aligned}|f_{\nu_1\nu_2}^{(k-2-\nu)}(\theta\lambda)| &\leq \int_{-\infty}^{\infty} Q_k(|x| + x^2 + \cdots + |x|^{2l-1}) \\ &\quad\cdot(1 + |z|^{(1+2s)(k-2-\nu)s^{-1}})w_{pq}(x, g(\theta\lambda))\,dx \\ &\leq \int_{-\infty}^{\infty} Q_k(|x| + x^2 + \cdots + |x|^{2l-1})(1 + |z|^{1+2(k-2)})e^{-\frac{1}{2}x^2}\,dx \\ &\leq Q_k(1 + |z|^{2k-3})\end{aligned}$$

Thus

$$C(z') \leq Q_k(1 + |z|^{2k-3}) \tag{36}$$

Lastly, an estimate of D is easy:

$$\begin{aligned}|F_1'(u)| &= \left|\frac{d}{du}\int_{-\infty}^{\infty}\int_{-\infty}^{u+L(x)}(w(x, y) + \gamma(x, y))\,dy\,dx\right| \\ &\leq \int_{-\infty}^{x}(w(x, u + L(x)) + |\gamma(x, u + L(x))|)\,dx \leq Q_k.\end{aligned} \tag{37}$$

Collecting the results of (24), (36), (37) we obtain

$$\Pr\{Y - L(X) \leq z'\} = B(z') + \Lambda_k((1 + |z|^{2k-3})n^{-\frac{1}{2}(k-2)} + (1 + z^2)n^{-\alpha_0}),$$

$$\alpha_0 = \min\left(\frac{k-2}{2}, \quad \frac{(k-1)\left[\frac{k}{2}\right]}{2\left(\left[\frac{k}{2}\right]+1\right)}\right). \tag{38}$$

Or, more simply,

$$(39) \qquad | Pr\{Y - L(X) \leqq z'\} - B(z') | \leqq Q_k(1 + |z|^{2k-3})n^{-\alpha_0},$$

where the first four terms of $B(z')$ are given by (35).

12. To return to $F(z)$. We see that $B(z')$ depends on the function $L(x)$. Recalling section 3 we now write B_m for the B corresponding to L_m, with $m = 2l - 1$ or $2l$.

Then by (4) the value of $F(z)$ lies between

$$\Pr\{Y - L_{2l-1}(X) \leqq z'\} \quad \text{and} \quad \Pr\{Y - L_{2l}(X) \leqq z'\}.$$

From the asymptotic expansion just obtained for either of them, we see that the absolute value of their difference does not exceed

$$| B_{2l-1}(z') - B_{2l}(z') | + Q_k(1 + |z|^{2k-3})n^{-\alpha_0}.$$

But

$$L_{2l}(x) = L_{2l-1}(x) - z'b_{2l}(\alpha_4 - 1)^l n^{-l} x^{2l} = L_{2l-1}(x) - b'_{2l}x^{2l} \text{ say,}$$

hence

$$| B_{2l-1}(z') - B_{2l}(z') | \leqq \int_{-\infty}^{\infty} \int_{z'+L_{2l-1}(x)-b'_{2l}x^{2l}}^{z'+L_{2l-1}(x)} |w(x, y) + \gamma(x, y)| \, dy \, dx$$

$$\leqq Q_k b'_{2l} \leqq Q_k |z| n^{-l} < Q_k |z| n^{-\alpha_0}.$$

Therefore

$$| Pr\{Y - L_{2l-1}(X) \leqq z'\} - Pr\{Y - L_{2l}(X) \leqq z'\} | \leqq Q_k n^{-\alpha_0}$$

and so we obtain

$$(40) \qquad F(z) = B(z') + \Lambda_k(1 + |z|^{2k-3})n^{-\alpha_0}$$

which is equivalent to (2) in the theorem stated in section 1.

Thus the theorem will be proved if the assertions regarding the form of $f(z)$ in (1) are shown to be true.

For this purpose we denote, as before, the terms of the order $n^{-\nu/2}$ in $\psi(it_1, it_2)$ and $\gamma(x, y)$ by ψ_ν, γ_ν respectively. Since the term in ψ_ν which yields a w_{pq} with the greatest q is U_3^ν, we have for every w_{pq} in γ_ν the condition $q \leqq 3\nu$. We expand $\int_{-\infty}^{\infty} \int_{-\infty}^{g(\lambda)} \gamma_\nu(x, y) dy dx$ to $k - 3 - \nu$ terms, in which $f_{pq}(0)$, $f'_{pq}(0)$, $\cdots$, $f_{pq}^{(k-3-\nu)}(0)$ occur. In the integrand of $f_{pq}^{(k-3-\nu)}(0)$, e.g., the coefficients of each $w_{pq}(x, z)$ are polynomials in z and x of a total degree in z and x not exceeding that of $(g'(0))^{k-3-\nu}$, i.e., $2(k - 3 - \nu)$. Hence the expansion of γ_ν will give rise to terms of the form

$$z^s I_{pq}^t, \qquad q \leqq 3\nu, \qquad s + t = 2(k - 3 - \nu).$$

Such a term will yield a term $z^s\Phi^{(q+t)}$, which in turn yields the terms $\Phi^{(r)}$ with

$$r \leqq s + q + t \leqq 3\nu + 2(k - 3 - \nu) \leqq 3(k - 3),$$

Equality holds only when $\nu = k - 3$ and $q = 3(k - 3)$. But when $\nu = k - 3$, the term in question is

$$f_{0,3(k-3)}(0) = I^0_{0,3k-10} = \Phi^{(3k-10)}$$

Next, we see that ψ_ν contains $U_3, \cdots, U_{\nu+2}$. Since $f^{(k-3-\nu)}(0)$ is a polynomial of the $(k - 3 - \nu)$th degree in x, the expansion of γ_ν will yield $I^0_{pq}, \cdots, I^{k-3-\nu}_{pq}$. But $I^{k-3-\nu}_{pq} = 0$ if $p > k - 3 - \nu$, hence $p \leqq k - 3 - \nu$. Thus in ψ_ν we need only take account of the terms $(it_1)^p(it_2)^q$ with $p \leqq k - 3 - \nu$. Now if $j < k - 3 - \nu$, in U_j only $\alpha_3, \cdots, \alpha_{2(k-3-\nu)}$ occur. If $j \geqq k - 3 - \nu$, in the coefficient of a term $(it_1)^p(it_2)^q$ with $p \leqq k - 3 - \nu$ the greatest index of α is

$$2(k - 3 - \nu) + j - (k - 3 - \nu) = j + k - 3 - \nu \leqq k - 1$$

since $j \leqq \nu + 2$. Hence in the expansion of every γ only $\alpha_3, \cdots, \alpha_{k-1}$ occur. The proof of the theorem is completed

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SOME IMPROVEMENTS IN SETTING LIMITS FOR THE EXPECTED NUMBER OF OBSERVATIONS REQUIRED BY A SEQUENTIAL PROBABILITY RATIO TEST

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Summary. Upper and lower limits for the expected number n of observations required by a sequential probability ratio test have been derived in a previous publication [1]. The limits given there, however, are far apart and of little practical value when the expected value of a single term z in the cumulative sum computed at each stage of the sequential test is near zero. In this paper upper and lower limits for the expected value of n are derived which will, in general, be close to each other when the expected value of z is in the neighborhood of zero. These limits are expressed in terms of limits for the expected values of certain functions of the cumulative sum Z_n at the termination of the sequential test.

In section 7 a general method is given for determining limits for the expected value of any function of Z_n

1. Introduction. Let x be a random variable and let $f(x, \theta)$ be the elementary probability law of x involving an unknown parameter θ. Let H_0 denote the hypothesis that $\theta = \theta_0$, and H_1 the hypothesis that $\theta = \theta_1$, where θ_0 and θ_1 are given specified values. The sequential probability ratio test for testing H_0 against H_1, as defined in [1], is given as follows: Put

$$z_i = \log \frac{f(x_i, \theta_1)}{f(x_i, \theta_0)} \tag{1.1}$$

where x_i denotes the i-th observation on x. Two constants, a and b are chosen where $a > 0$ and $b < 0$. At each stage of the experiment, at the m-th trial for each positive integral value m, the cumulative sum

$$Z_m = z_1 + \cdots + z_m \tag{1.2}$$

is computed. Experimentation is continued as long as $b < Z_m < a$. The first time that Z_m does not lie between b and a, experimentation is terminated. The hypothesis H_1 is accepted if $Z_m \geqq a$, and H_0 is accepted if $Z_m \leqq b$.

Let n denote the smallest value of m for which Z_m does not lie between b and a. Then n is the number of observations required by the sequential test. The expected value of n is a function of the true parameter value θ and is denoted by $E_\theta(n)$.

Upper and lower limits for $E_\theta(n)$ have been derived in section 4 of [1]. These limits, however, are of little practical value when the expected value of

$$z = \log \frac{f(x, \theta_1)}{f(x, \theta_0)} \tag{1.3}$$

is in the neighborhood of zero, for they converge to $+\infty$ and $-\infty$, respectively, as the expected value of z approaches zero. It can be shown that the expected value of z is negative when $\theta = \theta_0$, and positive when $\theta = \theta_1$ [1] Thus, if the expected value of z is a continuous function of θ, there will be a value θ' between θ_0 and θ_1 such that the expected value of z is zero when $\theta = \theta'$. Hence, the limits for $E_\theta(n)$, as given in [1], are of no practical value when θ is near θ'

The purpose of this paper is to derive upper and lower limits for $E_\theta(n)$ which will be, in general, close to each other when θ is in the neighborhood of θ' Thus, it will generally be possible to obtain close limits for $E_\theta(n)$ over the whole range of θ, if the limits given here are used for values in a certain small interval containing θ', and the limits given in [1] are used when θ is outside this interval.

2. Notation. We shall use the following notations throughout the paper. For any random variable u, the symbol $E_\theta(u)$ will denote the expected value of u when θ is the true value of the parameter. The conditional expected value of u, under the restriction that some relationship R is fulfilled will be denoted by $E_\theta(u \mid R)$. The symbol $P(R \mid \theta)$ will denote the probability that the relationship R holds when θ is true.

The cumulative distribution function of z will be denoted by $F(z, \theta)$ when θ is the true value of the parameter. The moment generating function of z, when θ is true, will be denoted by $\varphi(t, \theta)$, i e

$$\varphi(t, \theta) = \int_{-\infty}^{\infty} e^{tz}\, dF(z, \theta). \tag{2 1}$$

3. Assumptions concerning the family of distribution functions $F(z, \theta)$. In this section we shall formulate two assumptions concerning $F(z, \theta)$ which will then be used to prove various lemmas and theorems Since we are interested in values of θ near θ', we shall restrict the domain of θ to a finite closed interval I containing θ' in its interior. It will be understood throughout the paper that any statements concerning θ refer to the domain I, even if this is not explicitly stated.

ASSUMPTION 1. *The moment generating function $\varphi(t, \theta)$ exists for any point t in the complex plane and any value θ, and is a continuous function of θ.*

ASSUMPTION 2. *There eists a positive δ such that $P(e^z > 1 + \delta \mid \theta)$ and $P(e^z < 1 - \delta \mid \theta)$ have positive lower bounds with respect to θ.*

4. *Proof that $\varphi(t, \theta)$ is continuous in t and θ jointly and that all moments of z are continuous functions of θ.*[2] In this section we shall prove the following theorem:

[1] This follows easily from Lemma 1 in [1], p 156

[2] The original proof of the author was somewhat lengthy The present proof was suggested by T E Harris

THEOREM 4.1. *It follows from Assumption 1 that $\varphi(t, \theta)$ is continuous in t and θ jointly and all moments of z are continuous functions of θ.*

PROOF: First we show that $\varphi(t, \theta)$ is a bounded function of t and θ in the domain $|t| \leqq t_0$, for any finite positive value t_0. Clearly,

$$0 \leqq |\varphi(t, \theta)| \leqq 2[\varphi(t_0, \theta) + \varphi(-t_0, \theta)] \tag{4.1}$$

for all values t for which $|t| \leqq t_0$. The boundedness of $\varphi(t_0, \theta)$ and $\varphi(-t_0, \theta)$ follows from Assumption 1. Hence $\varphi(t, \theta)$ is a bounded function of θ and t over any bounded t-domain.

Let $\{t_m, \theta_m\}$ ($m = 1, 2, \cdots$, ad inf.) be a sequence of pairs converging to the pair (t', θ'). We have

$$\varphi(t_m, \theta_m) - \varphi(t', \theta') = [\varphi(t_m, \theta_m) - \varphi(t', \theta_m)] + [\varphi(t', \theta_m) - \varphi(t', \theta')]. \tag{4.2}$$

The second expression in brackets converges to zero by continuity in θ. Thus the first part of Theorem 4.1 is proved if we show that

$$\lim_{m \to \infty} [\varphi(t_m, \theta_m) - \varphi(t', \theta_m)] = 0. \tag{4.3}$$

It follows from Assumption 1 that for any given θ, $\varphi(t, \theta)$ is an analytic function with no singularities in any finite t-domain. Hence we can expand $\varphi(t_m, \theta_m)$ in a Taylor series around $t = t'$, i.e.

$$\varphi(t_m, \theta_m) - \varphi(t', \theta_m) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{\partial^k \varphi(t, \theta_m)}{\partial t^k} \bigg|_{t=t'} \right) (t_m - t')^k. \tag{4.4}$$

Let r be a given positive value. Because of the boundedness of $\varphi(t, \theta)$ in any finite t-domain, there exists a constant M such that $|\varphi(t, \theta)| < M$ for all θ and for all t in the domain $|t - t'| \leqq r$. From the Cauchy integral formula for an analytic function it follows that

$$\frac{1}{k!} \left| \frac{\partial^k \varphi(t, \theta_m)}{\partial t^k} \bigg|_{t=t'} \right| \leqq \frac{M}{r^k}. \tag{4.5}$$

From (4.4) and (4.5) we obtain

$$|\varphi(t_m, \theta_m) - \varphi(t', \theta_m)| \leqq M \sum_{k=1}^{\infty} \frac{|t_m - t'|^k}{r^k}. \tag{4.6}$$

Equation (4.3) is an immediate consequence of (4.6). This proves the first half of Theorem 4.1.

Let C be a circle in the complex t-plane with finite radius and center at the origin. According to the Cauchy integral formula we have

$$\frac{1}{2\pi i} \int_C \frac{\varphi(t, \theta)}{t^{k+1}} \, dt = \frac{1}{k!} \frac{\partial^k \varphi(t, \theta)}{\partial t^k} \bigg|_{t=0} = \frac{1}{k!} E_\theta(z^k). \tag{4.7}$$

Since $\varphi(t, \theta)$ is continuous in t and θ jointly, the integral on the left hand side of (4.7) is a continuous function of θ. This proves the second half of Theorem 4.1.

5. Some lemmas. In this section we shall prove several lemmas which will then be used to derive the results contained in sections 6 and 8.

LEMMA 5.1. *It follows from assumptions* 1 *and* 2 *that for any given* θ *the equation in* t

$$\varphi(t, \theta) = 1 \tag{5.1}$$

has exactly two real roots, one of which is zero. The other real root is different from zero if $E_\theta(z) \neq 0$. *If* $E_\theta(z) = 0$, *both roots are equal to zero, i.e., zero is a double root of* (5.1).

This lemma is essentially the same as Lemma 2 in [2] and the proof is therefore omitted.[3]

Let $h(\theta)$ denote the non-zero root of (5.1), if $E_\theta(z) \neq 0$. If $E_\theta(z) = 0$, we put $h(\theta) = 0$.

In what follows the variable t will be restricted to real values, unless the contrary is explicitly stated.

LEMMA 5.2. *It follows from assumptions* 1 *and* 2 *that* $h(\theta)$ *is a continuous function of* θ.

PROOF: It follows from assumption 2 that

$$\lim_{t\to\pm\infty} \varphi(t, \theta) = +\infty \tag{5.2}$$

uniformly in θ. Hence, since by definition

$$\varphi[h(\theta), \theta] = 1$$

identically in θ, $h(\theta)$ must be a bounded function of θ.

Let $\{\theta_m\}$ be a sequence of parameter values which converges to θ^*. From Theorem 4.1 it follows that

$$\lim_{m\to\infty} [\varphi(t, \theta_m) - \varphi(t, \theta^*)] = 0 \tag{5.3}$$

uniformly in t over any finite interval. Since $h(\theta)$ is bounded, we obtain from (5.3)

$$\lim_{m\to\infty} \{\varphi[h(\theta_m), \theta_m] - \varphi[h(\theta_m), \theta^*]\} = 0. \tag{5.4}$$

Since $\varphi[h(\theta_m), \theta_m] = 1$, it follows from (5.4) that

$$\lim_{m\to\infty} \varphi[h(\theta_m), \theta^*] = 1.$$

It follows from assumption 1 that for any limit point h of the bounded sequence $\{h(\theta_m)\}$ ($m = 1, 2, \cdots$, ad inf.) we have

[3] Condition IV of Lemma 2 in [2] is not postulated here, since the validity of this condition is implied by assumption 1 Condition IV could have been omitted also in [2], since it follows from condition III.

$$\varphi(h, \theta^*) = 1 \tag{5.5}$$

If $h(\theta^*) = 0$, then equation $\varphi(t, \theta^*) = 1$ has the only root $t = 0$. Consequently, all limit points of $\{h(\theta_m)\}$ must be equal to zero, that is

$$\lim_{m\to\infty} h(\theta_m) = 0 \quad \text{if} \quad h(\theta^*) = 0. \tag{5.6}$$

Now let us assume that $h(\theta^*) \neq 0$. Since the second derivative of $\varphi(t, \theta)$ with respect to t is positive, it can be seen that $\varphi(t, \theta) < 1$ for values t in the open interval $(0, h(\theta))$, and $\varphi(t, \theta) > 1$ for any t outside the closed interval $[0, h(\theta)]$. Hence, $\varphi(t, \theta) < 1$ implies that $|h(\theta)| > |t|$ and $h(\theta)$ and t have the same sign. Now let t_0, be a value in the open interval $(0, h(\theta^*))$. Then we have

$$\varphi(t_0, \theta^*) < 1 \tag{5.7}$$

It follows from assumption 1 that

$$\varphi(t_0, \theta_m) < 1 \tag{5.8}$$

for sufficiently large m. Hence $h(\theta_m)$ and t_0 have the same sign and

$$|h(\theta_m)| > |t_0| \tag{5.9}$$

Inequality (5.9) implies that zero cannot be a limit point of the sequence $\{h(\theta_m)\}$. Since $\varphi(t, \theta^*) = 1$ has only the roots $t = 0$ and $t = h(\theta^*)$, it follows from (5.5) that the sequence $\{h(\theta_m)\}$ cannot have a limit point different from $h(\theta^*)$. Thus,

$$\lim_{m\to\infty} h(\theta_m) = h(\theta^*) \tag{5.10}$$

and Lemma 5.2 is proved.

LEMMA 5.3. *It follows from assumption 1 that for any given t, $E_\theta(e^{|zt|})$ is a bounded function of θ.*

PROOF: We have

$$E_\theta(e^{|tz|}) \leqq E_\theta(e^{tz} + e^{-tz}) = \varphi(t, \theta) + \varphi(-t, \theta) \tag{5.11}$$

It follows from assumption 1 that $\varphi(t, \theta)$ and $\varphi(-t, \theta)$ are bounded functions of θ. Hence Lemma 5.3 is proved.

LEMMA 5.4. *Let θ' be a value of θ such that $E_{\theta'}(z) = 0$, but $E_\theta(z) \neq 0$ for all $\theta \neq \theta'$ in an open interval containing θ'. It follows from assumptions 1 and 2 that*

$$\lim_{\theta\to\theta'} \left(-\frac{2E_\theta(z)}{h(\theta)}\right) = E_{\theta'}(z^2). \tag{5.12}$$

PROOF: We have

$$e^{h(\theta)z} = 1 + h(\theta)z + \frac{[h(\theta)]^2}{2} z^2 + \frac{[h(\theta)]^3}{6} z^3 e^{uh(\theta)z} \tag{5.13}$$

where $0 \leqq u \leqq 1$. Hence

$$(5.14)\quad E_\theta(e^{h(\theta)z}) = 1 + h(\theta)E_\theta(z) + \frac{[h(\theta)]^2}{2} E_\theta(z^2) + \frac{[h(\theta)]^3}{6} E_\theta(z^3 e^{uh(\theta)z}).$$

Since $E_\theta(e^{h(\theta)z}) = 1$, we obtain from (5.14)

$$(5.15)\qquad h(\theta)E_\theta(z) + \frac{[h(\theta)]^2}{2} E_\theta(z^2) + \frac{[h(\theta)]^3}{6} E_\theta(z^3 e^{uh(\theta)z}) = 0.$$

We shall consider only values θ for which $h(\theta) \neq 0$. For such values of θ, also $E_\theta(z) \neq 0$. Dividing (5.15) by $h(\theta)E_\theta(z)$, we obtain

$$(5.16)\qquad 1 + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta(z^2) + \frac{h(\theta)}{3} E_\theta(z^3 e^{uh(\theta)z})\right] = 0.$$

Let t_0 be an upper bound of $|h(\theta)|$ with respect to θ. Then for a suitably chosen constant C we have

$$(5.17)\qquad |z^3 e^{uh(\theta)z}| < Ce^{|t_0 z|}$$

From this and Lemma 5.3 it follows that $E_\theta(z^3 e^{uh(\theta)z})$ is a bounded function of θ.

Because of the continuity of $h(\theta)$ we have

$$(5.18)\qquad \lim_{\theta \to \theta'} h(\theta) = 0.$$

Lemma 5.4 follows from (5.16), (5.18), the boundedness of $E_\theta(z^3 e^{uh(\theta)z})$ and the fact that $E_\theta(z^2)$ is a continuous function of θ and $E_{\theta'}(z^2) > 0$.

Lemma 5.5. *From assumptions 1 and 2 it follows that for any given t, $E_\theta(e^{|tZ_n|})$ exists and is a bounded function of θ.*

Proof: It is sufficient to show that $E_\theta(e^{tZ_n})$ is a bounded function of θ for any t, since

$$(5.19)\qquad e^{|tZ_n|} \leqq e^{tZ_n} + e^{-tZ_n}$$

Clearly, e^{tZ_n} lies between $e^{bt+z_n t}$ and $e^{at+z_n t}$ Hence Lemma 5.5 is proved if we show that $E_\theta(e^{z_n t})$ is a bounded function of θ.

It follows from Assumption 2 that there exists a positive integer k and a positive constant g such that

$$(5.20)\qquad P(|z_1 + \cdots + z_k| \geqq a - b \mid \theta) \geqq g$$

for all θ. For any positive integer m and for any real values $\lambda_1 < \lambda_2$ we have

$$(5.21)\qquad \frac{P[(m-1)k < n \leqq mk \mid \theta]}{P[(m-1)k < n \mid \theta]} \geqq g \qquad (m = 1, 2, \cdots, \text{ad inf.})$$

and

$$(5.22)\qquad \frac{P[(m-1)k < n \leqq mk \;\&\; \lambda_1 \leqq z_n < \lambda_2 \mid \theta]}{P[(m-1)k < n \mid \theta]} \leqq 1 - [1 - P(\lambda_1 \leqq z < \lambda_2 \mid \theta)]^k.$$

Hence

$$(5.23)\quad \frac{P[(m-1)k < n \leqq mk \,\&\, \lambda_1 \leqq z_n < \lambda_2 \mid \theta]}{P[(m-1)k < n \leqq mk \mid \theta]} \leqq \frac{1-[1-P(\lambda_1 \leqq z < \lambda_2 \mid \theta]^k}{g}.$$

Multiplying (5.23) by $P[(m-1)k < n \leqq mk \mid \theta]$ and summing with respect to m we obtain

$$(5.24)\quad P(\lambda_1 \leqq z_n < \lambda_2 \mid \theta) \leqq \frac{1-[1-P(\lambda_1 \leqq z < \lambda_2 \mid \theta)]^k}{g}.$$

From (5.24) it follows readily that

$$(5.25)\quad \frac{P(\lambda_1 \leqq z_n < \lambda_2 \mid \theta)}{P(\lambda_1 \leqq z < \lambda_2 \mid \theta)}$$

is a bounded function of λ_1, λ_2 and θ. Let A be an upper bound of the ratio (5.25). Then

$$(5.26)\quad E_\theta(e^{tz_n}) \leqq AE_\theta(e^{tz}) = A\varphi(t, \theta).$$

Because of Assumption 1, $\varphi(t, \theta)$ is a bounded function of θ. Hence also $E_\theta(e^{tz_n})$ is bounded and Lemma 5.5 is proved.

6. The limiting value of $E_\theta(n)$ when θ approaches a value θ' for which $E_{\theta'}(z) = 0$. In this section we shall prove the following theorem:

THEOREM 6.1. *Let θ' be a value of θ such that $E_{\theta'}(z) = 0$, but $E_\theta(z) \neq 0$ for all $\theta \neq \theta'$ in an open interval containing θ'. If assumptions 1 and 2 hold, we have*

$$(6.1)\quad \lim_{\theta \to \theta'} \left[E_\theta(n) - \frac{E_\theta(Zn^2)}{E_{\theta'}(z^2)} \right] = 0.$$

PROOF: Consider the Taylor expansion

$$(6.2)\quad e^{h(\theta)Z_n} = 1 + h(\theta)Z_n + \frac{[h(\theta)]^2}{2} Z_n^2 + \frac{[h(\theta)]^3}{6} Z_n^3 e^{\lambda h(\theta) Z_n}$$

where $0 \leqq \lambda \leqq 1$. It was shown in [2] (p. 286) that

$$(6.3)\quad E_\theta e^{h(\theta) Z_n} = 1.$$

Hence, taking expected values on both sides of (6.2), we obtain

$$(6.4)\quad h(\theta)E_\theta(Z_n) + \frac{[h(\theta)]^2}{2} E_\theta(Z_n^2) + \frac{[h(\theta)]^3}{6} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n}) = 0.$$

We consider only values of θ for which $E_\theta(z) \neq 0$. For such values, also $h(\theta) \neq 0$. Thus, we can divide both sides of (6.4) by $h(\theta)E_\theta(z)$. We then obtain

$$\frac{E_\theta(Z_n)}{E_\theta(z)} + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta Z_n^2 + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right] = 0. \tag{6.5}$$

It was shown in [1] (p. 142) that

$$E_\theta(n) = \frac{E_\theta(Z_n)}{E_\theta(z)}. \tag{6.6}$$

Hence

$$E_\theta(n) + \frac{h(\theta)}{2E_\theta(z)}\left[E_\theta(Z_n^2) + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right] = 0. \tag{6.7}$$

Let t_0 be an upper bound of $|h(\theta)|$. Then for a properly chosen constant C we have

$$|Z_n^3 e^{\lambda h(\theta) Z_n}| \leqq Ce^{|t_0 Z_n|} \tag{6.8}$$

From this and Lemma 5.5 it follows that $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ is a bounded function of θ. Since $\lim_{\theta\to\theta'} h(\theta) = 0$ and $E_\theta(Z_n^2)$ has a positive lower bound, Theorem 6.1 follows from 6.7, Lemma 5.4 and Theorem 4.1.

If $\lim_{\theta\to\theta'} E_\theta Z_n^2 = E_{\theta'} Z_n^2$, Theorem 6.1 gives[4]

$$E_{\theta'}(n) = \frac{E_{\theta'}(Z_n^2)}{E_{\theta'}(z^2)}. \tag{6.9}$$

Limits for $E_{\theta'}(n)$ can be obtained by computing limits for $E_\theta(Z_n^2)$. In the next section we shall give a general method for obtaining limits for $E_\theta[\psi(Z_n)]$, where $\psi(Z_n)$ is any function of Z_n.

7. Determination of lower and upper limits for the expected value of any function of Z_n. Let $\psi(Z_n)$ be a function of Z_n. Limits for $E_\theta[\psi(Z_n)]$ may be determined as follows: First we determine limits for $E_\theta[\psi(Z_n) \mid Z_n \geqq a]$. Let r be a positive variable. Clearly, for any given value r we have

$$E_\theta(\psi Z_n) \mid Z_{n-1} = a - r \text{ and } Z_n \geqq a] = E_\theta[\psi(a - r + z) \mid z \geqq r] \tag{7.1}$$

From (7.1) we obtain the limits

$$\begin{aligned} \underset{0<r<a-b}{\text{g.l.b.}}\ E_\theta[\psi(a - r + z) \mid z \geqq r] &\leqq E_\theta[\psi(Z_n) \mid Z_n \geqq a] \\ &\leqq \underset{0<r<a-b}{\text{l.u.b.}}\ E_\theta[\psi(a - r + z) \mid z \geqq r]. \end{aligned} \tag{7.2}$$

Limits for $E_\theta[\psi(Z_n) \mid Z_n \leqq b]$ can be obtained in a similar way. Again, let r be a positive variable. For any value of r we have

$$E_\theta[\psi(Z_n) \mid Z_n \leqq b \text{ and } Z_{n-1} = b + r] = E_\theta[\psi(b + r + z) \mid z \leqq -r \tag{7.3}$$

Hence we obtain the limits

[4] The validity of (6 9) was shown by the author [3] using an entirely different method.

$$
\begin{aligned}
\text{(7.4)} \quad & \underset{0<r<a-b}{\text{g.l.b.}}\ E_\theta[\psi(b+r+z) \mid z \leqq -r] \leqq E_\theta[\psi(Z_n) \mid Z_n \leqq b] \\
& \qquad \leqq \underset{0<r<a-b}{\text{l.u.b.}}\ E_\theta[\psi(b+r+z) \mid z \leqq -r].
\end{aligned}
$$

Since

$$
\text{(7.5)} \quad E_\theta[\psi(Z_n)] = P(Z_n \geqq a)E_\theta[\psi(Z_n) \mid Z_n \geqq a] + P(Z_n \leqq b)E_\theta[\psi(Z_n) \mid Z_n \leqq b],
$$

a lower (upper) limit for $E_\theta[\psi(Z_n)]$ can be obtained, by replacing the conditional expected values on the right hand side of (7 5) by their lower (upper) limits given in (7.2) and (7.4).

8. Limits for $E_\theta(n)$ when $h(\theta)$ is near but unequal to zero. Let θ' be a value of θ for which $h(\theta') = 0$. In this section we shall derive limits for $E_\theta(n)$ which will generally be close to each other for values θ in a small neighborhood of θ'.

From equation (6.7) we obtain

$$
\text{(8.1)} \qquad E_\theta(n) = -\frac{h(\theta)}{2E_\theta(z)}\left[E_\theta Z_n^2 + \frac{h(\theta)}{3} E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})\right]
$$

where $0 \leqq \lambda \leqq 1$ Thus, limits for $E_\theta(n)$ can be obtained by deriving limits for $E_\theta Z_n^2$ and $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$. Limits for $E_\theta Z_n^2$ can be obtained by using the method described in section 7.

If θ is near θ', any crude limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ will serve the purpose, since, as has been shown in section 6, $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ is bounded and $\lim_{\theta \to \theta'} h(\theta) = 0$.

Limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$ can be obtained as follows: For simplicity, let us assume that $h(\theta) > 0$. Then

$$
\text{(8.2)} \qquad Z_n^3 \leqq Z_n^3 e^{\lambda h(\theta) Z_n} \leqq Z_n^3 e^{h(\theta) Z_n} \qquad (h(\theta) > 0)
$$

Thus, to determine limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$, it is sufficient to determine a lower limit for $E_\theta(Z_n^3)$ and an upper limit for $E_\theta(Z_n^3 e^{h(\theta) Z_n})$. The latter limits may be derived by using the method given in section 7.

If $h(\theta) < 0$, we have

$$
\text{(8 3)} \qquad Z_n^3 \geqq Z_n^3 e^{\lambda h(\theta) Z_n} \geqq Z_n^3 e^{h(\theta) Z_n}
$$

and a similar procedure will yield the desired limits for $E_\theta(Z_n^3 e^{\lambda h(\theta) Z_n})$.

It should be emphasized that the limits of $E_\theta(n)$, as given in this section, can be expected to be close only if $h(\theta)$ is near zero. For values of θ for which $h(\theta)$ is not near zero, the limits of $E_\theta(n)$ given in [1] can be used.

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THE EFFICIENCY OF THE MEAN MOVING RANGE

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Summary. In studying the variation of a variable subject to erratic trend effects, it is customary to employ as a measure of variation a statistic that eliminates most of such effects It is shown in this paper that the statistic $w = \sum_1^{n-1} | x_{i+1} - x_i | \sqrt{\pi}/2(n - 1)$ is nearly as efficient as the statistic $\delta^2 = \sum_1^{n-1} (x_{i+1} - x_i)^2/(n - 1)$ that is customarily employed. The asymptotic variance of w is obtained by integration techniques, the proof of the asymptotic normality of w is based upon a theorem of S. Bernstein on the asymptotic distribution of sums of dependent variables. The method of proof is sufficiently general to prove the asymptotic normality of w, and of δ^2, for x having a distribution for which the third absolute moment exists.

1. Introduction. Let $x_1, x_2, \cdots, x_n$ denote a random sample of size n from a population with a continuous distribution function $f(x)$. If a measure of the variability of x is desired, it is customary to select the familiar statistic

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1}, \tag{1}$$

or its positive square root s, as an estimate of the corresponding theoretical measure of variability.

If, however, it is known that the variable x is subject to trend effects and that $f(x)$ represents the distribution of x without such effects, then s^2 will not serve as a satisfactory measure of variability about the trend. In order to eliminate the influence of trends, it is helpful to employ statistics that capitalize on the time order relationships of the observations. There are several statistics of this type available, although most of them make no pretense of completely eliminating trend effects, even if the trend is linear.

Perhaps the best known among statistics of the desired type is the mean square successive difference,

$$\delta^2 = \frac{\sum_{i=1}^{n-1} (x_{i+1} - x_i)^2}{n - 1} \tag{2}$$

This measure of variation has been studied extensively in recent years. Among the results of these investigations is a determination [1] of the efficiency of $\delta^2/2$ as an estimate of σ^2 for a normally distributed variable when no trend exists.

A closely related measure of variation that is not so well known is the mean moving range of successive pairs of observations,

$$\text{(3)} \qquad \mathrm{w} = \frac{\sum_{i=1}^{n-1} |x_{i+1} - x_i|}{n-1}.$$

Although w appears [1] to have been used by ballisticians, very little seems to be known concerning the relative merits of δ^2 and w. Since w is considerably easier to calculate than δ^2, it would be preferred to δ^2 for applications in which computational advantages are important. However, one would hardly allow such advantages to dominate a choice unless δ^2 and w were about equally efficient as estimates of variation.

The purpose of this paper is to determine the efficiency of w and to study efficiency properties of generalizations of w.

2. Definition of efficiency. The definition that will be used in this paper [2] may be stated in the following manner. Let θ be a parameter, or a function of parameters, of the distribution function $f(x)$. Let T be a statistic for which there exists a number μ such that

$$t = \sqrt{n}\,(T - \theta)$$

is asymptotically normally distributed with zero mean and variance μ^2. Let T' be any other statistic for which there exists a number μ' such that

$$t' = \sqrt{n}\,(T' - \theta)$$

is asymptotically normally distributed with zero mean and variance μ'^2. Then T is said to be an efficient estimate of θ provided that $\mu \leq \mu'$ for all possible choices of T', and the efficiency of any particular T' is defined to be

$$\text{(4)} \qquad \mathcal{E}_{T'} = \left(\frac{\mu}{\mu'}\right)^2.$$

In order to determine the efficiency of a statistic, it is therefore necessary to first demonstrate its asymptotic normal distribution and then calculate its asymptotic variance. This order of procedure will be reversed in the following determination of the efficiency of w.

3. Variance of w. Let x be normally distributed with zero mean and unit variance. Then the mean of w, where w is given by (3), may be evaluated as follows:

$$\begin{aligned} E(\mathrm{w}) &= E\,|x_2 - x_1| \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int |x_2 - x_1|\, e^{-\frac{1}{2}(x_1^2+x_2^2)}\, dx_1\, dx_2 \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\left[\int_{-\infty}^{x_2} (x_2 - x_1)e^{-(x_1^2/2)}\, dx_1 + \int_{x_2}^{\infty} (x_1 - x_2)e^{-(x_1^2/2)}\, dx_1\right] dx_2 \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\left[x_2\int_{-\infty}^{x_2} e^{-(x_1^2/2)}\,dx_1 - x_2\int_{x_2}^{\infty} e^{-(x_1^2/2)}\,dx_1 + 2e^{-(x_2^2/2)}\right]dx_2$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\cdot 2\left[x_2\int_{0}^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-(x_2^2/2)}\right]dx_2\,.$$

If integration by parts is performed on the first integral with

$$u = \int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 \qquad \text{and} \qquad dv = x_2\,e^{-(x_2^2/2)}\,dx_2\,,$$

the uv term will vanish at both limits and $E(w)$ will reduce to

$$E(\mathrm{w}) = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{-x_2^2}\,dx_2 = \frac{2}{\sqrt{\pi}}\,. \tag{5}$$

This result could have been obtained more easily by other methods, but some of the integrals involved will be needed later.

For the purpose of computing the second moment of w, it is convenient to separate the independent and dependent product terms of w^2. Since there are $2(n-2)$ of the latter, $E(\mathrm{w}^2)$ may be expressed in the form

$$(n-1)^2E(\mathrm{w}^2) = (n-1)E\,|x_2-x_1|^2 + 2(n-2)E\,|x_2-x_1|\,|x_3-x_2| \\ + (n-2)(n-3)E^2\,|x_2-x_1|.$$

But

$$E\,|x_2-x_1|^2 = E(x_2-x_1)^2 = E(x_2^2) + E(x_1^2) = 2.$$

Consequently, because of (5),

$$(n-1)^2E(\mathrm{w}^2) = 2(n-2)E\,|x_2-x_1|\,|x_3-x_2| + 2(n-1) \\ + 4(n-2)(n-3)/\pi. \tag{6}$$

Now consider the evaluation of the product term

$$E\,|x_2-x_1|\,|x_3-x_2| = (2\pi)^{-\frac{3}{2}}\int\int_{-\infty}^{\infty}\int |x_2-x_1|\,|x_3-x_2|\,e^{-\frac{1}{2}(x_1^2+x_2^2+x_3^2)}\,dx_1\,dx_2\,dx_3\,.$$

By means of the expressions that were used to give (5), this triple integral may be reduced in the following manner:

$$E\,|x_2-x_1|\,|x_3-x_2| = (2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty}\int |x_3-x_2|\,e^{-\frac{1}{2}(x_2^2+x_3^2)} \\ \cdot 2\left[x_2\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-(x_2^2/2)}\right]dx_3\,dx_2$$

$$= (2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\cdot 4\left[x_2\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-(x_2^2/2)}\right]^2 dx_2$$

$$= 4(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty} e^{-(x_2^2/2)}\left[x_2^2\left(\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1\right)^2 \right. \\ \left. + 2x_2e^{-(x_2^2/2)}\int_0^{x_2} e^{-(x_1^2/2)}\,dx_1 + e^{-x_2^2}\right]dx_2\,.$$

These three integrals, without their constant factors, will be denoted by I_1, I_2, and I_3, respectively. I_1 may be evaluated by integrating by parts with

$$u = x_2 \left(\int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 \right)^2 \quad \text{and} \quad dv = x_2 e^{-(x_2^2/2)}\, dx_2 .$$

The uv term will vanish at both limits, consequently

$$(7) \quad \begin{aligned} I_1 &= \int_{-\infty}^{\infty} e^{-(x_2^2/2)} \left[2x_2 e^{-(x_2^2/2)} \int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 + \left(\int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 \right)^2 \right] dx_2 \\ &= 2 \int_{-\infty}^{\infty} x_2 e^{-x_2^2} \int_0^{x_2} e^{-(x_1^2/2)}\, dx_1\, dx_2 + \int_{-\infty}^{\infty} e^{-(x_2^2/2)} \left(\int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 \right)^2 dx_2 . \end{aligned}$$

The first of these two integrals may be evaluated in the same manner as the first integral preceding (5). The second integral may be evaluated by making the change of variable

$$u = \int_0^{x_2} e^{-(x_1^2/2)}\, dx_1 .$$

As a result of such manipulations,

$$I_1 = \frac{\sqrt{6\pi}}{3} + \frac{\pi\sqrt{2\pi}}{6} .$$

It will be observed that I_2 is the same as the first integral of (7) and that I_3 is available in tables; hence

$$(8) \quad \begin{aligned} &E\,|\,x_2 - x_1\,||\,x_3 - x_2\,| \\ &\quad = 4(2\pi)^{-\frac{3}{2}} \left[\frac{\sqrt{6\pi}}{3} + \frac{\pi\sqrt{2\pi}}{6} + \frac{\sqrt{6\pi}}{3} + \frac{\sqrt{6\pi}}{3} \right] = \tfrac{1}{3} + \frac{2\sqrt{3}}{\pi} . \end{aligned}$$

If (8) is substituted in (6), $E(\mathrm{w}^2)$ will reduce to

$$(9) \quad E(\mathrm{w}^2) = \frac{2(n-2)}{(n-1)^2} \left[\tfrac{1}{3} + \frac{2\sqrt{3}}{\pi} \right] + \frac{2}{n-1} + \frac{4(n-2)(n-3)}{\pi(n-1)^2} .$$

Since $\sigma_{\mathrm{w}}^2 = E(\mathrm{w}^2) - E^2(\mathrm{w})$, (9) and (5) will yield the following desired variance of w,

$$(10) \quad \sigma_{\mathrm{w}}^2 = \frac{2}{(n-1)^2} \left[\left(\tfrac{4}{3} + \frac{2\sqrt{3} - 6}{\pi} \right) n + \left(\frac{10 - 4\sqrt{3}}{\pi} - \tfrac{8}{3} \right) \right] .$$

4. Efficiency of w. Now let x be normally distributed with mean m and variance σ^2. Then the mean of w as given by (5) will be multiplied by σ and the variance of w as given by (10) will be multiplied by σ^2; consequently $z = \mathrm{w}\sqrt{\pi}/2$ will serve as an unbiased estimate of σ In the next section it will be shown that

$$t' = \sqrt{n}(z - \sigma)$$

possesses an asymptotic normal distribution From (10) and section 2, it therefore follows that the asymptotic variance, μ'^2, that is needed to determine the efficiency of z is given by

$$\mu'^2 = \frac{\pi}{4}\, 2\left(\tfrac{4}{3} + \frac{2\sqrt{3} - 6}{\pi}\right) = \frac{2\pi}{3} + \sqrt{3} - 3.$$

Now it is known that for x normally distributed s, as defined by (1), is an efficient estimate of σ with $\mu^2 = \frac{1}{2}$, consequently, because of (4), the efficiency of z as an estimate of σ is given by

$$\mathcal{E}_z = \frac{1}{2\left(\frac{2\pi}{3} + \sqrt{3} - 3\right)} \doteq .605. \tag{11}$$

In [1] it was shown that for x normally distributed $\delta^2/2$ was an unbiased estimate of σ^2 and, assuming the normality of its asymptotic distribution, that the efficiency of $\delta^2/2$ as an estimate of σ^2 was 2/3. Thus, $z = \mathrm{w}\sqrt{\pi}/2$ possesses very nearly the same efficiency as a measure of variation of a normal variable as $\delta^2/2$ does.

5. Asymptotic distribution of mean moving ranges. Although the efficiency obtained in the preceding section requires for its validity merely a demonstration that for x normally distributed w possesses an asymptotic normal distribution, it will be shown in this section that general mean moving ranges of a continuous variable x possess asymptotic normal distributions provided only that x possesses a third absolute moment.

Let r_i denote the range of the observations from x_i to x_{i+k-1}. Then the variable

$$W = \frac{r_1 + r_2 + \cdots + r_{n-k+1}}{n - k + 1} \tag{12}$$

will represent a generalized mean moving range, of which w will be a special case when $k = 2$.

A proof of the asymptotic property of W can be constructed as an application of a general theorem of S. Bernstein [3]. Since his theorem is long and involves much explanation of notation, a simplified version of it that is sufficient to cover this application, and indeed many similar applications, will be given.

Let $y_1, y_2, \cdots, y_m$ denote m variables for which the third absolute moments are bounded and let

$$S_m = y_1 + y_2 + \cdots + y_m .$$

Then Bernstein's theorem implies that if there exist constants c_1, c_2, c_3, and c_4 such that

$$c_1 m < \sigma_{S_m}^2 < c_2 m, \tag{a}$$

and

(b) y_i and y_{i+g} are independently distributed for

$$g > c_3 m^{c_4}, \; c_4 < \tfrac{1}{2},$$

then

$$\frac{S_m - E(S_m)}{\sigma_{S_m}}$$

possesses an asymptotic normal distribution with zero mean and unit variance.

Consider the application of this theorem to $R = (n - k + 1)W$. The variance of R may be expressed in compact form by means of the techniques of section 3. Since r_i is the range of k consecutive observations, it is clear that

$$E(r_i r_{i+g}) = E^2(r_1)$$

if $g \geq k$. Furthermore, for subscripts for which it is defined, $E(r_i r_{i+g})$ will be independent of i. These two properties may be used to collect terms in the expansion of $E(R^2)$ to give

$$E(R^2) = (n - k + 1)E(r_1^2) + 2\sum_{i=0}^{k-2}(n - k - i)E(r_1 r_{2+i}) + (n - 2k + 1)(n - 2k + 2)E^2(r_1).$$

Consequently,

$$\sigma_R^2 = (n - k + 1)E(r_1^2) + 2\sum_{i=0}^{k-2}(n - k - i)E(r_1 r_{2+i}) + [n(1 - 2k) + (k - 1)(3k - 1)]E^2(r_1). \tag{13}$$

From the definition of the correlation coefficient and the fact that a correlation coefficient cannot exceed one, it follows that

$$\begin{aligned} E(r_1 r_{2+i}) &\leq E(r_1)E(r_{2+i}) + \sigma_{r_1}\sigma_{r_{2+i}} \\ &\leq E^2(r_1) + \sigma_{r_1}^2 . \end{aligned}$$

If this inequality is applied to (13),

$$\begin{aligned} \sigma_R^2 &\leq (n - k + 1)E(r_1^2) + (k - 1)(2n - 3k + 2)[E^2(r_1) + \sigma_{r_1}^2] + [n(1 - 2k) + (k - 1)(3k - 1)]E^2(r_1) \\ &\leq (n - k + 1)[E(r_1^2) - E^2(r_1)] + (k - 1)(2n - 3k + 2)\sigma_{r_1}^2 \\ &\leq [n(2k - 1) - (k - 1)(3k - 1)]\sigma_{r_1}^2 \\ &\leq 2k\sigma_{r_1}^2(n - k + 1). \end{aligned}$$

Thus, for a fixed k the right inequality in (a) of Bernstein's modified theorem is satisfied.

For the purpose of demonstrating that the left inequality in (a) is also satisfied, consider the following application of Schwarz's inequality. Let

$$G(x_p, \cdots, x_k) = \int \cdots \int r_p f(x_{k+1}) \cdots f(x_{k+p-1})\, dx_{k+1} \cdots dx_{k+p-1}, \tag{14}$$

where $f(x)$ denotes the distribution function of the variable x and the range of integration in this and subsequent integrals is from $-\infty$ to ∞. Since r_p and f are continuous non-negative functions, this integral is a positive function of the indicated variables. Then, denoting $G(x_p, \cdots, x_k)$ by G, it follows from Schwarz's inequality that

$$\begin{aligned} I &= \left[\int \cdots \int r_1 f(x_1) \cdots f(x_k)\, dx_1 \cdots dx_k\right]^2 \\ &= \left[\int \cdots \int \{r_1 f(x_1) \cdots f(x_k) G\}^{\frac{1}{2}} \{r_1 f(x_1) \cdots f(x_k) G^{-1}\}^{\frac{1}{2}}\, dx_1 \cdots dx_k\right]^2 \\ &\leq \int \cdots \int r_1 f(x_1) \cdots f(x_k) G\, dx_1 \cdots dx_k \int \cdots \int r_1 f(x_1) \cdots f(x_k) G^{-1} \\ &\qquad dx_1 \cdots dx_k. \end{aligned} \tag{15}$$

The two integrals of this inequality will be denoted by I_α and I_β, respectively. If the value of G given by (14) is substituted in I_α, it will be observed that

$$I_\alpha = \int \cdots \int r_1 r_p f(x_1) \cdots f(x_{k+p-1})\, dx_1 \cdots dx_{k+p-1}. \tag{16}$$

Now I_β may be written in the form

$$I_\beta = \int \cdots \int f(x_p) \cdots f(x_k) G^{-1}\left[\int \cdots \int r_1 f(x_1) \cdots f(x_{p-1})\, dx_1 \cdots dx_{p-1}\right] dx_p \cdots dx_k.$$

Since the x_i possess the same distribution function and r_1 is the range of the variables from x_1 to x_k, the integral in brackets is equivalent to the integral defining G in (14); hence

$$I_\beta = \int \cdots \int f(x_p) \cdots f(x_k) G^{-1} G\, dx_p \cdots dx_k = 1. \tag{17}$$

If (16) and (17) are applied to inequality (15), they will yield the inequality

$$\left[\int \cdots \int r_1 f(x_1) \cdots f(x_k)\, dx_1 \cdots dx_k\right]^2 \leq \int \cdots \int r_1 r_p f(x_1) \cdots f(x_{k+p-1})\, dx_1 \cdots dx_{k+p-1}.$$

In statistical language, this inequality states that

$$E^2(r_1) \leq E(r_1 r_p),$$

or, what is equivalent, that

$$E^2(r_1) \leq E(r_i r_j). \tag{18}$$

If (18) is applied to (13),

$$\sigma_R^2 \geq (n - k + 1)E(r_1^2) + (k - 1)(2n - 3k + 2)E^2(r_1) + [n(1 - 2k) + (k - 1)(3k - 1)]E^2(r_1)$$

$$\geq (n - k + 1)[E(r_1^2) - E^2(r_1)]$$

$$\geq \sigma_{r_1}^2(n - k + 1).$$

Thus, for a fixed k the left inequality in (a) of the theorem is also satisfied, and it merely remains to be shown that condition (b) is satisfied.

For k fixed, r_i and r_{i+g} will be independently distributed provided that $g \geq k$. But if $c_3 > k$, then $c_3(n - k + 1)^{c_4} > k$ for $0 < c_4 < \frac{1}{2}$ because $n - k + 1 > 1$; consequently r_i and r_{i+g} will be independently distributed for $g > c_3(n - k + 1)^{c_4}$, where $0 < c_4 < \frac{1}{2}$. Thus, conditions (a) and (b) are both satisfied by R. Since $R = (n - k + 1)W$, it therefore follows that

$$\frac{W - E(W)}{\sigma_W} \tag{19}$$

possesses an asymptotic normal distribution with zero mean and unit variance provided only that x possesses a continuous distribution function for which the third absolute moment exists. The existence of the third absolute moment for x insures the existence of the same moment for r_i.

If $k = 2$, W reduces to w, and therefore the validity of (11) is assured.

6. Other asymptotic distributions. The only property of the range employed in the proof of the preceding section was its positive nature; consequently the proof is applicable to moving means of other dependent statistics that are positive and possess third absolute moments.

For example, the preceding proof can be applied to δ^2 to show that δ^2 possesses an asymptotic normal distribution provided only that the sixth moment of x exists. In the study [1] of the efficiency of δ^2 for x normally distributed, no proof was given of its asymptotic property. The preceding proof could be used in studying the efficiency of δ^2, or obvious generalizations of it, as measure of variation for non-normal populations. The normality of the asymptotic distribution of the serial correlation coefficient could also be verified by means of this proof.

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CONFIDENCE LIMITS FOR THE FRACTION OF A NORMAL POPULATION WHICH LIES BETWEEN TWO GIVEN LIMITS[1]

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Summary. Let μ and σ^2 be the unknown mean and variance, respectively, of a normally distributed population on which N independent observations $x_1, \cdots, x_N$ have been made. Let L_1 and L_2, $L_1 < L_2$, and α, $0 < \alpha < 1$, be given constants. We define the following symbols:

(a) $$\gamma = (\sqrt{2\pi}\,\sigma)^{-1} \int_{L_1}^{L_2} \exp\left\{-\frac{1}{2}\frac{(y-\mu)^2}{\sigma^2}\right\} dy$$

(b) $$\bar{x} = N^{-1}\Sigma x_i$$

(c) $$s^2 = (N-1)^{-1}\Sigma(x_i - \bar{x})^2$$

(d) $\chi^2_{1-\alpha}$ as that number for which $P\{\chi^2 < \chi^2_{1-\alpha}\} = 1 - \alpha$ where χ^2 has $N - 1$ degrees of freedom.

(e) $$w = \sqrt{N-1}\,\frac{s}{\chi_{1-\alpha}}$$

(f) $$D = (2\pi)^{-1/2} \int_{(L_1-\bar{x})/w}^{(L_2-\bar{x})/w} \exp\left\{-\frac{1}{2}y^2\right\} dy$$

It is proved that, under restrictions stated precisely below, and before the observations are made, the probability that $D \leq \gamma$ differs from α by a number which can be made arbitrarily small by making N sufficiently large. Thus an approximate (large sample) lower confidence limit for γ is obtained. Similar methods can be applied to obtain upper and two-sided confidence limits.

A problem raised by the present paper (but not attacked here) is to investigate the rapidity of approach to α of $P\{D \leq \gamma\}$. It would perhaps be useful to obtain a series for the latter in powers of $N^{-\frac{1}{2}}$, the first term of such an expansion is obtained here

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[1] Formula (5 1) of the present paper was given without proof by the author in July, 1945, in solution of a problem put to him by Dr M A Girshick At the time, both were members of the Statistical Research Group, formed in the Division of War Research of Columbia University under contract with the National Defense Research Committee of the Office of Scientific Research and Development The validation of formula (5.1) in all rigor as it is given in the present paper was constructed by the author after he was no longer a member of the Statistical Research Group

In January, 1945, Professor A Wald, then a consultant to the Statistical Research Group, and the present author jointly submitted to the Group an unpublished memorandum (#410) entitled "Acceptance Regions Which Involve the Normal Distribution and Large Sample Sizes" While this memorandum dealt with a different problem, its ideas were logically antecedent to formula (5 1) The present author wishes to express his indebtedness to this memorandum and to his colleague Professor Wald.

1. The problem. Let μ and σ^2 be the unknown mean and variance, respectively, of a normally distributed population on which the N independent observations $x_1, x_2, \cdots, x_N$ have been made. Let L_1 and L_2 be given constants with $L_1 < L_2$. We then have that

$$\gamma = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{L_1}^{L_2} \exp\left\{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right\} dy$$

is the fraction of the normal population which lies between L_1 and L_2. The problem considered in this paper is to construct a lower confidence limit for the unknown γ, when N is large. An upper confidence limit or two-sided confidence limits may be constructed in a manner very similar to that described in the present paper. Since the construction of a lower limit is the problem which occurs most often in practice the discussion will be centered on it.

A lower (confidence) limit on γ with confidence coefficient α is a function $D(x_1, \cdots, x_N)$ of the observations $x_1, \cdots, x_N$ with the property that, *before* the observations are made, the probability is α that $D(x_1, \cdots, x_N) \leq \gamma$. In any specific application it is unknown whether this last inequality holds, because γ is unknown. However, one who proceeds as if this inequality were true is using a procedure which will give correct results $100\alpha\%$ of the time in the long run.

When either $L_1 = -\infty$ or $L_2 = +\infty$ the solution, by use of the non-central t distribution, is well known. For a description of the procedure and necessary tables the reader is referred to [1].

2. Acceptance regions. Let γ_0 be any value of the parameter γ. To γ_0 there correspond infinitely many couples (μ, σ) with the property that the normal distributions characterized by these couples all have a fraction γ_0 lying between L_1 and L_2; we may write this symbolically by saying that the couples (μ, σ) satisfy

$$\gamma(\mu, \sigma) = \gamma_0. \tag{2.1}$$

The construction of confidence regions is equivalent to the construction, for every γ_0, of an acceptance region $R(\gamma_0)$ in the N-dimensional Euclidean space, with the property that every normal distribution whose parameters μ and σ satisfy (2.1) assigns to $R(\gamma_0)$ the constant probability α. While this property of similarity (cf. [2]) is sufficient for the construction of confidence regions, additional properties of the acceptance regions $R(\gamma_0)$ are needed in order that the confidence region be an interval or that the upper confidence limit be always one (i.e., that the confidence limits turn out to be a lower limit only), or to insure other features deemed desirable.

It is easy to construct acceptance regions which will fulfill the condition of similarity. As an example, consider the case $N = 3$ for convenience. Let b_1, b_2, b_3 be a number triple such that $b_1 + b_2 + b_3 = 0$. Let $R(\gamma_0)$, for any given γ_0, $0 < \gamma_0 < 1$, consist of all the points x_1, x_2, x_3 which are such that the absolute value of the angle $\psi(-\pi \leq \psi \leq \pi)$ between the vector (b_1, b_2, b_3) and the vector

$(x_1 - \bar{x}, x_2 - \bar{x}, x_3 - \bar{x})$ does not lie between $\pi\alpha\gamma_0$ and $\pi + \pi\alpha(\gamma_0 - 1)$. (We define, in general, $\sum_1^N x_i = N\bar{x}$. The points (x_1, x_2, x_3) for which $x_1 = x_2 = x_3$ may be disregarded, since their probability is zero when the distribution is continuous.) One readily verifies that the probability of $R(\gamma_0)$ for any γ_0 is α, no matter what μ and σ are, and hence this is true in particular for the pairs which satisfy (2.1).

The above method of constructing acceptance regions yields confidence regions which, while they cover the unknown γ with confidence coefficient α, are not very meaningful otherwise. The fact that the probability of $R(\gamma_0)$ is α whether or not (μ, σ) satisfies (2.1) is already indicative of their lack of discrimination. Since $\bar{x}$ and s (where s is defined by

$$ns^2 = \sum_1^N (x_i - \bar{x})^2$$

and $n = N - 1$) are sufficient estimates of μ and σ, which in turn determine γ, it is clear that desirable confidence regions should be functions only of $\bar{x}$ and s. Consequently our first task must be to construct the acceptance regions $R(\gamma_0)$ in the $\bar{x}$, s plane. In the present paper we construct in the $\bar{x}$, s plane regions $R(\gamma_0)$ which have the property that their probability, under any normal distribution whose parameters satisfy (2.1), differs from the prescribed α by a quantity which is bounded in absolute value for all γ_0, in such a way that the bound approaches zero as N increases. Thus when the sample number is sizeable we can obtain confidence regions for γ which correspond to a confidence coefficient which differs little from α. Finally, the acceptance regions $R(\gamma_0)$ which we shall construct will be such that the confidence region will be always an interval, and the upper limit will always be 1, i.e., we will construct a lower confidence limit for γ.

3. Construction of regions $R(\gamma_0)$ in the $\bar{x}$, s plane. First we describe two assumptions which we shall make. It is believed that these are reasonable from the practical standpoint and are satisfied in most actual investigations where the present problem arises. Mathematically their purpose is to enable us to secure a *uniform* bound on the difference between α and the probability of $R(\gamma_0)$ (for *all* γ_0) under all couples (μ, σ) which satisfy (2.1).

ASSUMPTION 1: *There exists a positive d such that*

$$L_1 + d < \mu < L_2 - d.$$

In most practical cases where the present problem will occur γ will be larger than $\frac{1}{2}$. If the latter is the case and μ were very near either L_1 or L_2, then σ would have to be very small. In that case other methods would have to be used in the solution of the practical problem. The present paper deals with the situation, unfortunately only too common in practice, where σ is not too small. Assumption 1 puts a lower bound on σ for any given value γ_0. (The bound is a function of γ_0).

ASSUMPTION 2: *The standard deviation σ is less than a positive number C.*

In most practical problems such an upper bound can reasonably be set. Naturally, the larger d and the smaller C the more a priori information is at our disposal, the closer are our approximations and the narrower our limits. The effect of Assumptions 1 and 2 is to place a lower limit G on γ where

$$G = \gamma(L_1 + d, C) = \gamma(L_2 - d, C).$$

Let γ_0 be any positive number such that $G < \gamma_0 < 1$. For an $\bar{x}$ such that $L_1 < \bar{x} < L_2$, let $r(\bar{x}, \gamma_0)$ be the positive number such that

$$\gamma(\bar{x}, r(\bar{x}, \gamma_0)) = \gamma_0 .$$

We define $\chi^2_{1-\alpha}$ to be that number for which

$$P(\chi^2 < \chi^2_{1-\alpha}) = 1 - \alpha,$$

where χ^2 has n degrees of freedom and P is the probability of the relation in parentheses. The number $\chi^2_{1-\alpha}$ may be found in tables of the χ^2-distribution if the value of α is one of those in common use. Finally define

$$\varphi(\bar{x}, \gamma_0) = r(\bar{x}, \gamma_0) \sqrt{\frac{\chi^2_{1-\alpha}}{n}}$$

The acceptance regions $R(\gamma_0)$, $G < \gamma_0 < 1$, which we shall employ, are defined as follows for any γ_0, $G < \gamma_0 < 1$:

$$L_1 \leq \bar{x} \leq L_2$$

$$s \geq \varphi(\bar{x}, \gamma_0).$$

4. Proof that $P\{R(\gamma_0)\} \sim \alpha$. This section will be devoted to a proof of the following:

THEOREM. *Let $R(\gamma_0)$ be as defined in Section 3 for $G < \gamma_0 < 1$. Let the assumptions 1 and 2 of Section 3 be fulfilled. Then the absolute value of the difference between α and the probability of $R(\gamma_0)$ under any couple (μ, σ) which satisfies (2.1) is less than any arbitrarily small positive ϵ when N is sufficiently large, i.e., when N is sufficiently large,*

$$| P\{R(\gamma_0)\} - \alpha | < \epsilon$$

uniformly for all (μ, σ) which satisfy (2.1) with $G < \gamma_0 < 1$, and which fulfill Assumptions 1 and 2.

LEMMA 1. $\dfrac{\partial r(\bar{x}, \gamma_0)}{\partial \bar{x}}$ *exists in the open interval $L_1 < \bar{x} < L_2$.*

PROOF: We have

$$\gamma_0 = \frac{1}{\sqrt{2\pi}r(\bar{x}, \gamma_0)} \int_{L_1}^{L_2} \exp\left\{-\frac{1}{2}\left(\frac{y - \bar{x}}{r(\bar{x}, \gamma_0)}\right)^2\right\} dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{(L_1-\bar{x})/r}^{(L_2-\bar{x})/r} \exp\left\{-\frac{1}{2} y^2\right\} dy.$$

Differentiating with respect to $\bar{x}$ we obtain, since $r > 0$,

$$e^{-R^2/2}\left(1 + R\frac{\partial r}{\partial \bar{x}}\right) = e^{-T^2/2}\left(1 + T\frac{\partial r}{\partial \bar{x}}\right)$$

with

$$R = \frac{L_2 - \bar{x}}{r} \qquad T = \frac{L_1 - \bar{x}}{r}.$$

Hence

$$\frac{\partial r}{\partial \bar{x}} = -\left(\frac{e^{-R^2/2} - e^{-T^2/2}}{Re^{-R^2/2} - Te^{-T^2/2}}\right). \tag{4.1}$$

Since $R > 0$ and $T < 0$ within the open interval $L_1 < \bar{x} < L_2$, it follows that $\frac{\partial r}{\partial \bar{x}}$ exists in the entire open interval.

Lemma 2. *In the open interval* $L_1 < \bar{x} < L_2$,

$$\frac{\partial P\{s \geq \varphi(\bar{x}, \gamma_0)\}}{\partial \bar{x}}$$

exists.

Proof: We have, with k a suitable constant,

$$P = k\int_{\sqrt{n}\varphi/\sigma}^{\infty} y^{n-1}e^{-y^2/2}\,dy = k\int_{(r(\bar{x},\gamma_0)/\sigma)\chi_{1-\alpha}}^{\infty} y^{n-1}e^{-y^2/2}\,dy.$$

Hence

$$\frac{\partial P}{\partial \bar{x}} = \frac{-k\chi_{1-\alpha}}{\sigma}\frac{\partial r}{\partial \bar{x}}\left(\frac{r\cdot\chi_{1-\alpha}}{\sigma}\right)^{n-1}\exp\left(\frac{-r^2\chi_{1-\alpha}^2}{2\sigma^2}\right). \tag{4.2}$$

Lemma 3. *Let δ be any arbitrarily small positive number. The function* $\left|\frac{\partial P}{\partial \bar{x}}\right|$ *of $\bar{x}$ and γ_0 is bounded for* $L_1 + \delta \leq \bar{x} \leq L_2 - \delta$, $G < \gamma_0 < 1$.

Proof: From (4.1) we have

$$\left|\frac{\partial r}{\partial \bar{x}}\right| < \frac{e^{-R^2/2} + e^{-T^2/2}}{Re^{-R^2/2} - Te^{-T^2/2}} \leq \max.\left(\frac{1}{R}, \frac{-1}{T}\right) = \max.\left(\frac{r}{L_2 - \bar{x}}, \frac{r}{\bar{x} - L_1}\right) \leq \frac{r}{\delta}.$$

Therefore from (4.2) we have that $\left|\frac{\partial P}{\partial \bar{x}}\right|$ is less than a constant multiplied by $\left(\frac{r}{\sigma}\right)^n \exp\left(\frac{-r^2\chi_{1-\alpha}^2}{2\sigma^2}\right)$ and is therefore bounded.

Proof of the theorem: From Lemma 3 and the Theorem of the Mean it follows that, in the closed interval

$$L_1 + \frac{d}{2} \leq \bar{x} \leq L_2 - \frac{d}{2},$$

the function $P\{s \geq \varphi(\bar{x}, \gamma_0)\}$ is uniformly continuous in $\bar{x}$ uniformly for all (μ, σ) which satisfy (2.1) with $G < \gamma_0 < 1$. Hence for every positive ϵ_1 there

exists a positive $\eta < \frac{d}{2}$ such that $| l_1 - l_2 | < \eta$,

$$L_1 + \frac{d}{2} \leq l_1, l_2 \leq L_2 - \frac{d}{2},$$

implies

$$| P\{s \geq \varphi(l_1, \gamma_0)\} - P\{s \geq \varphi(l_2, \gamma_0)\} | < \epsilon_1 .$$

For fixed arbitrary $\epsilon_2 > 0$ we have, when N is sufficiently large,

$$P\{ | \bar{x} - \mu | < \eta\} > 1 - \epsilon_2 ,$$

from Assumption 2 and the stochastic convergence of $\bar{x}$. Now

$$P\{s \geq \varphi(\mu, \gamma_0)\} = \alpha.$$

Hence, when N is sufficiently large,

$$| P\{R(\gamma_0)\} - \alpha | \leq \epsilon_1(1 - \epsilon_2) + \epsilon_2 \leq \epsilon_1 + \epsilon_2 .$$

Since ϵ_1 and ϵ_2 are arbitrarily small, this proves the desired result.

5. Construction of large sample confidence regions. The acceptance regions $R(\gamma_0)$ whose size never differs from α by more than a uniform bound which approaches zero as N increases, readily yield a lower confidence limit for γ (within the approximation involved). The confidence region consists of all the γ_0 for which $R(\gamma_0)$ contains the observed $\bar{x}$, s. Our acceptance regions $R(\gamma_0)$ are so constructed that, if $\gamma_1 < \gamma_2$, $R(\gamma_1)$ is entirely contained within $R(\gamma_2)$. Hence the confidence region is an interval, one end of which is always unity, as was desired. The rule for constructing the lower confidence limit D is, therefore, as follows:

a) if $\bar{x} < L_1$ or $\bar{x} > L_2$, then $D = G$

b) if $L_1 \leq \bar{x} \leq L_2$, then

$$D = \frac{1}{\sqrt{2\pi}} \int_{(L_1 - \bar{x})/w}^{(L_2 - \bar{x})/w} \exp\{-\tfrac{1}{2}y^2\}\, dy \tag{5.1}$$

where

$$w = \sqrt{N - 1} \cdot \frac{s}{\chi_{1-\alpha}}.$$

(The value of D may be found in a table of the normal distribution. It is easy to see that $s = \varphi(\bar{x}, D)$, i.e., D is the smallest value of γ_0 for which $\bar{x}$, s will still lie in $R(\gamma_0)$).

If the statement $D \leq \gamma$ is made in a large number of cases, where the assumptions are fulfilled and the sample size is large, the proportion of correct statements will be close to α.

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NOTES

This section is devoted to brief research and expository articles on methodology and other short items.

ON SEQUENTIAL BINOMIAL ESTIMATION

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The present note, written after a reading of the very interesting paper by Girshick, Mosteller, and Savage [1], is for the purpose of adding a few remarks in the nature of a supplement. For the sake of brevity the notation and terminology of [1] are adopted in toto.

Theorem 1 below generalizes Theorem 1 of [1]. In Theorem 2′ we formulate explicitly the fact which lies at the basis of the GSM method of estimation. Parts of the proofs of Theorems 3 and 4 of [1] are simply proofs of special cases of this (e.g., equation (2) of [1]). We then use this fact repeatedly in proving Theorem 3, which states that the Girshick-Mosteller-Savage estimate is the only proper unbiased estimate for sequential tests defined by regions which we shall call doubly simple.

A doubly simple region is defined precisely below. Intuitively we may describe such a region as the one between two curves $y = f_1(x)$ and $x = f_2(y)$, where $f_1(x)$ is defined and monotonically non-decreasing for all non-negative x, $f_2(y)$ is defined and monotonically non-decreasing for all non-negative y, $f_1(0) > 0$, $f_2(0) > 0$. If the two curves intersect, the region is finite, and the values of the functions f_1 and f_2 beyond the point of intersection are of no interest This description is of course purely heuristic, because in actual fact only integral values of the variables come into play, and intersection of the curves, for example, is not needed to make the region finite. Since the question of finite regions is completely settled by [1], Theorem 7, only non-finite regions remain to be discussed, and the precise definition given below is such as to imply that the region is not finite. It seems to the present writer that at least many of the non-finite sequential tests which may be developed for meaningful statistical problems will require doubly simple regions. The Wald sequential binomial test [2] defines such a region, which also falls within the scope of Theorem 6 of [1]. It is easy to see that there exist closed regions which are doubly simple and do not satisfy the conditions of this theorem.

By a "proper" estimate $p(\alpha)$ we shall mean an estimate such that $0 \leq p(\alpha) \leq 1$ for every α. It is difficult to see how any estimate which is not proper can make much sense.

THEOREM 1. *A sufficient condition that a region R be closed is that* $\liminf_{n\to\infty} \frac{A(n)}{\sqrt{n}} < \infty$, *where $A(n)$ is the number of accessible points of index n.*

PROOF: The hypothesis of the theorem implies that there exist a positive number H and an increasing sequence of positive integers $n_1, n_2, n_3, \cdots$, with the following properties:

a) $n_{i+1} > 2n_i$ ($i = 1, 2, \cdots$ ad inf.)

b) $A(n_i) < H\sqrt{n_i}$.

For n_i sufficiently large, the conditional probability of reaching the accessible points on $x + y = n_{i+1}$, when an accessible point on $x + y = n_i$ has been reached, is $< K < 1$ by the normal approximation to the binomial distibution, where K is constant (and depends on H). Hence the probability of passing through accessible points on all members of the set $x + y = n_i$ ($i = 1, 2, \cdots, L$) approaches zero as $L \to \infty$, so that the region is closed.

THEOREM 2. *Let R be any region, B its boundary, and $t = (a, b)$; any accessible point in R. Let $l_t(\alpha)$ be the number of paths from t to $(x, y) = \alpha \in B$. Let $Q(t)$ be the conditional probability that a path, which has reached t, will reach the boundary B. Then*

$$\sum_{\alpha \in B} l_t(\alpha)p^y q^x = Q(t)p^b q^a.$$

THEOREM 2'. (Corollary to Theorem 2)

If R is closed, then

$$\sum_{\alpha \in B} l_t(\alpha)p^y q^x = p^b q^a. \tag{1}$$

PROOF: Let $k(t)$ be the number of paths in R from the origin to t. The probability of reaching $\alpha \in B$ by a path which passes through t is $k(t)l_t(\alpha)p^y q^x$. The probability of reaching t from the origin is $k(t)p^b q^a$, and hence the probability of reaching the boundary via t is $Q(t)k(t)p^b q^a$. From this the desired result follows.

We now define a doubly simple region. The boundary of the region consists of the two infinite sequences of points

$$(0, a_0), (1, a_1), (2, a_2), \cdots$$

and

$$(b_0, 0), (b_1, 1), (b_2, 2), \cdots$$

where $a_0, a_1, a_2, \cdots$ and $b_0, b_1, b_2, \cdots$ are two infinite non-decreasing sequences of positive integers. The accessible points of the region are all points which can be reached by a path from the origin which does not contain a boundary point. (It is to be noted that since a boundary point is, by definition, a point not in the region which can be reached by a path in the region, the above definition implies that a doubly simple region is not finite. The reason for making this so has been given above.)

THEOREM 3. Let R be a closed doubly simple region. Then $\hat{p}(\alpha)$ is the unique proper unbiased estimate of p.

PROOF: Suppose there were two proper unbiased estimates $p_1(\alpha)$ and $p_2(\alpha)$. Writing $m(\alpha) = p_1(\alpha) - p_2(\alpha)$, we would have

$$\sum_{\alpha \in B} m(\alpha)k(\alpha)p^y q^x = 0 \tag{2}$$

with

$$| m(\alpha) | \leq 1 \tag{3}$$

First we prove

LEMMA 1. *If* $a_0 > 1$, *then* $m(b_0, 0) = 0$.

PROOF: Let $k^*(\alpha)$ denote the number of paths in R from the point (0, 1) to the boundary point α. For all points $\alpha \in B$ except $(b_0, 0)$ we have

$$b_0 k^*(\alpha) \geq k(\alpha). \tag{4}$$

From (1), (2), (3), and (4) we have, since $k(b_0, 0) = 1$,

$$\begin{aligned} | m(b_0, 0) | q^{b_0} &= \Big| \sum_{\alpha \in B, \alpha \neq (b_0, 0)} m(\alpha)k(\alpha)p^y q^x \Big| \\ &\leq \sum_{\alpha \in B, \alpha \neq (b_0, 0)} k(\alpha)p^y q^x \leq b_0 \sum_{\alpha \in B} k^*(\alpha)p^y q^x = b_0 p. \end{aligned} \tag{5}$$

Now as $p \to 0$, the left member of the inequality (5) approaches $| m(b_0, 0) |$, and the right member approaches zero. This proves Lemma 1.

LEMMA 2. *For every* $z < a_0 - 1$, $m(b_z, z) = 0$.

PROOF: In view of Lemma 1 it is sufficient to prove the following:

If $Z \leq a_0 - 2$, and if $m(b_z, z) = 0$ for $z = 0, 1, \cdots, Z - 1$, then $m(b_Z, Z) = 0$. Let $k_{Z+1}(\alpha)$ denote the number of paths in R from $(0, Z + 1)$ to the boundary point α. For any point $\alpha \in B$ whose ordinate is $\geq Z + 1$ we have

$$b_0 b_1 \cdots b_Z k_{Z+1}(\alpha) \geq k(\alpha). \tag{6}$$

From (1), (2), (3), and (6) we have

$$\begin{aligned} | m(b_Z, Z) | k(b_Z, Z)p^Z q^{b_Z} &= | \Sigma m(\alpha)k(\alpha)p^y q^x | \leq \Sigma k(\alpha)p^y q^x \\ &\leq b_0 b_1 \cdots b_Z \Sigma k_{Z+1}(\alpha)p^y q^x = b_0 b_1 \cdots b_Z p^{Z+1} \end{aligned} \tag{7}$$

where the summations take place over all boundary points whose ordinates are $\geq Z + 1$. Hence

$$| m(b_Z, Z) | k(b_Z, Z)q^{b_Z} \leq b_0 b_1 \cdots b_Z p.$$

and letting $p \to 0$ we obtain the desired result.

LEMMA 3. $m(b_{a_0-1}, a_0 - 1) = 0$.

PROOF: Let s be the smallest integer such that (s, a_0) is an accessible point.

We proceed as in Lemma 2, with (s, a_0) playing the role of $(0, Z + 1)$, and eventually obtain the following inequality:

$$\begin{aligned} | m(b_{a_0-1}, a_0 - 1) | k(b_{a_0-1}, a_0 - 1)p^{a_0-1}q^{b_{a_0-1}} &= | \textstyle\sum_0 m(\alpha)k(\alpha)p^y q^x | \\ &\leq p^{a_0}\left(\sum_{i=0}^{s-1} k(i, a_0)q^i + b_0 b_1 \cdots b_{a_0-1} q^s\right), \end{aligned} \tag{8}$$

where Σ_0 denotes summation over all boundary points with ordinate $\geq a_0$. The desired result follows.

LEMMA 4. *Let $h(\geq a_0)$ be the smallest ordinate for which at least one boundary point (w^*, h) exists such that $m(w^*, h) \neq 0$ (If no such h exists the theorem is proved). Of all such points let w be the one with the smallest abscissa. Then the point (w, h) is a member of the sequence*

$$(0, a_0)\ (1, a_1),\ (2, a_2),\ \cdots$$

PROOF: If the lemma is not true, then for all boundary points α with ordinate h, $m(\alpha) = 0$, except that $m(b_h, h) \neq 0$. Let W be that accessible point of R whose ordinate is $h + 1$ and whose abscissa v is a minimum. Let $k_w(\alpha)$ be the number of paths in R from W to the boundary point α. For boundary points α accessible from W we have

$$b_0 b_1 \cdots b_h k_W(\alpha) \geq k(\alpha). \tag{9}$$

From (1), (2), (3), and (9) we have

$$\begin{aligned} | m(b_h, h) | k(b_h, h) p^h q^{b_h} = | \Sigma_1(m(\alpha)k(\alpha)p^y q^x | &\leq \Sigma_2 k(\alpha) p^{h+1} q^x \\ &+ b_0 b_1 \cdots b_h p^{h+1} q^v = K^* p^{h+1}, \end{aligned} \tag{10}$$

where:

a) Σ_1 denotes summation over all $\alpha \epsilon B$ for which $y > h$

b) Σ_2 denotes summation over all boundary points α of ordinate $h + 1$ and abscissa $< v$.

c) K^* denotes a constant.

From this it easily follows that $m(b_h, h) = 0$, in contradiction to the definition of h. This proves Lemma 4.

PROOF OF THEOREM 3: Let (w, h) be as defined in the statement of Lemma 4. From Lemma 4 it follows that, if any other boundary points with abscissa w exist, they must be members of the sequence $(b_0, 0)$, $(b_1, 1)$, $(b_2, 2)$, $\cdots$ and hence their ordinates are $< h$. From the definition of (w, h) and from Lemma 4 it follows that for any $\alpha \epsilon B$ whose abscissa is $< w$, $m(\alpha) = 0$.

Now in the proofs of Lemmas 1–4 the roles of x and y are not symmetrical. However, symmetry of course exists, and analogous lemmas follow. In particular, the analogue to Lemma 4 has as a consequence that, since w is the smallest abscissa such that $m(\alpha) = 0$ when abscissa of $\alpha < w$, and $m(w, h) \neq 0$, there exists a boundary point (w, h'), such that $m(w, h') \neq 0$ and (w, h') is a member of $(b_0, 0)$, $(b_1, 1)$, $(b_2, 2)$, $\ldots$ Then $h' < h$. But this contradicts the definition of h and proves the theorem.

It is easy to see that, if the boundary points of a closed region constitute a single "curve" instead of two "curves" as in a doubly simple region, the estimate $\hat{p}(\alpha)$ will be the only proper unbiased estimate of p.

It is interesting to consider some of the consequences of Theorem 3 for all unbiased estimates (not necessarily proper) for doubly simple regions. An

examination of the proof of Theorem 3 shows that it would go through with little change if equation (3) were replaced by the requirement that $|m(\alpha)|$ be bounded. We therefore obtain the following result: If for a doubly simple region there exists an unbiased estimate $p(\alpha)$ of p, not identically equal to $\hat{p}(\alpha)$, then not only is $p(\alpha)$ not proper, but also, no matter how large M, there exists a boundary point α such that $|p(\alpha)| > M$. The uselessness of such an estimate is manifest.

The author is of the opinion that freedom from bias is not necessarily an indispensable characteristic of an optimum estimate. In general there is no reason for requiring the first moment of the estimate rather than any other moment to be the unknown parameter. The justification in any particular case must be based on special conditions of the problem.

The author is indebted to Mr. Howard Levene for reading the present paper and making valuable suggestions.

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DIFFERENTIATION UNDER THE EXPECTATION SIGN IN THE FUNDAMENTAL IDENTITY OF SEQUENTIAL ANALYSIS

BY ABRAHAM WALD

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1. Introduction. Let $\{z_\alpha\}$ ($\alpha = 1, 2, \cdots$, ad inf.) be a sequence of random variables which are independently distributed with identical distributions. Let a be a positive, and b a negative constant. For each positive integral value m, let Z_m denote the sum $z_1 + \cdots + z_m$. Denote by n the smallest integral value for which Z_n does not lie in the open interval (b, a). For any random variable u, let the symbol $E(u)$ denote the expected value of u. The following identity, which plays a fundamental role in sequential analysis, has been proved in [1].

$$E[e^{Z_n t}\varphi(t)^{-n}] = 1, \tag{1.1}$$

where

$$\varphi(t) = E(e^{zt}) \tag{1.2}$$

and the distribution of z is equal to the common distribution of $z_1, z_2, \cdots$, etc. Identity (1.1) holds for all points t in the complex plane for which $\varphi(t)$ exists and $|\varphi(t)| \geq 1$.

The purpose of this paper is to formulate conditions under which we may differentiate (1.1) with respect to t under the expectation sign. This is of interest, since various results in sequential analysis can easily be established by differentiating (1.1) under the expectation sign. For example, the formula for $E(n)$ can immediately be obtained by differentiating (1.1) at $t = 0$. The derivative of $e^{Z_n t}\varphi(t)^{-n}$ at $t = 0$ is given by

$$(1.3) \qquad Z_n - \frac{\varphi'(0)}{\varphi(0)} n = Z_n - E(z)n$$

where $\varphi'(t)$ denotes the derivative of $\varphi(t)$. Hence, if we may differentiate (1.1) under the expectation sign, we obtain the basic formula

$$(1.4) \qquad E(Z_n) = E(z)E(n).$$

If $E(z) \neq 0$, the above equation has been used [2] to derive lower and upper limits for $E(n)$. If, however, $E(z) = 0$, formula (1.4) is of little value. It will be shown in section 3 that

$$(1.5) \qquad E(n) = \frac{E(Z_n^2)}{E(Z^2)} \quad \text{when} \quad E(z) = 0.$$

This result is obtained, as will be seen in section 3, by differentiating identity (1.1) twice at $t = 0$.

2. A sufficient condition for the differentiability of (1.1) under the expectation sign. In what follows, the parameter t in (1.1) will be restricted to real values, even if this is not stated explicitly. For any random variable u and any relation R, the symbol $E(u \mid R)$ will denote the conditional expected value of u under the restriction that R holds. In this section we shall establish the following theorem.

THEOREM 2.1. *If $\varphi(t)$ exists for all real values t, identity (1.1) may be differentiated under the expectation sign any number of times with respect to t at any value t in the domain $\varphi(t) \geq 1$.*

PROOF: First we shall derive an upper bound for $E(e^{tZ_n} \mid n = m)$ for any given integral value m. Consider the case when $t > 0$. Then

$$(2.1) \qquad E(e^{tZ_n} \mid n = m) \leq E(e^{tZ_n} \mid Z_n \geq a,\ n = m) \qquad (t > 0).$$

Clearly,

$$(2.2) \qquad E(e^{tZ_n} \mid Z_n \geq a, n = m, e^{tZ_{n-1}} = \rho e^{at}) = e^{at}\rho E\left(e^{zt} \mid e^{zt} \geq \frac{1}{\rho}\right).$$

Let $l(t)$ denote the least upper bound of the expression

$$(2.3) \qquad \rho E\left(e^{zt} \mid e^{zt} \geq \frac{1}{\rho}\right)$$

with respect to ρ over the interval $(e^{-(a-b)|t|}, 1)$. The existence of $\varphi(t)$ implies that $l(t)$ is finite. It follows from (2.1) and (2.2) that

$$E(e^{tZ_n} \mid n = m) \leq e^{at}l(t) \qquad (t > 0) \tag{2.4}$$

and, therefore, also

$$E(e^{tZ_n}) \leq e^{at}l(t) \qquad (t > 0). \tag{2.5}$$

If $t < 0$, one can show in a similar way that

$$E(e^{tZ_n} \mid n = m) \leq e^{bt}l(t) \qquad (t < 0) \tag{2.6}$$

and

$$E(e^{tZ_n}) \leq e^{bt}l(t) \qquad (t < 0). \tag{2.7}$$

To prove Theorem 2.1, it is sufficient to show that the following two propositions hold.[1]

PROPOSITION 2.1. *All derivatives of $e^{Z_n t}\varphi(t)^{-n}$ with respect to t exist in the domain $\varphi(t) \geq 1$.*

PROPOSITION 2.2. *For any positive integral value r and for any finite interval I in which $\varphi(t) \geq 1$, it is possible to find a function $D(Z_n, n)$ such that*

$$D(Z_n, n) \geq \left| \frac{d^r}{dt^r} [e^{Z_n t} \varphi(t)^{-n}] \right| \tag{2.8}$$

for all values t in I and

$$E[D(Z_n, n)] < \infty. \tag{2.9}$$

Proposition 2.1 is clearly true, if all derivatives of $\varphi(t)$ exist. The existence of these derivatives follows from the existence of $\varphi(t)$ for all values t.

Since $\dfrac{d^r}{dt^r} e^{Z_n t}\varphi(t)^{-n}$ is equal to the sum of a finite number of terms of the type $Z_n^{r_1} n^{r_2} e^{Z_n t}\varphi(t)^{-n}$, Proposition 2.2 is proved if we can show that for any given integral values r_1 and r_2 there exists a function $D_{r_1 r_2}(Z_n, n)$ such that

$$D_{r_1 r_2}(Z_n, n) \geq | Z_n^{r_1} n^{r_2} e^{Z_n t}\varphi(t)^{-n} | \tag{2.10}$$

for all t in I and

$$E[D_{r_1 r_2}(Z_n, n)] < \infty. \tag{2.11}$$

Clearly, since $\varphi(t) \geq 1$ in I,

$$| Z_n^{r_1 r_2} e^{Z_n t}\varphi(t)^{-n} | \leq | Z_n^{r_1} | n^{r_2} e^{| Z_n | t_0} \tag{2.12}$$

where t_0 is an upper bound of $|t|$ in I. Let t_1 be a value $> t_0$. Then for a properly chosen constant C we have

$$| Z_n^{r_1} | e^{| Z_n | t_0} < C e^{| Z_n | t_1}. \tag{2.13}$$

[1] See, for example, E. J. McShane, *Integration*, Princeton University Press (1944), p. 216, 217 and 276.

Hence, it follows from (2.12) and (2.13) that

$$|Z_n^{r_1} n^{r_2} e^{Z_n t} \varphi(t)^{-n}| \leq C n^{r_2} e^{|Z_n| t_1} \leq C n^{r_2} (e^{Z_n t_1} + e^{-Z_n t_1}) \tag{2.14}$$

for all t in I.

We put

$$D_{r_1 r_2}(Z_n, n) = C n^{r_2} (e^{Z_n t_1} + e^{-Z_n t_1}). \tag{2.15}$$

We have

$$E[D_{r_1 r_2}(Z_n, n)] = C \sum_{m=1}^{\infty} p_m m^{r_2} [E(e^{Z_n t_1} \mid n = m) + E(e^{-Z_n t_1} \mid n = m)] \tag{2.16}$$

where p_m denotes the probability that $n = m$.

Hence, because of (2.4) and (2.6), we obtain

$$\begin{aligned} E[D_{r_1 r_2}(Z_n, n)] &\leq C(e^{a t_1} l(t_1) + e^{-b t_1} l(-t_1))[\Sigma p_m m^{r_2}] = \\ &= C[e^{a t_1} l(t_1) + e^{-b t_1} l(-t_1)] E(n^{r_2}). \end{aligned} \tag{2.17}$$

Since all moments of n are finite,[2] Proposition 2.2 is proved. This completes the proof of Theorem 2.1.

3. The expected value of n when $E(z) = 0$. It will be shown in this section that

$$E(n) = \frac{E(Z_n^2)}{E(z^2)} \quad \text{when} \quad E(z) = 0, \tag{3.1}$$

if identity (1.1) can be differentiated twice under the expectation sign at $t = 0$. The second derivative of $e^{tZ_n}\varphi(t)^{-n}$ with respect to t is given by

$$\left\{ \left[Z_n - n \frac{\varphi'(t)}{\varphi(t)} \right]^2 - n \frac{\varphi''(t)\varphi(t) - [\varphi'(t)]^2}{[\varphi(t)]^2} \right\} e^{Z_n t} \varphi(t)^{-n} \tag{3.2}$$

where $\varphi'(t)$ denotes the first, and $\varphi''(t)$ the second derivative of $\varphi(t)$.

Since $\varphi(0) = 1$, $\varphi'(0) = E(z) = 0$ and $\varphi''(0) = E(z^2)$, putting $t = 0$, expression (3.2) becomes

$$Z_n^2 - n\varphi''(0) = Z_n^2 - nE(z^2) \tag{3.3}$$

Hence, if (1.1) may be differentiated twice under the expectation sign at $t = 0$, we obtain

$$E[Z_n^2 - nE(z^2)] = 0 \tag{3.4}$$

from which (3.1) follows.

An approximate value of $E(n)$ can be obtained from (3.1) by neglecting the excess of Z_n over the boundaries. Then Z_n can take only the values a and b. Hence

$$E(Z_n^2) \sim a^2 P(Z_n \geq a) + b^2 P(Z_n \leq b) \tag{3.5}$$

where the sign $\sim$ denotes approximate equality.

---

[2] See the paper by C. Stein, "A note on cumulative sums," in this issue of the *Annals of Mathematical Statistics*.

It was shown in [1] (equation 28) that neglecting the excess of Z_n over the boundaries, the approximation formula

$$P(Z_n \geq a) \sim \frac{1 - e^{bh}}{e^{ah} - e^{bh}} \tag{3.6}$$

holds, where h is the non-zero root of the equation $\varphi(t) = 1$. This formula was derived there under the assumption that $E(z) \neq 0$. If $E(z)$ approaches zero, $h \to 0$ and the right hand member of (3.6) converges to $\frac{-b}{a - b}$.

Putting $P(Z_n \geq a) = \frac{-b}{a - b}$ and $P(Z_n \leq b) = 1 - \frac{-b}{a - b} = \frac{a}{a - b}$, we obtain from (3.5)

$$E(Z_n^2) \sim a^2 \left(\frac{-b}{a - b}\right) + b^2 \frac{a}{a - b} = -ab. \tag{3.7}$$

Hence[3]

$$E(n) \sim \frac{-ab}{E(z^2)}. \tag{3.8}$$

Limits for $E(n)$ can be obtained by deriving limits for $E(Z_n^2)$. Let r be a non-negative real variable. One can verify that

$$a^2 \leqq E(Z_n^2 \mid Z_n \geq a) \leqq \underset{0<r<a-b}{\text{l.u.b.}} E[(a - r + z)^2 \mid z \geq r] \tag{3.9}$$

and

$$b^2 \leqq E(Z_n^2 \mid Z_n \leqq b) \leq \underset{0<r<a-b}{\text{l.u.b.}} E[(b + r + z)^2 \mid z + r \leq 0]. \tag{3.10}$$

We have

$$E(Z_n^2) = P(Z_n \geq a)E(Z_n^2 \mid Z_n \geq a) + P(Z_n \leq b)E(Z_n^2 \mid Z_n \leq b). \tag{3.11}$$

Limits for $E(Z_n^2)$ can be obtained by replacing the conditional expected values in the right hand member of (3.11) by their limits given in (3.9) and (3.10).

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[3] This approximation formula was obtained also by W. A. Wallis independently of the author. It is included in the publication of the Statistical Research Group of Columbia Univ., *Techniques of Statistical Analysis*, Chapter 17, Section 7.2, McGraw Hill, New York (1946)

A NOTE ON CUMULATIVE SUMS

By Charles Stein

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Let $\{Z_i\}$ be a denumerable sequence of identical independent real-valued random variables Two constants $a > 0 > b$ are chosen and the random variable n defined as the smallest integer for which one of the inequalities $\sum_1^n Z_i \geq a$ $\sum_1^n Z_i \leq b$ holds. For any events E_1 and E_2, $P\{E_1\}$ will denote the probability of the event E_1 and $P\{E_1 \mid E_2\}$ the conditional probability of the event E_1 given that E_2 has occurred.

It will be shown that there exists $t_0 > 0$ such that the moment generating function, Ee^{nt} exists for any complex number t whose real part is less than or equal to t_0, and as an immediate consequence that n has finite moments of all orders.

If d is any constant satisfying $b < d < a$, then, for fixed m,

$$P\left\{b < \sum_1^m Z_i + d < a\right\} \leq P\left\{\left|\sum_1^m Z_i\right| < c\right\} \tag{1}$$

where $c = |a| + |b|$. We exclude the case $P\{Z_i = 0\} = 1$. Then there exists $\epsilon > 0$ such that either

$$\delta_1 = P\{Z_i \geq \epsilon\} > 0 \quad \text{or} \quad \delta_2 = P\{Z_i \leq -\epsilon\} > 0.$$

Taking, for example, the former alternative with $m_1 = \left[\frac{c}{\epsilon}\right] + 1$,

$$P\left\{\left|\sum_1^{m_1} Z_i\right| \geq c\right\} \geq P\left\{Z_i \geq \epsilon \quad \text{for} \quad i = 1, \cdots, m_1\right\} = \delta_1^{m_1} > 0 \tag{2}$$

where $[w]$ denotes the largest integer less than or equal to w. For any poitive integer k,

$$\begin{aligned}\frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} &= P\{n > km_1 \mid n > (k-1)m_1\} \\ &\leq P\left\{b < \sum_1^{km_1} Z_i < a \middle| b < \sum_1^s Z_i < a \quad \text{for} \quad s = 1, \cdots, (k-1)m_1\right\}\end{aligned}$$

since $n > km$ implies $b < \sum_1^{km_1} Z_i < a$.

But $\sum_1^{km_1} Z_i = \sum_1^{(k-1)m_1} Z_i + \sum_{(k-1)m_1+1}^{km_1} Z_i$ and the second sum on the right hand side is independent of all terms in the first sum.

Thus the distribution of $\sum_{1}^{km_1} Z_i$ given $\sum_{1}^{s} Z_i$ for $s = 1, \cdots, (k-1)m_1$ depends only on $\sum_{1}^{(k-1)m_1} Z_i$ so that

$$
\begin{aligned}
\frac{P\{n > km_1\}}{P\{n > (k-1)m_1\}} &\leq P\left\{b < \sum_{(k-1)m_1+1}^{km_1} Z_i + \sum_{1}^{(k-1)m_1} Z_i < a \,\middle|\, b < \sum_{1}^{(k-1)m_1} Z_i < a\right\} \\
&\leq P\left\{\left|\sum_{(k-1)m_1+1}^{km_1} Z_i\right| < c\right\} \leq 1 - \delta_1^{m_1} \quad \text{by (1) and (2).}
\end{aligned} \tag{3}
$$

Consequently, by induction on k,

$$
P\{n > m\} \leq P\left\{n > \left[\frac{m}{m_1}\right] m_1\right\} \leq (1 - \delta_1^{m_1})^{\left[\frac{m}{m_1}\right]}. \tag{4}
$$

Let t_0 be any positive number less than $-\frac{1}{m_1}\log(1 - \delta_1^{m_1})$.

Then

$$
\begin{aligned}
Ee^{nt_0} &= \sum_{m=1}^{\infty} e^{mt_0} P\{n = m\} \\
&\leq \sum_{k=1}^{\infty} e^{km_1t_0} P\{(k-1)m_1 < n \leq km_1\} \\
&\leq \sum_{k=1}^{\infty} e^{km_1t_0} P\{n > (k-1)m_1\} \\
&\leq \sum_{k=1}^{\infty} e^{km_1t_0} (1 - \delta_1^{m_1})^{k-1} \\
&= \frac{1}{1 - \delta_1^{m_1}} \sum_{k=1}^{\infty} \{e^{m_1t_0}(1 - \delta_1^{m_1})\}^k.
\end{aligned} \tag{5}
$$

But this is a geometric series with decreasing terms, and is consequently convergent. Thus for any t whose real part $R(t) \leq t_0$, the moment generating function Ee^{nt} exists. Since, for all positive l, $m^l < e^{mt_0}$ for sufficiently large m, n has finite moments of all orders.

ABSTRACTS OF PAPERS

Presented on August 21, 1946, at the Cornell meeting of the Institute

1. A Test of Randomness in Two Dimensions. Howard Levene, Columbia University.

A square of side N is divided into N^2 unit cells, and each cell takes on the characteristics A or B with probabilities p and $q = 1 - p$ respectively, independently of the other cells. A cell is an "upper left corner" if it is A and the cell above and cell to the left are not A. Let V_1 be the total number of upper left corners and let V_2, V_3, V_4 be the number of similarly defined upper right, lower right, and lower left corners respectively. Let $V = (V_1 + V_2 + V_3 + V_4)/4$. It is proved that V is normally distributed in the limit with $E(V) = p(Nq + p)^2$ and $\sigma^2(V) \sim N^2pq^2(4 - 20p + 45p^2 - 27p^3)/4$. The conditional limit distribution of V when p is estimated from the data, and the limit distribution of a related quadratic form are also obtained These statistics are in a sense a generalization of the run statistics used for testing randomness in one dimension.

2. Asymptotic Distribution of Moments from a System of Linear Stochastic Difference Equations. Herman Rubin, Cowles Commission for Research in Economics.

Let $\sum_{\tau=0}^{\infty} B_\tau y'_{t-\tau} + \Gamma z'_t = u'_t$, $(t = 1, 2, \cdots)$, be a complete system of linear stochastic difference equations determining y_{ti} (the coordinates of y_t), $t > 0$, in terms of y_{ti}, $t \leq 0$, and z_{tk} (the coordinates of z_t), which are assumed to be fixed variates, and the random variables u_{ti} (the coordinates of u_t) Such a system is called a stable if for every bounded set of fixed variates, and $E(u'_tu_t)$ uniformly bounded, $E(y'_ty_t)$ is uniformly bounded. This condition is shown to be equivalent to $\sum | h_{ij\tau} |$ finite, where $y'_t = \sum_{\tau=0}^{\infty} H_\tau(u'_{t-\tau} - \Gamma z'_{t-\tau}) + \sum_{\nu=0}^{\infty} J_{t,\nu}y'_{-\nu}$ is the solution of the above difference equation. Let Q_t be an infinite quadratic form in $y_{t-\tau,i}$ and $z_{t-\nu,k}$ $(\tau, \nu = 0, 1, \cdots)$ with coefficients depending only on i, k, τ, and ν. Such a quadratic form is called convergent if the sum of the absolute values of the coefficients is finite. It is shown under fairly general conditions that the mean of a convergent quadratic form is asymptotically normally distributed with variance $0\left(\frac{1}{T}\right)$.

3. Conditional Expectation and Unbiased Sequential Estimation. David Blackwell, Howard University.

It is shown that $E[f(x_\alpha)E_\alpha y] = E(fy)$ whenever $E(fy)$ is finite, and that $\sigma^2(E_\alpha y) \leq \sigma^2(y)$, with equality holding only if $E_\alpha y = y$, where $E_\alpha y$ denotes the conditional expectation of y with respect to the family of chance variables x_α. These results imply that whenever there is a sufficient statistic u and an unbiased estimate t, not a function of u only, for a parameter p, the function E_ut, which is a function of u only, is an unbiased estimate for p with variance smaller than that of t. A sequential unbiased estimate for a parameter is obtained, such that when the sequential test terminates after i observations, the estimate is a function of a sufficient statistic for the parameter with respect to these observations. A special case of this estimate is that obtained by Girshick, Mosteller, and Savage (*Annals of Math. Stat.*, Vol. XVII (1946), pp. 13–23) for the parameter of a binomial distribtion.

4. A Discussion of the Ehrenfest Model. Preliminary report. Mark Kac, Cornell University.

A particle moves along a straight line in steps Δ, the duration of each step being τ. The probabilities that the particle at $k\Delta$ will move to the right or left are $(1/2)(1 - k/R)$

and $(1/2)(1 + k/R)$ respectively. R and k are integers and $|k| \leq R$. M. C. Wang and G. E. Uhlenbeck in their paper *On the theory of Brownian motion II* (*Rev Mod. Phys.* Vol. 17 (1945), pp 323–342) discuss this random walk problem and state several unsolved problems. In answer to some of the questions raised the following results are obtained Let $(1 - z)^{R-j} \cdot (1 + z)^{R+j} = \Sigma C_k^{(j)} z^k$ (j an integer) then, the probability $P(n, m \mid s)$ that a particle starting from $n\Delta$ will come to $m\Delta$ after time $t = s\tau$ is equal to $2^{-2R}(-1)^{R+n}\Sigma(j/R)^s C_{R+j}^{(-n)} C_{R+m}^{(j)}$, where the summation is extended over all j such that $|j| \leq R$ Also, if R is even the probability $P'(n, 0 \mid s)$ that the particle starting from $n\Delta$ will come to 0 at $t = s\tau$ for the first time is calculated. For $n = 0$ this gives a solution of the so-called recurrence time problem first studied on simpler models by Smoluchowski. Through a limiting process in which $\tau \to 0$, $\Delta \to 0$, $\Delta^2/2\tau \to D$, $1/R\tau \to \beta$, $n\Delta \to x_0$, $m\Delta \to x$, $s\tau = t$, one is led to fundamental distributions concerning the velocity of a free Brownian particle In particular, $P(n, m \mid s)$ approaches the well-known Ornstein-Uhlenbeck distribution

5. Sampling from Contaminated Distributions. Preliminary report. JOHN W. TUKEY, Princeton University.

A contaminated distribution is a nearly normal distribution in which extreme observations are more frequent than in a normal distribution. By studying the bias and variability of several measures of dispersion when applied to samples from particular one-parameter families of contaminated distributions it is shown that (i) for nearly normal distributions, the mean deviation is often better than the standard deviation; (ii) small changes in the underlying distribution may increase the sampling variance of the standard deviation by a factor of three This suggests that, in a broad class of cases, the mean deviation is safer than the standard deviation when a single dispersion is estimated from a set of data This conclusion need not apply in an analysis of variance situation.

6. On the Class of Functions Defined by the Difference Equation $(x + 1)f(x + 1) = (a + bx)f(x)$. LEO KATZ, Wayne University

The difference equation defines only three discrete functions the binomial, the Poisson and the Pascal functions, the first and third have one parameter (N) slightly generalized. It is shown that the Pascal function with this generalization is identical with the Polya-Eggenburgher distribution, which is a very useful form of the Compound Poisson Law and has been used to explain probability situations involving contagion Areas for all functions in the class are given in terms of existing tables of the incomplete γ and β-functions Observed distributions are fitted by two moments. As Carver (*Handbook of Mathematical Statistics*) pointed out, the advantages of fitting by difference equations are many, not the least is the fact that it is unnecessary to discriminate among the various functions in fitting an observed distribution. The problem of discrimination, posed by Frisch (*Metron*, Vol. 10) and others, may be resolved in terms of the sampling distribution of variances for the Poisson function, since the three functions correspond to situations where the variance is less than, equal to, or greater than the mean, respectively

7. Retention of Decimal Places in Matrix Calculations. FRANKLIN E. SATTERTHWAITE, Aetna Life Insurance Company. (Read by title)

The accumulation of errors in matrix calculations has been studied by the author and others for special types of matrices and for special methods of calculation In the present paper, error formulae are developed for the standard Doolittle and Waugh-Dwyer Compact routines. These formulae do not place any restrictions on the matrices involved and do not require any extra calculations or initial approximations. Simple rules are developed which give for each step in the calculations the number of decimal places which must be retained. These rules are efficient in the sense that the retention of fewer places will,

except for good fortune in balancing of errors, lead to results less accurate than those specified The rules also assist in choosing that arrangement of the calculations which will lead to the smallest average number of significant figures which must be retained for the calculation as a whole.

8. The Efficiency of the Mean Moving Range. PAUL G. HOEL, University of California at Los Angeles. (Read by title)

The statistic $w = \sum_1^{n-1} | x_{i+1} - x_i | \sqrt{\pi}/2(n-1)$ is studied as an estimate of σ for a normal variable subject to trend effects. It is shown that the efficiency of w compares favorably with that of the mean square successive difference, δ^2. The proof that w, and also δ^2, is asymptotically normally distributed is made to depend upon a general result that can be derived from a theorem of S. Bernstein on dependent variables.

9. Some Basic Theorems for Developing Tests of Fit for the Case of the Non-Parametic Probability Distribution Function. BRADFORD F. KIMBALL, New York State Department of Public Service. (Read by title)

Given a universe with C D.F. $P[X \leq x] = F(x)$. Consider a random sample of n values x_i which have been ordered so that $x_i \leq x_{i+1}$. The successive differences of the true c.d.f. values at $X = x_i$ are denoted by u_i. Thus

$$u_1 = f(x_1)$$

$$u_i = F(x_i) - F(x_{i-1}), 2 \leq i \leq n$$

$$u_{n+1} = 1 - F(x_n).$$

THEOREM 1. *The product power moments*

$$E(u_r^p u_s^q u_t^w \cdots)$$

for any or all different indices from 1 to $n + 1$, where the powers are real numbers greater than minus one, are given by

$$E(u_r^p u_s^q u_t^w \cdots) = \frac{\Gamma(n+1)\,\Gamma(p+1)\,\Gamma(q+1)\,\Gamma(w+1)\cdots}{\Gamma(n+1+p+q+w+\cdots)}$$

COROLLARY. *If a range $R(k, m)$ is defined by*

$$R(0, m) = F(x_m),\ R(n+1, m) = 1 - F(x_{n+1-m})$$

$$R(k, m) = F(X_{k+m}) - F(x_k)$$

where k and m are positive integers such that $m \leq n$ and $k + m \leq n$, its probability distribution is independent of k, and hence equal to that of $F(x_m)$.

THEOREM 2. *Given a test funtion of u_i*

$$Y = \sum_m u_i^p$$

where p is a real positive number, and the sum is for m indices chosen at random on the range 1 to $n + 1$. Let $\bar{Y}$ and σ^2 denote the mean and variance of this test function. Establish a convention for increasing the indices included in the above sum for increasing m as n increases, such that $[m/(n + 1)] =$ constant, to nearest multiple of $1/(n + 1)$. Then the asymptotic distribution of $(Y - \bar{Y})/\sigma$ for inreasing n, subject to the above condition, is the normal distribution with zero mean and unit variance, except in the trivial case $m = n + 1$, $p = 1$.

10. Confidence Limits for the Fraction of a Normal Population which Lies between Two Given Limits. Jacob Wolfowitz, Columbia University. (Read by title)

Let $x_1, \cdots, x_N$ be N independent observations from a normal population with mean μ and variance σ^2, both unknown. Let $N\bar{x} = \Sigma x_i$ and $(N-1)s^2 = \Sigma(x_i - \bar{x})^2$ define $\bar{x}$ and s^2 Let L_1 and L_2 be given constants with $L_1 < L_2$, and let

$$\gamma = (\sqrt{2\pi}\sigma)^{-1} \int_{L_1}^{L_2} \exp -\tfrac{1}{2}\left\{\frac{y-\mu}{\sigma}\right\}^2 dy$$

By a lower confidence limit on γ with confidence coefficient α is meant a function $D(x_1, \cdots, x_N)$ such that the probability is α that $D \leq \gamma$ Since $\bar{x}$ and s^2 are sufficient estimates of μ and σ^2 the restriction that D be a function of $\bar{x}$ and s only is imposed It is assumed that there exist a) a positive d such that $L_1 + d < \mu < L_2 - d$; b) a positive C such that $\sigma < C$. From these it follows that there exists a lower bound $G = G(d, C)$ on γ. Let $\chi^2_{1-\alpha}$ be that number for which $P\{\chi^2 < \chi^2_{1-\alpha}\} = 1 - \alpha$, where χ^2 has $N - 1$ degrees of freedom, and let $w = \dfrac{\sqrt{N-1}s}{\chi_{1-\alpha}}$ It is shown that if D be defined as follows:

1) if $L_1 \leq \bar{x} \leq L_2$,

$$D = (2\pi)^{-\frac{1}{2}} \int_{L_1 - \bar{x}/w}^{L_2 - \bar{x}/w} \exp\ \{-\tfrac{1}{2}y^2\}\, dy$$

2) $D = G$ otherwise, then $| P\{D \leq \gamma\} - \alpha |$ approaches zero as $N \to \infty$. Thus D is a large sample lower confidence limit. The extension to upper and two-sided limits presents no difficulty.

11. The Consolidated Doolittle Technique. Paul Boschan, Econometric Institute (Read by title)

The quadratic matrix notation is interpreted as a segment in a sequence of matrices wherein each successor matrix is augmented by a bordering row and column. Extension theorems based on this idea date back into the last century. The step from the original concept to one of higher order is also fruitful in discussing inverse matrices, specifically the inverse of a symmetric matrix The symmetry of the matrix of normal equations for a set of multiple regression coefficients is restored by adding the transpose of the column on the right side of the equations, i e. the co-variances with the dependent variable and the variance of the dependent variable itself. The inverse of this matrix can be constructed as partial sum over a series of matrices. Each individual element of this series is in itself meaningful. The solution for the set of multiple regression coefficients relating the k-th variable to the preceding $(k - 1)$ variables is a column matrix. The product of this matrix with its transpose expressed in terms of the residual variance forms the k-th term in the matrix series. The summation of the first n products yields the inverse matrix. This characteristic of the inverse can be used to great advantage in the standardization of elementary computational steps

12. Estimation of Structural Equations through Linear Transformation of Regression Coefficients. Theodore W. Anderson and Herman Rubin, Cowles Commission for Research in Economics.

A method is presented for estimating the coefficients of a single structural equation in a system $By'_t + \Gamma z'_t = u'_t$ $(t = 1, 2, \cdots, T)$, where B and Γ are matrices of coefficients, y_t is a row vector of G observed jointly dependent variables, z_t of K observed predetermined

variables and u_t of G random elements. Given the distribution of the random elements, the equations define the distribution of the y_t. Some coordinates of z_t may be coordinates of y_{t-1}, etc. It is assumed that the structural equation to be estimated has at least $G - 1$ coefficients prescribed zero. The part of the population regression matrix corresponding to the predetermined variables with zero coefficients has rank one less than the number of jointly dependent variables with non-zero coefficients. The maximum likelihood estimate of this matrix is a linear transformation of the unrestricted sample regression matrix. The estimated vector of coefficients of y_t is the vector annihilated by this matrix. The vector of coefficients of z_t is estimated by means of this vector and the regression matrix. These estimates are consistent and asymptotically normally distributed. For z_t fixed, small sample confidence regions are given for the coefficients.

NEWS AND NOTICES

Readers are invited to submit to the Secretary of the Institute news items of interest

Personal Items

Dr. Armen A. Alchian, who has been discharged from the Army with the rank of Captain, is now an Assistant Professor in the Economics Dept at the University of California at Los Angeles.

Dr. Franz L. Alt is now Assistant Director of Research at the Econometric Institute.

Colonel Dinsmore Alter is on terminal leave after more than four years' service in the Transportation Corps of the Army, and has returned to his duties as Director of the Griffith Observatory in Los Angeles. During these years Colonel Alter traveled approximately 250,000 miles on the ocean as a Transport Commander, visiting each continent except the Antarctic

Dr Theodore W. Anderson, formerly with the Cowles Commission, is now an Instructor in the Dept of Math. Statistics at Columbia University, and plans to be on a Guggenheim Fellowship beginning in June 1947.

Mr Herbert Barkan has been appointed to an Instructorship in the Newark College of Engineering

Mr. Robert E. Bechhofer, formerly a statistician with The Kellex Corporation, is a graduate student at Columbia University this year.

Mr. Stanley G. Behrends is now Cost Accountant with the California Wire Cloth Corporation, in Oakland.

Messrs. Carl A. Bennett, Jack I. Northam, and Max A. Woodbury have all returned from various types of war service to the University of Michigan as graduate students in statistics. Mr. Bennett was with the Manhattan Engineering District for over two years, first at the Metallurgical Lab., University of Chicago, and then at Oak Ridge, Tenn. Mr. Northam was recently discharged from the Army with the rank of Lieutenant, having served with the Signal Corps for four years in the Pacific area. Mr. Woodbury was discharged from the Army with the rank of Captain, having been in the Meteorology service for five years, most of which time was spent in the European theater.

Mr. Richard Berger has received his discharge from the Navy, and is employed as a Research Analyst with Dun and Bradstreet.

Dr. Archie Blake, formerly at Aberdeen Proving Ground, is now Senior Statistician in the Office of the Army Surgeon General

Dr. Ernest E. Blanche, who had been teaching in one of the European Army University Centers, is now Principal Administrative Analyst in the Plans and Policy Office of the War Department General Staff, and is also Lecturer at American University.

Mr Royal F. Bloom has resigned the position which he held for a short time with the Psychology Dept. of Iowa State College after his release from the

Navy last March, and has returned to the Navy Department as Assistant Head of the Classification Research Division, Bureau of Naval Personnel.

Mr Earl K. Bowen has been appointed to an Instructorship in statistics at Babson Institute of Business Administration

Mr Albert H. Bowker is enrolled this year as a graduate student at the University of North Carolina.

Mr. Charles R. Brearty has joined the Technical Staff of Bell Telephone Laboratories, Inc.

Mr Clyde A. Bridger is on leave from his position at the University of Utah, and is spending the year at the Institute of Statistics in Raleigh, North Carolina.

Mr. Arthur W Brown, formerly with the Columbia University Division of War Research, is now with the Standard Oil Company of New Jersey.

Dr George W. Brown, formerly connected with the RCA Laboratories at Princeton as Research Engineer, has accepted a position as Research Associate Professor in the Statistical Laboratory at Iowa State College.

Mr. Richard H. Brown has been appointed to a Lectureship in Mathematics at Columbia University.

Dr Richard S. Burington, Director of the Evaluation and Analysis Groups of the Research and Development Division of the Bureau of Ordnance, Navy Department, has been named Chief Mathematician, Bureau of Ordnance.

Mr. Roy A. Chapman, who has been Silviculturist at the Hitchiti Experimental Forest, Round Oak, Georgia, is now with the U S. Forest Service in Washington, D. C.

Dr. Way Ming Chen has been appointed to an Instructorship in mathematics at Brown University.

Dr John M. Clarkson has been promoted to a professorship at North Carolina State College.

Mr. S. Lee Crump has been promoted to an Assistant Professorship at Iowa State College.

Dr. Joseph F. Daly, formerly an Instructor at Catholic University, and more recently a Lieutenant in the Navy Department, is now Statistician with the Bureau of the Census.

Dr. Daniel B. DeLury has been promoted to a professorship in statistics at Virginia Polytechnic Institute.

Dr. Acheson J. Duncan has been appointed to an associate professorship of political economy at The Johns Hopkins University.

Dr Jack W. Dunlap, formerly at Rochester University and more recently a Lieutenant Commander in the U. S. Navy, is now Director of the Division of Biomechanics of the Psychological Corporation.

Mr. Francis B. Elmore has been discharged from the Army and is Quality Control Engineer at the Union Bag and Paper Company, in Savannah, Ga.

Mr. Mark W. Eudey has returned from service to his former position with the Statistical Laboratory at the University of California.

Mr. Charles D Ferris, formerly at Aberdeen Proving Ground, is now Quality Control Engineer with the General Electric Company, in Bridgeport, Conn.

Mr Lester R. Frankel is now a statistician with Dun and Bradstreet

Mr John E. Freund has accepted a position as assistant professor of mathematics at Alfred University.

Dr Bernard Friedman has been promoted to an assistant professorship at New York University.

Dr. Milton Friedman has been appointed to an associate professorship in the Department of Economics at the University of Chicago.

Mr. G Rupert Gause is now with the Technical Staff of the Bell Telephone Laboratories

Professor Edwin L Godfrey has been appointed Head of the Department of Mathematics and Astronomy at Defiance College

Dr Casper Goffman has been appointed to an assistant professorship in the Department of Mathematics at the University of Kentucky

Mr Harry H. Goode is now a Mathematician in the Office of Research and Inventions, U. S. Navy

Mr. Robert D Gordon is a Teaching Assistant in Mathematics at Indiana University.

Mr Bert A Gottfried has returned from the service and is Research Analyst with Dun and Bradstreet.

Dr. Clyde H. Graves, formerly at Pennsylvania State College, is now Operations Branch Chief of the Office of Price Board Management, OPA.

Dr. Joseph A Greenwood has recently been separated from active duty with the Navy and is now a statistician in the Bureau of Aeronautics.

Mr. Harris T Guard has returned to Colorado A. and M as an Instructor in the Department of Mathematics.

Dr Joy P. Guilford has returned to his former position as Professor of Psychology at the University of Southern California

Prof. Emil J. Gumbel, formerly with the New School of Social Research, has been appointed to a Special Lectureship in Statistics at Newark College of Engineering

Mr. Lee S. Gunlogson has been discharged from the Navy and is now in the statistical department of the Lumbermens Mutual Casualty Company, Chicago

Dr Paul R. Halmos has been appointed to an assistant professorship in mathematics at the University of Chicago.

Professor Preston C. Hammer has returned to his former position at Oregon State College.

Mr. Joseph O. Harrison, Jr. is now employed as a mathematician for the Harvard University Automatic Sequence Controlled Calculator Project in Cruft Laboratory

Mr. Millard Hastay, formerly with the Statistical Research Group at Columbia University, is now Research Associate at the National Bureau of Economic Research.

Mr. Bernard Hecht has been promoted from Chief Quality Control Engineer to Manager of the Quality Control Department of the International Resistance Company, Philadelphia.

Mr. Joseph L. Hodges, Jr. has been appointed to a teaching assistantship in mathematics at the University of California

Dr. Paul G. Hoel has been promoted to an associate professorship in mathematics at the University of California at Los Angeles.

Mr Richard A. Hornseth has been appointed to an instructorship in the Department of Sociology and Anthropology at the University of Wisconsin.

Mr. Harry M. Hughes has been appointed to a teaching assistantship at the University of California.

Mr. Leonid Hurwicz, formerly with the Cowles Commission, has been appointed to an associate professorship at Iowa State College.

Mr. Joseph B. Jeming has been separated from service with the Air Forces, and is now a Financial and Economic Consultant in New York City.

Mr Paul Johner has been discharged from the Army and is now in the Industrial Engineering Division of the Aluminum Company of America, New Kensington, Pa.

Miss Margaret Kampschaefer, who is a statistician in the War Department, is now serving in the Supply Division of the Air Force Service Command in Erlangen, Germany.

Dr. Leo Katz has been appointed to an assistant professorship at Michigan State College.

Mr. Frederick G. King has been discharged from the Army and is now a civilian instructor in the Anti-Aircraft Artillery School at Fort Bliss.

Dr. Tjalling Koopmans has been appointed Associate Professor of Economics at the University of Chicago.

Mr. Paul J. Kopp has been discharged from the Army and is now with the Patent Department of the Gulf Oil Corporation, Washington, D. C.

Dr. Carl F. Kossack has accepted a position as mathematician with the Joint Army-Navy Air Intelligence in the Strategic Vulnerability Branch.

Dr Waclaw Kozakiewicz has been promoted to an assistant professorship in mathematics at the University of Saskatchewan.

Professor Rafael Laguardia has returned to Uruguay as Director of the Institute de Matematica y Estadistica, Facultad de Ingenieria.

Dr. Charles R. Langmuir, formerly with the Psychological Corporation, is now Secretary-Treasurer and Lab. Director of the Bennett and Langmuir Development Corporation, Mamaroneck, N. Y

Mr. Charles M. Larson has accepted a position as mathematician with the Pacific Mutual Life Insurance Company, Los Angeles.

Miss Lucy A. LaSala, formerly with the research group at Columbia University, is now teacher of mathematics at East New York Vocational High School.

Dr. Richard A. Leibler is now a Member of the Institute for Advanced Study, Princeton.

Miss Grace L Lesser, formerly with the research group at Columbia University, is now employed as a statistician with the Econometric Institute.

Miss Myra Levine has accepted a position as statistician with the Socony-Vacuum Oil Company, in New York City.

Dr Jerome C R. Li has been appointed to an instructorship at Oregon State College.

Professor William T. Martin has accepted a professorship in the Department of Mathematics at Massachusetts Institute of Technology.

Miss Ethelyne L. McBee, formerly with the U. S. Department of Agriculture, is now teaching science and mathematics at the Falls Church High School, Falls Church, Virginia

Dr. Paul W. McGann has been appointed to an assistant professorship in economics at American University.

Dr Max F. Millikan has been appointed to a research associateship at Yale University

Mr. Probodh C. Mittra has accepted a position as consulting statistician with the United Nations Economic and Social Council.

Dr. Marjorie E Moore has transferred from her position as statistician with the Social Security Administration, to one as Program Analyst in the Office of Vocational Rehabilitation, Federal Security Agency.

Miss Judith Moss, who was with the research group at Columbia University, is now research assistant with the National Bureau of Economic Research

Dr Frederick Mosteller has been appointed to a lectureship and research associateship in the Department of Social Relations at Harvard University.

Mr. James E. Myers, formerly with the Naval Research Laboratory at Anacostia Station, is now with the research group of the Moore School of Electrical Engineering, University of Pennsylvania

Mr. Stanley W Nash is a graduate student this year at the University of California

Professor J. Neyman is on leave from his position at the University of California for the fall semester, and is Visiting Professor of Mathematical Statistics at Columbia University

Mr. Russell T. Nichols has been discharged from the Army, and is a graduate student at the University of Chicago

Mr Harold Nisselson has been discharged from the Navy, and is now a statistician in the Bureau of the Census, where he was formerly employed.

Professor Nilan Norris has been separated from his service with the Army, with the rank of Major, and has returned to his position in the Department of Economics at Hunter College

Dr. Guy H. Orcutt, formerly at Massachusetts Institute of Technology, has accepted a research position in the Department of Applied Economics, Cambridge University. This new department is to be modelled somewhat along the lines of the Cowles Commission at the University of Chicago, and is to be under the direction of Dr. J. R N. Stone.

Mr. Warren H Page has been separated from service with the Army, and is now a graduate student at Columbia University.

Mr. Nicholas Pastore has been appointed to an instructorship at Union Junior College, Cranford, New Jersey.

Mr. I. B. Perrott has been demobilized from the British Army with the rank of Major.

Mr George W Petrie, III has accepted a position as Special Engineer with the Bethelehem Steel Company.

Dr. Harry S. Pollard has been promoted to a professorship at Miami University.

Dr. G. Baley Price, Professor of Mathematics at the University of Kansas, has been awarded a Post-Service Guggenheim Fellowship, beginning September 1, 1946.

Mr. Robert J Randall has been discharged from the Army and is now a graduate student at Columbia University

Professor Lowell J. Reed, of the School of Hygiene and Public Health, The Johns Hopkins University, has been appointed Vice-President of the University.

Dr. Francis Regan has been promoted to a professorship at St. Louis University.

Mrs. Kathryn B. Rolfe, formerly at the University of California at Berkeley, has accepted a position as associate in mathematics at the University of California College of Agriculture, at Davis.

Mr Frank Saidel is a graduate student in mathematical statistics this year at Columbia University.

Dr. Leonard J. Savage has been awarded a Special Rockefeller Fellowship, beginning September 1946.

Professor Henry Scheffé of the University of California at Los Angeles has been awarded a Guggenheim Fellowship, and is spending the year at the University of California at Berkeley

Professor Andrew S. Schultz, Jr. has been separated from service with the Army and has returned to Cornell University with the rank of associate professor.

Dr Saul B. Sells, formerly with the OPA, has accepted a position as Assistant to the President of the A. B. Frank Company, San Antonio.

Mr Lawrence W. Shaw is now a statistician with the U S. Public Health Service in Bethesda.

Dr. Ronald W. Shephard has been appointed to a lectureship at the University of California, Berkeley.

Mr Clifford R. Simms has accepted a position as manager of the Cleveland office of the B. E Wyatt Company.

Mr George B Simon has been separated from Army service with the rank of major, and has accepted a civilian position as chief of the Analysis and Research Unit, Psychological Section, Office of Surgeon, Barksdale Field.

Mr. Herbert Solomon has been appointed to an instructorship at the College of the City of New York.

Mr. Melvin D. Springer has returned to the University of Illinois and has been appointed to an assistantship.

Mr. Andrew P. Stergion has been discharged from the Army, and is now Statistical and Quality Control Engineer with the Corning Glass Works.

Mr Milton S. Stevens has been discharged from the Navy, and has accepted a position as Director of Special Projects with Time, Inc

Dr. George J. Stigler has been appointed to a professorship in economics at Brown University.

Mr Alexander L Stott has been discharged from the Navy, and is now a staff assistant in the Treasury Department of the American Telephone and Telegraph Company

Dr. L V Toralballa has accepted a teaching position at Fordham University

Dr. Walter R. Van Voorhis has returned to Fenn College, with the rank of associate professor.

Mr. Edward H Van Winkle has been appointed to a professorship of business statistics at Rensselaer Polytechnic Institute

Dr Charles W. Vickery has been appointed to an associate professorship at Ohio State University.

Mr. David F Votaw, Jr. has been separated from service with the Navy and has returned to Princeton University as Research Associate

Mr. W. Allen Wallis has been appointed to a professorship at the University of Chicago.

Mr. Ralph E. Wareham is now managing director of the National Photocolor Corporation

Dr Jacob Wolfowitz has been appointed to an associate professorship in mathematical statistics at Columbia University.

Mr John F. Wyckoff, formerly at Trinity College, has accepted a position in the Research Division of the Actuarial Department, Connecticut General Life Insurance Company, Hartford.

Mr. Earl K. Yost, Jr. has been appointed to a graduate assistantship in mathematics at the University of Oregon.

A conference on applied mathematical statistics was held at Lake Junaluska, North Carolina, August 4–9, 1946 under the sponsorship of the Institute of Statistics of the University of North Carolina The following individuals attended the conference: C I. Bliss, W. G Cochran, Gertrude M. Cox, D. B. Duncan, C Eisenhart, R. A Fisher, Carl F Kossack, Frederick Mosteller, H. W Norton, Paul Peach, Charles F. Roos, Walter A. Shewhart, Frederick Stephan, Gerhard Tintner, John W. Tukey, S. S. Wilks, C. P. Winsor, and J. Wolfowitz

Newark College of Engineering is sponsoring a series of conferences on Industrial Statistics. The first of these, on Acceptance Sampling, began on September 27 and ran for eleven four-hour Friday sessions. Among the members of the Advisory Panel on Industrial Statistics are Institute members S. B. Littauer, A. I. Peterson, W. A Shewhart, and S. S Wilks.

New Members

The following persons have been elected to membership in the Institute:

Anscombe, F. J. Rothamsted Experimental Station, Harpenden, Herts, Eng.

Back, Kurt W., M.A (California at L. A.) Stat , Surveillance Branch, Ballistic Res Lab , Aberdeen Proving Gd., Md.

Bresnahan, Maurice F. Stat., U. S. Bur. of Labor Statistics, Wash., D. C., *Apt. 305, 1015 N St , N.W., Wash. 1*

Chung, Kai-Lai, M.A. (Princeton) Graduate Coll., Princeton Univ., Princeton, N. J.

Clarke, P. C. Asst Gen. Mgr., Hunter Pressed Steel Co., Lansdale, Pa., *Line Lexington, Pa.*

Coon, Helen J., M.A. (Southern Methodist) Ballistic Res. Lab., Aberdeen Proving Gd., Md.

Copp, Warren F., B.S. (Ohio State) Supv., Quality Control Dept., Wheeling Steel Corp., Yorkville Works, Yorkville, Ohio

Divatia, Vasishtha V., B.Sc. (Bombay) Student in Math. Stat., Columbia Univ. *#724 John Jay Hall, Columbia Univ., N. Y. City*

Fanshaw, Hugh L., M.S (Manitoba) Standards Supv , Canadian Indus. Ltd., General Chemicals Div., Hamilton, Ont., Can., *120 St. Clair Ave.*

Feriet, Kampe de, Dr Sci. (Paris) Professeur a la Faculte des Sci. de l'Universite de Lille, 16 rue des Jardins, Lille, France

Fine, Clarence B., B.S.S. (C C.N.Y.) Economist, OPA, Wash., D. C., *1388 Tuckerman St., N.W., Wash. 11*

Golub, Abraham, B.A. (Brooklyn) Math., Ballistic Res. Lab., Aberdeen Proving Gd., Md , *Men's Dorm*

Gomberg, William, Ph.D. (Columbia) Dir. of Mgt., Engr. Dept , International Ladies Garment Workers Union, 1710 Broadway, N. Y., N. Y., *444 Beach 142nd St., Neponsit, L I.*

Halton, Frederick J., Jr. Asst. to Pres., John Deere & Co , 230 S. Clark St., Chicago, Ill., *1314 Westview Rd., Highland Park*

Hanson, Robert H., M.S. (Iowa) Stat., Bur. of the Census, Wash., D. C., *3148 Westover Dr., Wash. 20*

Hardy, Philip H., B.S. (Rice) Quality Engr., General Elec. Co.; on leave, Cpl. US Army, 4000 BU Sq. S., Wright Field, Dayton, Ohio

Hasty, Willis L., Jr , B.C.S. (Benjamin Franklin) Capt., Signal Corps, *3475 South Wakefield St., Arlington, Va.*

Hess, Ida I., A.B. (Indiana) Stat., Population Div., Bur. of the Census, Wash., D. C., *1425 Rhode Island Ave., N.W.*

Jacobson, Jack J., M.B.A. (Chicago) Stat., Spiegel, Inc., Chicago, Ill., *3562 W. Palmer St.*

Janko, Prof. Jaroslav, Technical Univ , Prague, Czechoslovakia, *Na bojisti 3, Praha II*

Karp, Abraham E., M S. (C C.N.Y.) Stat., Aberdeen Proving Gd , Md., *25 Aberdeen Ave.*

Keefe, David P., B.S. (St. Thomas) Supv., Raw Material Testing, Minn. Mining and Mfg. Co., St. Paul 6, Minn., *590 Holly Ave., St Paul*

Kellogg, Lester S., M.A. (Northwestern) Chief, Prices and Cost of Living Branch, Bur. of Labor Statistics, U. S. Dept. of Labor, Wash., D. C., *404 Shady Lane, Falls Church, Va.*

Kindig, Fred E., B.S. (Pennsylvania State) Industrial Math., Westinghouse Elec. Co., Braddock Ave., E. Pittsburgh, Pa., *53 Nantucket Dr., R.D. 6, Pleasant Hills, Pittsburgh 10*

Kosambi, D. D., ScB. (Harvard) Dept. of Math., Tata Inst. of Fundamental Res., 53 Pedder Rd., Bombay, India

Kruskal, William, M.S. (Harvard) *150 W. 80th St., N. Y., N. Y.*

Lacey, Prof. Oliver L., Ph.D. (Cornell) Head, Dept. of Psychology, Univ. of Ala., University, Ala.

Martin, Cyrus A., B.S. (Iowa State) Stat, Headquarters Fifth Army, *7817 South Shore Dr., Chicago 49, Ill.*

Maxwell, Pat, Jr., B S (Fordham) Stat, Surveillance Br., Ballistic Res. Lab, Aberdeen Proving Gd, Md, *Apt. A6-3, Baldwin Manor*

Mills, Vicente, P O. Box 2090, Manila, Philippine Islands

Newman, Doris, M.A (Michigan) Instr. in Stat., School of Bus. Adm, Crosby Hall, Univ. of Buffalo, Buffalo 14, N Y

Ott, Asso. Prof. Ellis R., Ph D (Illinois) Dept. of Math., Rutgers Univ, New Brunswick, N J, *199 Sterling Dr., Orange*

Paull, Allan E., B A (Manitoba) 114 Mangum, U of N C, Chapel Hill, N. C.

Pollard, A. H. The Mutual Life & Citizens' Ass. Co Ltd, Martin Place and Castlereath St., Sydney, N S W, Australia

Quenouille, Maurice H., B A. (Cambridge) Stat., Rothamsted Experimental Station, Harpenden, Herts, Eng, *8 Chipstead St, Fulham, London, S W. 6*

Ryan, Asso. Prof. Thomas A., Ph D. (Cornell) Dept of Psychology, Morrill Hall, Cornell Univ, Ithaca, N. Y

Sachs, Rose, B.A. (Goucher) Stat., *1428 R St, N W., Wash. 9, D. C*

Sandelius, D. Martin, Fil kand. (Stockholm) Baltzar von Platensg. 5, VI, Stockholm, Sweden

Schoenbaum, Prof. Emil, Ph D (Prague) Dept. of Applied Math, School of Natural Science, Univ of Charles IV, Prague, Czechoslovakia, *Praha XIX, Terronska 26*

Schweitzer, Morton D., Ph D. (Columbia) Stat. and econ consultant, 60 West 10th St., N. Y. 11, N. Y

Shybekay, Derso S., Ph D. (Budapest) Pres, Indus. Res. Council, P. O. Box 324, Minneapolis 1, Minn

Stein, Irving, B.S. (Mass Inst. Tech.) Qual. Control Engr, Polaroid, Inc., *10 Forest Park Ave., Adams, Mass*

Stephens, William H., M.Sc. (Queen's) British Commonwealth Scientific Office, *1785 Massachusetts Ave., Wash., D. C, 3807 Military Rd., N W.*

Strieby, J. Glenn, B S (Iowa Wesleyan) Tech Dept, Kimberly-Clark Corp., Neenah, Wis., *616 E. Circle St, Appleton*

Thompson, Juanita, Qual. Control Inspector, *4117 West 25th St, Chicago 23, Ill.*

Thompson, Sidney L., M S (Tulane) Ala. Polytechnic Inst., Auburn, Ala.

Tompkins, Haldan M. Head, Works Control Lab, National Carbon Co, Fostoria, Ohio, *P O Drawer 191*

Tosti, Carlo R., M A (Stanford) Capt., AC, Adm. Engr., Tech. Control Officer, Wright Field, Dayton, Ohio, *501 Forest Ave., Dayton 5*

Toulouse, Julian H., Ph D. (Iowa State) Chief Engr., Quality and Specifications Sec, Owens-Ill. Glass Co, Toledo, Ohio, *P. O. Box 1035–1036, Toledo 1*

Truksa, Ladislav, Ph D (Charles Univ.) Dir., Lecturer in Math. Stat., Charles Univ., Praha XI-1800, Czechoslovakia

Urie, Frank D., A.B. (Michigan) Supt of Inspection, Elgin National Watch Co., Elgin, Ill.

Vajda, Stefan, Ph D. (Vienna) Stat., Admiralty, Whitehall, London, Eng., *54 Chapel Way, Epsom, Surrey*

Weiss, Lionel, M A (Columbia) Gottsberger Fellow, Columbia Univ., N. Y., N. Y., *2205 Davidson Ave., N. Y. 53*

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REPORT ON THE ITHACA MEETING OF THE INSTITUTE

The Ninth Summer Meeting of the Institute of Mathematical Statistics was held at Cornell University, Ithaca, New York, on Thursday, August 21, and Saturday, August 23, 1946. The meeting was held in conjunction with the summer meetings of the American Mathematical Society and the Mathematical Association of America. The following 71 members of the Institute attended the meeting:

P. L. Alger, C. B Allendoerfer, T. W. Anderson, Jr., J L. Barnes, E. E. Blanche, Paul Boschan, A H. Bowker, A. E. Brandt, R S Burington, W. G. Cochran, E P. Coleman, H. B. Curry, J. H Curtiss, J. L. Doob, J Dutka, P. S Dwyer, B Epstein, Will Feller, C. D Ferris, R. M Foster, J. E. Freund, M. A Girschick, A. A Goodman, Louis Guttman, W. W. Gutzman, P R Halmos, T. E Harris, Bertha I. Hart, E. H C. Hildebrandt, P. G. Hoel, R. H. Hoskins, Harold Hotelling, W W. Jacobs, T. J. Jaramillo, Evan Johnson, Jr., H. L. Jones, Mark Kac, Irving Kaplansky, Leo Kart, Tjalling Koopmans, C. F. Kossack, M. M. Lavin, Walter Leighton, Jr., Howard Levene, M. S Macphail, J. W. Mauchly, P. J. McCarthy, E. C. Molina, Margaret E. Moore, J. E. Morton, L F. Nanni, P. M. Neurath, E. G. Olds, G. B Price, C. J. Rees, Selby Robinson, Herman Rubin, P J. Rulon, Arthur Sard, F. E. Satterthwaite, I. E. Segal, G. R. Seth, Andrew Sobczyk, Herbert Solomon, C M. Stein, F. F. Stephan, A. P. Stergion, A. W. Tucker, J. W. Tukey, J L. Ullman, Abraham Wald, S S. Wilks.

The first session, a joint session with the American Mathematical Society, was held on Thursday morning, and was devoted to contributed papers. Professor W. G. Cochran, President of the Institute, presided. The following seven papers were presented:

1. *A Test of Randomness in Two Dimensions.*
   Mr. Howard Levene, Columbia University.
2. *Asymptotic Distribution of Moments from a System of Linear Stochastic Difference Equations*
   Mr. Herman Rubin, Cowles Commission for Research in Economics.
3. *Conditional Expectation and Unbiased Sequential Estimation.*
   Professor David Blackwell, Howard University.
4. *A Discussion of the Ehrenfest Model* Preliminary report.
   Professor Mark Kac, Cornell University
5. *Sampling from Contaminated Distributions.* Preliminary report.
   Professor John W. Tukey, Princeton University.
6. *On the Class of Functions Defined by the Difference Equation* $(x + 1) f(x + 1) = (a + bx) f(x)$.
   Dr Leo Katz, Wayne University.
7. *Retention of Decimal Places in Matrix Calculations*
   Dr. Franklin E. Satterthwaite, Aetna Life Insurance Company.

The following four papers were presented by title.

8. *The Efficiency of the Mean Moving Range.*
   Professor Paul G. Hoel, University of California at Los Angeles.
9. *Some Basic Theorems for Developing Tests of Fit for the Case of the Non-Parametric Probability Distribution Function.*
   Mr. Bradford F. Kimball, N. Y. State Department of Public Service, New York City.

10 *Confidence Limits for the Fraction of a Normal Population Which Lies Between Two Given Limits*
Professor Jacob Wolfowitz, Columbia University.

11 *The Consolidated Doolittle Technique*
Dr Paul Boschan, The Econometric Institute, Inc.

Abstracts of all these papers appear elsewhere in this issue of the *Annals*.

At two o'clock on Thursday afternoon there was a joint session with the American Mathematical Society which featured the invited address of Professor J. L. Doob of the University of Illinois on *Probability in Function Space.* This address was followed by a business meeting of the Institute which featured reports by the President, the Secretary-Treasurer, the Editor, and Professor Feller, who spoke for the recently created committee on the distribution of the *Annals* in the war areas.

On Thursday evening there was a joint dinner with the American Mathematical Society and the Mathematical Association of America.

The meeting closed with a session on Friday morning devoted to the topic, *Multivariate Analysis for Non-Experimental Data* Professor Will Feller, of Cornell University, presided. Professor T. Koopmans, of the Cowles Commission for Research in Economics, presented a paper entitled *Statistical Inference in Dynamic Economic Models.* Dr T W. Anderson, Jr. presented a paper written by himself and Mr. Herman Rubin entitled *Estimation of Structural Equations through Linear Transformation of Regression Coefficients.* The meeting concluded with a discussion of these papers

P. S. Dwyer,
Secretary.

REPORT OF THE PRINCETON MEETING OF THE INSTITUTE

The twenty-third meeting of the Institute of Mathematical Statistics was held in Princeton, New Jersey on Friday, November 1, 1946, in connection with the year-long Celebration of the Bicentennial of Princeton University. The meeting was devoted entirely to *Analysis of Variance*. The meeting was attended by 118 persons including the following 96 members of the Institute:

Adam Abruzzi, Forman S. Acton, R. L. Anderson, T. W. Anderson, Jr., M. S. Bartlett, Robert Bechofer, Gilbert W. Beebe, J. H. Bigelow, Archie Blake, C I. Bliss, A. E. Brandt, Burton H. Camp, George C. Campbell, A. George Carlton, Kai Lai Chung, W G. Cochran, Gertrude Cox, Harold Cramér, S. Lee Crump, J. H. Curtiss, Joseph F. Daly, Besse B. Day, D B DeLury, V V. Divatia, J. Dutka, Churchill Eisenhart, B. Epstein, H. L. Fanshaw, Nicholas Fattu, Will Feller, Merrill M. Flood, Bernard Friedman, Hilda Geiringer, H. H Goldstine, Joseph A. Greenwood, E. J. Gumbel, Margaret Gurney, L. Gutmann, T. E. Harris, Millard Hastay, Irwin S. Hoffer, C J. Kirchen, B. F. Kimball, Lila F. Knudsen, H. S. Konijn, Jack Laderman, J. D Maddrill, Sophie Marcuse, H. C. Mathisen, J. W. Mauchly, Margaret Merrell, Elmer B Mode, Margaret E. Moore, J E. Morton, Judith Moss, F Mosteller, Charles M. Mottley, Ray B. Murphy, P. M. Neurath, Hugo Nilson, Gottfried E Noether, Monroe L. Norden, H. W. Norton, C O. Oakley, P. S. Olmstead, J. G Osborne, Ellis R Ott, C. J Rees, W. A. Reynolds, A. C. Rosander, David Rosenblatt, Ernest Rubin, P. U. Rulon, Frank Saidel, Marian M. Sandomire, Walter A. Shewhart, James G. Smith, Milton Sobel, Herbert Solomon, Mortimer Spiegelmon, Charles M. Stein, G. R. Stibitz, John R. Tomlinson, Marion M. Torrey, John W. Tukey, D. F. Votaw, Jr., F. M. Wadley, Alton J. Wadman, A. Wald, Robert M. Walter, Lionel Weiss, Frank Wilcoxon, S. S. Wilks, C. P. Winsor, J. Wolfowitz, and W. J. Youden.

At the morning session the following program was presented with Professor S. S. Wilks of Princeton University as chairman:

Topic: *Mathematical Approaches to the Analysis of Variance*

Papers: *Two Probability Models for the Analysis of Variance*
Professor A. Wald, Columbia University
Applications of Analysis of Variance
Professor M. S. Bartlett, Cambridge University and The University of North Carolina

Discussion: Professor S. L. Crump, Iowa State College
Dr. J. F. Daly, Bureau of the Census
Professor J. W. Tukey, Princeton University
Professor C. P. Winsor, Johns Hopkins University
Professor J Wolfowitz, Columbia University

The program for the afternoon session, under the chairmanship of Professor Will Feller of Cornell University, was as follows:

Topic: *Multivariate Problems in the Analysis of Variance*

Papers: *Analysis of Covariance*
Professor W. G. Cochran, The University of North Carolina
Vector Methods
Professor J W. Tukey, Princeton University

Discussion. Professor T W Anderson, Columbia University
Professor C I. Bliss, Yale University
Professor Harold Cramér, The University of Stockholm and Princeton University
Professor D. B. DeLury, Virginia Polytechnic Institute
Professor P L. Hsu, The University of North Carolina

The evening session consisted of round table discussion on *Unsolved Problems of the Analysis of Variance*, with Professor Gertrude M. Cox as chairman.

Members of the Institute and others who attended the meeting were guests of the Institute for Advanced Study at tea in Fuld Hall from 4 to 6 P.M. Those attending the evening session were guests of Princeton members of the Institute for refreshments in Fine Hall from 10 to 11 P.M.

P. S. DWYER,
Secretary.

MEMBERS OF THE INSTITUTE OF MATHEMATICAL STATISTICS*

(As of October 1, 1946)

(The names of Fellows of the Institute are designated by * and Life Members by †)

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